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ON A NEW SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY RAFID-OPERATOR

R. VIJAYA[1] AND T.V. SUDHARSAN[2]

[1]Department of Mathematics, S.D.N.B. Vaishnav College, Chennai - 600 044

[2]Department of Mathematics, SIVET College, Chennai - 600 073, India

ABSTRACT. In this paper, we define a new subclass of analytic and univalent functions with negative coefficients in the open unit disc $\Delta = \{z : |z| < 1\}$ defined by Hadamard product (or convolution) with Rafid-operator and obtain coefficient bounds. Extreme points, growth and distortion thoerem using fractional calculus techniques and Hadamard product are also obtained for this subclass.

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1. INTRODUCTION

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.1}$$

that are analytic and univalent in the disc $\Delta = \{z : |z| < 1\}$ and normalized by $f(0) = 0$, $f'(0) = 1$.

Let S be the subclass of A containing of analytic and univalent functions of the form (1.1).

Let T be the class of analytic functions with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0), z \in \Delta \tag{1.2}$$

If $f \in T$ is given by (1.2) and $g \in T$ given by

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad (b_n \geq 0), z \in \Delta$$

then the Hadamard product (or convolution) $f * g$ of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z) \tag{1.3}$$

Recently Waggas Galib Atshan and Rafid Habib Buti [7], Shanmugam et al. [3] have studied certain subclasses of the class A using Rafid-operator and investigated certain properties.

We now state the following lemmas used for our study.

Lemma 1.1. *The Rafid-operator of* $f \in T$ *for* $0 \le \theta \le 1$, $0 \le \mu < 1$ *is denoted by* R_μ^θ *and defined as follows*

$$\begin{aligned} R_\mu^\theta(f(z)) &= \frac{1}{(1-\mu)^{(1+\theta)}\Gamma(\theta+1)} \int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt \\ &= z - \sum_{n=2}^{\infty} K(\theta,\mu,n) a_n z^n \end{aligned} \tag{1.4}$$

where $K(\theta,\mu,n) = \frac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)}$.

Lemma 1.2. [1] *Let* $w = u+iv$. *Then* $Re\, w \ge \sigma$ *if and only if* $|w-(1+\sigma)| \le |w+(1-\sigma)|$.

Lemma 1.3. [1] *Let* $w = u+iv$ *and* σ, γ *are real numbers. Then* $Re\, w > \sigma|w-1|+\gamma$ *if and only if* $Re\{w(1+\sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$.

Motivated by the above study on Rafid-operator, in this paper, we define a new subclass $SV(\lambda,\beta,\alpha,\mu,\theta)$ of analytic univalent functions.

2. The Class $SV(\lambda,\beta,\alpha,\mu,\theta)$

Definition 2.1. *A function* $f(z) \in T$, $z \in \Delta$ *is said to be in the class* $SV(\lambda,\beta,\alpha,\mu,\theta)$ *if and only if the following inequality is satisfied.*

$$\begin{aligned} &Re\left\{\frac{z(R_\mu^\theta((f*g)(z)))' + (2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''}{4(\lambda-\lambda^2)z + (2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))' + (2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))} - \alpha\right\} \\ &> \beta\left|\frac{z(R_\mu^\theta((f*g)(z)))' + (2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''}{4(\lambda-\lambda^2)z + (2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))' + (2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))} - 1\right| \end{aligned} \tag{2.1}$$

where $0 \le \alpha < 1$, $\beta \ge 0$, $0 \le \lambda \le 1$, $0 \le \mu < 1$ *and* $0 \le \theta \le 1$.

The main object of this paper is to obtain necessary and sufficient conditions for functions in the class $SV(\lambda,\beta,\alpha,\mu,\theta)$. Further we investigate extreme points and also obtain distortion theorem using fractional calculus techniques and a result on Hadamard product.

3. Coefficient Bounds and Extreme Points

We now obtain a necessary and sufficient condition for functions $f(z)$ to be in the class $SV(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 3.1. *The function $f(z) \in T$ defined by (1.2) is in the class $SV(\lambda,\beta,\alpha,\mu,\theta)$ if and only if*

(3.1)

$$\sum_{n=2}^{\infty}[n(1+\beta)+(2\lambda-1)((n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda))]K(\theta,\mu,n)a_nb_n \leq 1-\alpha$$

where $0 \leq \alpha < 1$*,* $\beta \geq 0$*,* $0 \leq \lambda \leq 1$*,* $0 \leq \mu < 1$ *and* $0 \leq \theta \leq 1$*.*

Proof. If $f(z) \in SV(\lambda,\beta,\alpha,\mu,\theta)$. Then

$$Re\left\{\frac{z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))}\right\}$$

$$\geq \beta\left|\frac{z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))}-1\right|+\alpha$$

Then by Lemma 1.3, we have

$$Re\left\{\frac{[z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''](1+\beta e^{i\phi})}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))}-\beta e^{i\phi}\right\}\geq\alpha,$$

$-\pi \leq \phi \leq \phi$

or equivalently,

(3.2)

$$Re\left\{\frac{\begin{array}{l}[z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''](1+\beta e^{i\phi})\\ -\beta e^{i\phi}[4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))]\end{array}}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))}\right\}\geq\alpha,$$

or, $Re\left\{\frac{F(z)}{E(z)}\right\} \geq \alpha$, where

$$F(z)=\begin{array}{l}[z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_\mu^\theta((f*g)(z)))''](1+\beta e^{i\phi})\\ -\beta e^{i\phi}[4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))]\end{array}$$

and

$$E(z)=4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_\mu^\theta((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_\mu^\theta((f*g)(z)))$$

Using Lemma 1.2, from equation (3.2) we have

$$|F(z)+(1-\alpha)E(z)| \geq |F(z)-(1+\alpha)E(z)|, \quad 0 \leq \alpha < 1$$

Now

$$|F(z)+(1-\alpha)E(z)| \geq (2-\alpha)|z|$$
$$-\sum_{n=2}^{\infty} K(\theta,\mu,n)[n+(2\lambda-1)(n\lambda^2-\alpha\lambda n-\alpha\lambda+\lambda+\alpha-1)]a_n b_n|z|^{n-1}$$
$$-\beta\sum_{n=2}^{\infty} K(\theta,\mu,n)[n+(2\lambda-1)(\lambda n^2-2\lambda n-\lambda+1]a_n b_n|z|^{n-1}$$

Also we have,

$$|F(z)-(1+\alpha)E(z)| \leq \alpha|z|$$
$$+\sum_{n=2}^{\infty} K(\theta,\mu,n)[n+\lambda n(n-1)(2\lambda-1)-(1+\alpha)(2\lambda-1)(\lambda n+\lambda-1)]a_n b_n|z|^{n-1}$$
$$+\beta\sum_{n=2}^{\infty} K(\theta,\mu,n)[n+(2\lambda-1)(\lambda n^2-2\lambda n-\lambda+1]a_n b_n|z|^{n-1}$$

Therefore,

$$|F(z)+(1-\alpha)E(z)|-|F(z)-(1+\alpha)E(z)|$$
$$\geq 2(1-\alpha)|z|-\sum_{n=2}^{\infty} K(\theta,\mu,n)[2n+(2\lambda-1)(2\lambda n^2-2\alpha\lambda n-2\lambda n-2\alpha\lambda+2\alpha)]a_n b_n|z|^{n-1}$$
$$-\beta\sum_{n=2}^{\infty} K(\theta,\mu,n)[2n+(2\lambda-1)(2\lambda n^2-4\lambda n-2\lambda+2]a_n b_n|z|^{n-1}$$
$$\geq 0$$

That is

$$\sum_{n=2}^{\infty} K(\theta,\mu,n)[n(1+\beta)+(2\lambda-1)(\lambda n^2-\alpha\lambda n-\lambda n-\alpha\lambda+\alpha)$$
$$+\beta(\lambda n^2-2\lambda n-\lambda+1)]a_n b_n \leq 1-\alpha$$

which is equivalent to

$$\sum_{n=2}^{\infty}[n(1+\beta)+(2\lambda-1)((n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda))]K(\theta,\mu,n)a_n b_n \leq 1-\alpha$$

Conversely, assume that (3.1) holds. That is

$$\sum_{n=2}^{\infty} K(\theta,\mu,n)[n(1+\beta)+(2\lambda-1)((n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda))]a_n b_n \leq 1-\alpha$$

then we must prove that

$$Re\left\{\begin{array}{c}\frac{\begin{array}{l}[z(R_{\mu}^{\theta}((f*g)(z)))'+(2\lambda^2-\lambda)z^2(R_{\mu}^{\theta}((f*g)(z)))''](1+\beta e^{i\phi})\\ -\beta e^{i\phi}[4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_{\mu}^{\theta}((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_{\mu}^{\theta}((f*g)(z)))]\end{array}}{4(\lambda-\lambda^2)z+(2\lambda^2-\lambda)z(R_{\mu}^{\theta}((f*g)(z)))'+(2\lambda^2-3\lambda+1)(R_{\mu}^{\theta}((f*g)(z)))}\\ \end{array}\right\}\geq\alpha,$$

Now choosing the value of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$Re\left\{\begin{array}{c}\frac{\begin{array}{l}(1-\alpha)-\sum_{n=2}^{\infty}K(\theta,\mu,n)[n(1+\beta e^{i\phi})+(2\lambda-1)(\lambda n^2-\alpha\lambda n-\lambda n-\alpha\lambda+\alpha)\\ +\beta e^{i\phi}(\lambda n^2-2\lambda n-\lambda+1)]a_nb_nz^{n-1}\end{array}}{1-\sum_{n=2}^{\infty}K(\theta,\mu,n)[(2\lambda-1)(n\lambda+\lambda-1)]a_nb_nz^{n-1}}\\ \end{array}\right\}\geq 0$$

Since $Re(-e^{i\phi}) \geq -|e^{i\phi}| = -1$, the above inequality becomes

$$Re\left\{\begin{array}{c}\frac{\begin{array}{l}(1-\alpha)-\sum_{n=2}^{\infty}K(\theta,\mu,n)(n(1+\beta)+(2\lambda-1)\\ \times[(n-1)(\beta\lambda n-\beta\lambda-\lambda n)(\alpha+\beta)(1-\lambda)])a_nb_nr^{n-1}\end{array}}{1-\sum_{n=2}^{\infty}K(\theta,\mu,n)[(2\lambda-1)(n\lambda+\lambda-1)]a_nb_nr^{n-1}}\\ \end{array}\right\}\geq 0$$

Letting $r \to 1^-$, we obtain the required result. □

Corollary 3.2. *Let* $f(z) \in SV(\lambda, \beta, \alpha, \mu, \theta)$*, then*

$$a_n \leq \frac{1-\alpha}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]\,K(\theta,\mu,n)b_n}$$

where $0 \leq \alpha < 1$*,* $\beta \geq 0$*,* $0 \leq \lambda \leq 1$*,* $0 \leq \mu < 1$*,* $0 \leq \theta \leq 1$ *then*

$$f(z) = z - \frac{1-\alpha}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]K(\theta,\mu,n)b_n}z^n$$

for $n = 2, 3, \ldots$.

Theorem 3.3. *Let* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{1-\alpha}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]K(\theta,\mu,n)b_n}z^n$$

where $n \geq 2$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, $\beta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \mu < 1$ *and* $0 \leq \theta \leq 1$. *Then* $f(z) \in SV(\lambda, \beta, \alpha, \mu, \theta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$$

where $\sigma_n \geq 0$ *and* $\sum_{n=1}^{\infty} \sigma_n = 1$.

Proof. Let $f(z) = \sum_{n=1}^{\infty} \sigma_n f_n(z)$, where $\sigma_n \geq 0$ and $\sum_{n=1}^{\infty} \sigma_n = 1$. Then

$$f(z) = z - \sum_{n=2}^{\infty} \frac{(1-\alpha)\sigma_n}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]K(\theta,\mu,n)b_n} z^n$$

That is

$$\begin{aligned} f(z) &= \left[\frac{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]b_n}{(1-\alpha)}\right] \\ &\quad \times \sum_{n=2}^{\infty}\left[\frac{(1-\alpha)}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]b_n}\sigma_n\right] \\ &= \sum_{n=2}^{\infty} \sigma_n \\ &= 1 - \sigma_1 \\ &\leq 1. \end{aligned}$$

Therefore by Theorem 3.1 $f(z) \in SV(\lambda, \beta, \alpha, \mu, \theta)$.

Conversely, suppose that $f(z)$ of the form (1.2) belongs to $SV(\lambda, \beta, \alpha, \mu, \theta)$, then we have

$$a_n \leq \frac{1-\alpha}{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]b_n}, n \in N, n \geq 2.$$

Set

$$\sigma_n = \frac{[n(1+\beta)+(2\lambda-1)[(n-1)(\beta\lambda n-\beta\lambda+\lambda n)+(\alpha+\beta)(1-\lambda)]]a_n b_n}{1-\alpha}$$

and $\sigma_1 = 1 - \sum_{n=2}^{\infty} \sigma_n$.

Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \sigma_n f_n(z) \\ &= \sigma_1 f_1(z) + \sum_{n=2}^{\infty} \sigma_n f_n(z) \end{aligned}$$

This completes proof of Theorem 3.1. □

4. Hadamard Product

Theorem 4.1. *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *and* $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ *belong to* $SV(\lambda, \beta, \alpha, \mu, \theta)$. *Then the Hadamard product of* $f(z)$ *and* $g(z)$ *given by* $(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$ *belongs to* $SV(\lambda, \beta, \alpha, \mu, \theta)$.

Proof. Since $f(z), g(z) \in SV(\lambda, \beta, \alpha, \mu, \theta)$, we have

$$\sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)}{1-\alpha} \right] a_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)}{1-\alpha} \right] b_n \leq 1$$

Applying Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n}$$

$$\leq \sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)a_n}{1-\alpha} \right]^{1/2}$$

$$\times \sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)b_n}{1-\alpha} \right]^{1/2}$$

Now we have to show that

$$\sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n} \leq 1.$$

Since

$$\sum_{n=2}^{\infty} \left[\frac{n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]}{1-\alpha} \right] K(\theta, \mu, n) a_n b_n$$

$$= \sum_{n=2}^{\infty} \left[\frac{[n(1+\beta) + (2\lambda - 1)[(n-1)(\beta\lambda n - \beta\lambda + \lambda n) + (\alpha + \beta)(1-\lambda)]]K(\theta, \mu, n)\sqrt{a_n b_n}}{1-\alpha} \right] \sqrt{a_n b_n}.$$

We obtain the required result. This completes the proof. □

5. Application of the Fractional Calculus

The operators of fractional calculus, such as fractional derivative and fractional integral have been studied by many researchers (see for example [4, 5, 6]). We now state the following definitions as given by Owa [2] for proving our theorems in this section.

Definition 5.1. *The fractional integral operator of order δ is defined for a function $f(z)$ by*

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(t)}{(z-t)^{1-\delta}} dt \quad (\delta > 0) \tag{5.1}$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)$ of $(z-t)^{\delta-1}$ is removed by requiring $log(z-t)$ to be real, when $(z-t) > 0$.

Definition 5.2. *The fractional derivative operator of order δ is defined for a function $f(z)$ by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\delta}} dt \quad (0 \leq \delta < 1) \tag{5.2}$$

where $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin and the multiplicity of $(z-t)^{\delta-1}$ is removed by requiring $log(z-t)$ to be real, when $(z-t) > 0$.

Definition 5.3. *The fractional derivative operator of order $k+\delta$ is defined by*

$$D_z^{k+\delta} f(z) = \frac{d^k}{dz^k} D_z^{\delta} f(z), \quad (0 \leq \delta < 1) \tag{5.3}$$

From Definitions 5.1 and 5.2, we obtain

$$D_z^{-\delta} f(z) = \frac{1}{\Gamma(2+\delta)} z^{\delta+1} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\delta)} a_n z^{n+\delta} \tag{5.4}$$

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} a_n z^{n-\delta} \tag{5.5}$$

Theorem 5.4. *Let $f(z) \in SV(\lambda, \beta, \alpha, \mu, \theta)$. Then*

$$\begin{aligned} |D_z^{-\delta} f(z)| &\leq \frac{1}{\Gamma(2+\delta)} |z|^{\delta+1} \\ &\times \left[1 + \frac{2(1-\alpha)}{(2+\delta)[2(1+\beta) + (2\lambda-1)[\lambda(\beta+2) + (\alpha+\beta)(\lambda-1)]](\theta+1)b_2} |z| \right] \end{aligned} \tag{5.6}$$

and

$$|D_z^{-\delta} f(z)| \geq \frac{1}{\Gamma(2+\delta)}|z|^{\delta+1}$$

$$\times\left[1-\frac{2(1-\alpha)}{(2+\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right] \tag{5.7}$$

The inequalities (5.6) and (5.7) are attained for the function given by

$$f(z) = z - \frac{1-\alpha}{2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]b_2}z^2$$

Proof. From Theorem 3.1, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)](\theta+1)b_2}z^2 \tag{5.8}$$

Using (5.4), we have

$$\Gamma(2+\delta)z^{-\delta}D_z^{-\delta} f(z) = z - \sum_{n=2}^{\infty} \ell(n,\delta)a_n z^n \tag{5.9}$$

where

$$\ell(n,\delta) = \frac{\Gamma(n+1)\Gamma(\delta+2)}{\Gamma(n+1+\delta)}, \quad n \geq 2.$$

Observe that $0 < \ell(n,\delta) \leq \ell(2,\delta) = \dfrac{2}{\delta+2}$.
Or using (5.8) and (5.9), we get

$$|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta} f(z)| \leq |z| + \ell(2,\delta)|z|^2 \sum_{n=2}^{\infty} a_n$$

$$\leq |z| + \frac{2(1-\alpha)}{(2+\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|^2,$$

which proves (5.6).
Also we have

$$|\Gamma(2+\delta)z^{-\delta}D_z^{-\delta} f(z)|$$

$$\geq |z| - \frac{2(1-\alpha)}{(2+\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|^2$$

Hence the proof. □

Theorem 5.5. *Let* $f(z) \in SV(\lambda,\beta,\alpha,\mu,\theta)$*. Then*

$$|D_z^{\delta} f(z)| \leq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}$$

$$\left[1+\frac{2(1-\alpha)}{(2-\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right] \tag{5.10}$$

and

$$|D_z^\delta f(z)| \geq \frac{1}{\Gamma(2-\delta)}|z|^{1-\delta}$$

(5.11) $$\left[1 - \frac{2(1-\alpha)}{(2+\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right]$$

The inequalities (5.10) and (5.11) are attained for the function given by

(5.12) $$f(z) = z - \frac{1-\alpha}{[2(1+\beta)+(2\lambda-1)[\lambda(2+\beta)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}z^n$$

Proof. By Theorem 3.1, we have

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}$$

By (5.5), we have

(5.13) $$\Gamma(2-\delta)z^\delta D_z^\delta f(z) = z - \sum_{n=2}^{\infty} m(n,\delta)a_n z^n$$

where $m(n,\delta) = \dfrac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(n+1-\delta)}$.
Observe that

$$m(n,\delta) \leq m(2,\delta) = \frac{2}{2-\delta}.$$

Therefore, using (5.8) and (5.13) we have

$$\begin{aligned}|\Gamma(2-\delta)z^\delta D_z^\delta f(z)| &\leq |z| + m(2,\delta)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{2(1-\alpha)}{(2-\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}z^2\end{aligned}$$

Similarly, we obtain

$$|\Gamma(2-\delta)z^\delta D_z^\delta f(z)| \geq |z| - \frac{2(1-\alpha)}{(2-\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}z^2$$

Hence the proof. □

Corollary 5.6. *Let $f(z) \in SV(\lambda,\beta,\alpha,\mu,\theta)$. From the Definition 5.2 and Theorem 5.4, we can obtain the following inequality by taking $\delta = 1$.*

$$\frac{|z|^2}{2}\left[1 - \frac{2(1-\alpha)}{3[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right]$$

$$\leq \left|\int_0^z f(t)dt\right| \leq \frac{|z|^2}{2}\left[1 + \frac{2(1-\alpha)}{3[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right]$$

and

$$|z|\left[1-\frac{1-\alpha}{[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right]$$
$$\leq |f(z)| \leq |z|\left[1+\frac{1-\alpha}{[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}|z|\right]$$

Corollary 5.7. *$D_z^{-\delta}f(z)$ and $D_z^{\delta}f(z)$ are included in the disc with centre at origin and radii*

$$\frac{1}{\Gamma(2+\delta)}\left[1+\frac{2(1-\alpha)}{(2+\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}\right]$$

and

$$\frac{1}{\Gamma(2-\delta)}\left[1+\frac{2(1-\alpha)}{(2-\delta)[2(1+\beta)+(2\lambda-1)[\lambda(\beta+2)+(\alpha+\beta)(\lambda-1)]](\theta+1)b_2}\right]$$

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