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ON A GENERALIZED PROPERTY OF CERTAIN SUBCLASSES OF UNIVALENT INTEGRAL OPERATOR

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ABSTRACT. For analytic function of the form $f_i(z) = z + \sum_{n=2} a_n^i z^n$, in the open unit disk, the study on class $\Gamma_\alpha(\zeta_1, \zeta_2; \gamma)$ is extended to the class of $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ and some of their properties in relation to the coefficient bounds, convex combination as well as convolution were discussed.

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Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic and normalized with $f(0) = f'(0) - 1 = 0$ in the open disk, $U = \{z \in C : |z| < 1\}$. Makinde and Oladipo [6] intoduced the class

(1) $$\Gamma_\alpha(\zeta_1, \zeta_2, \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\}$$

for some complex ζ_1, ζ_2, α and for some real $\gamma, 0 \leq |\zeta_1| \leq 1, 0 < |\zeta_2| \leq 1, |\alpha| \leq 1$ and $0 < \gamma \leq 1$. See also, [1,2,3,4,5,7,8].

In this paper, we investigated the extension of the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma$ to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ Defined by

(2) $$\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma) = \left\{ f_i \in A \left| \frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2} \right| \leq \gamma \right\}$$

for the integral operator

(3) $$F_\alpha, (z) = \int_0^z \prod_{t=1}^{k} \left(\frac{D^n f_i(s)}{s} \right)^{1/\alpha} ds, \alpha \in C$$

Where

$$D^n f_i(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, k = 1, 2, 3, ... \tag{4}$$

Moreover,
Let $D^n f_i(z) = z+\sum_{n=2}^{\infty} n^k a_n^i z^n$ and $D^n g_i(z) = z+\sum_{n=2}^{\infty} n^k b_n^i z^n$, we define the convolution of $D^n f_i(z)$ and $D^n g_i(z)$ by

$$D^n f_i(z) * D^n g_i(z) = (D^n f_i * D^n g_i)(z) = z + \sum_{n=2}^{\infty} n_k a_n^i b_n^i z^n \tag{5}$$

We shall now present our main results.

Main Results

Theorem 1 Let $D^n f_i$ be in A. Then $D^n f_i$ is in the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ if and only if

$$\sum_{i=1}^{k} \sum_{n=2}^{\infty} \left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right] |a_n^i| \le \gamma|\zeta_1 + \alpha\zeta_2| - |1 - \alpha| \tag{6}$$

$$\gamma|\zeta_1 + \alpha\zeta_2| > |1 - \alpha|$$

Proof

$$\begin{aligned}
\left|\frac{G(z) + \frac{1}{\alpha} - 1}{\zeta_1(G(z) + \frac{1}{\alpha}) + \zeta_2}\right| &= \left|\frac{\sum_{i=1}^{k} \frac{1}{\alpha}\left(\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1\right) + \frac{1}{\alpha} - 1}{\zeta_1\left(\sum_{i=1}^{k} \frac{1}{\alpha}\left(\frac{D^{n+1}f_i(z)}{D^n f_i(z)} - 1\right) + \frac{1}{\alpha} + \zeta_2\right)}\right|, k \ge 1 \\
&= \left|\frac{1 - \alpha + \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n-\alpha)a_n^i z^n}{\zeta_+\alpha\zeta_2 + \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n\zeta_1 + \alpha\zeta_2)a_n^i z^n}\right| \\
&\le \frac{|1 - \alpha| + \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n-\alpha)|a_n^i|}{|\zeta_+\alpha\zeta_2| - \sum_{i=1}^{k}\sum_{n=2}^{\infty} n^k(n\zeta_1 + \alpha\zeta_2)|a_n^i|} \\
&\le \gamma
\end{aligned}$$

Let $D^n f_i(z)$ satisfy the inequality (6) above, then $D^n f_i(z)$ belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$
Coversely, Let $D^n f_i(z) \in \Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$, then inequality

$$\sum_{i=1}^{k} \sum_{n=2}^{\infty} \left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right] |a_n^i| \le \gamma|\zeta_1 + \alpha\zeta_2| - |1 - \alpha|$$

is satisfied. **Corollary 1** If $D^n f_i(z) \in \Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ then

$$|a_n^i| \leq \frac{\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|}{n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)}$$

Theorem 2 Let the function $D^n f_i(z) \in \Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ and the function $D^n g_i(z)$ defined by

$$D^n g_i(z) = z + \sum_{n=2}^{\infty} n^k b_n^i z^n$$

be in the same $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$. Then the function $H_i(z)$ defined by

$$H_i(z) = (1-\lambda)D^n f_i(z) + \lambda D^n g_i(z) = z + \sum_{n=2}^{\infty} n^k C_n^i z^n$$

also belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$
where $C_n^i = (1-\lambda)a_n^i + \lambda b_n^i,\ 0 \leq \lambda \leq 1$
Proof Suppose that both $D^n f_i(z)$ and $D^n g_i(z)$ belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$.
Then by (6), we have

$$\begin{aligned}
& \sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right]|C_n^i| \\
= & \sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right]|(1-\lambda)a_n^i + \lambda b_n^i| \\
= & (1-\lambda)\sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right]|a_n^i| \\
+ & \lambda\sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right]|b_n^i| \\
\leq & (1-\lambda)\{\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|\} + \lambda\{\gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|\} \\
= & \gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|
\end{aligned}$$

Thus, the convex combination of $D^n f_i(z)$ and $D^n g_i(z)$ also belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$
Theorem 3 Let the function $D^n f_{i1}(z) \in \Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ and the function $D^n f_{i2}(z)$ belong to the class $\Gamma_{D\alpha}(\beta_1, \beta_2; \gamma)$. Then $D^n f_{i1}(z) * D^n f_{i2}(z) = (D^n f_{i1} * D^n f_{i2})(z)$ belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma) \subset \Gamma_{D\alpha}(\beta_1, \beta_2; \gamma)$
Proof $D^n f_{i1}(z) \in \Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma)$ implies that

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1 + 1) + \alpha n^k(\gamma\zeta_2 - 1)\right]|a_{n1}^i| \leq \gamma|\zeta_1 + \alpha\zeta_2| - |1-\alpha|$$

and $D^n f_{i2}(z) \in \Gamma_{D\alpha}(\beta_1, \beta_2; \gamma)$ implies that

$$\sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\beta_1+1)+\alpha n^k(\gamma\beta_2-1)\right]|a_{n2}^i| \leq \gamma|\beta_1+\alpha\beta_2|-|1-\alpha|$$

where $a_{n1}^i,\ a_{n2}^i < 1$

Now,

$$\begin{aligned}(D^n f_{i1} * D^n f_{i2})(z) &= \sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\beta_1+1)+\alpha n^k(\gamma\beta_2-1)\right]|a_{1n}^i||a_{2n}^i| \\ &\leq \sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n^{k+1}(\gamma\zeta_1+1)+\alpha n^k(\gamma\zeta_2-1)\right]|a_{1n}^i| \\ &\leq \gamma|\zeta_1+\alpha\zeta_2|-|1-\alpha|\end{aligned}$$

This implies that $(D^n f_{i1} * D^n f_{i2})(z)$ belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma) \subset \Gamma_{D\alpha}(\beta_1, \beta_2; \gamma)$

Corollary 2 Let $(D^n f_{i1} * D^n f_{i2})(z)$ belong to the class $\Gamma_{D\alpha}(\zeta_1, \zeta_2; \gamma) \subset \Gamma_{D\alpha}(\beta_1, \beta_2; \gamma)$. Then

$$|a_{1n}^i||a_{2n}^i| \leq \frac{\gamma|\zeta_1+\alpha\zeta_2|-|1-\alpha|}{n^{k+1}(\gamma\zeta_1+1)+\alpha n^k(\gamma\zeta_2-1)}$$

References

[1] O. Altintas, and S. Owa, On some subclasses of univalent functions with negative coefficient, Pusan Dyongnam Math. J, **4**(1988), 41-46.

[2] S. Kanas and F. Ronning, Uniformly Starlike and convex functions and other related classes of univalent functions. Ann University, Marie Curie-SklodowskaSection A, **53**(1999), 95-105.

[3] H. S. Kim and S. K. Lee, On some classes of univalent functions, Math Japan, **32**(987),781-796.

[4] D. O. Makinde and T. O. Opoola, On sufficient condition for starlikenesss, General Mathematics **18**(2010), 30-35.

[5] D. O. Makinde, On a certain family of meromorphic functions with positive coefficients, Acta Univasitatis Appulensis (submitted).

[6] D.O. Makinde and A.T. Oladipo (2013), Some properties of certain subclasses of univalent integral operators, Scientia Magna 9(1) 80-88.

[7] N. Seenivasagan, Sufficient conditions for univalence, Applied Mathemaitics E-notes, **8**(2008), 30-35.

[8] Xiao-Fei Li and An-Ping Wang, Some properties of xome subclassess of univalent functions, Applied Mathematical Sciences, **6**(2012), 2687-2693.