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SOME RESULTS ON 2-BEST COAPPROXIMATION IN QUOTIENT GENERALIZED 2-NORMED SPACES

MAJID ABRISHAMI-MOGHADDAM

Department of Mathematics, Islamic Azad University, Birjand Branch, Birjand, Iran

Abstract. In this paper we study the concept of $2-$best coapproximation in generalized 2-normed spaces. We introduce the notions of $2-$best coapproximation, $2-$coproximinal sets and $2-$cochebyshev sets and prove some interesting theorems to characterization of $2-$best coapproximation elements in quotient spaces.

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1. Introduction

As a generalization of normed spaces is 2-normed spaces that play a very important role in functional analysis. The concept of linear 2-normed spaces was introduced by Gähler [4] in 1965 as an interesting non-linear generalization of a normed linear space, which was developed extensively in different subjects by others. During 1999-2006, Lewanwodska has published a series of papers on 2-normed sets and generalized 2-normed spaces (see [6]-[11]). Some others carry on the development of this concept to 2-functional and approximation in 2-normed spaces. (see [2], [5], [16]).
What we offer in this paper is to study another kind of Approximation, called best coapproximation, in generalized 2-normed spaces. The concept of coapproximation, was introduced by Franchettei and Furi [3] in 1972. Some results on best coapproximation theory in metric and linear normed spaces have been obtained by P.L. Papini, I. Singer, T.D. Narang, and others (see [12]-[15]). In this paper, we shall introduce the notions of 2-best coapproximation in quotient generalized 2-normed spaces and we give some results in this field.

2. Preliminaries

Definition 2.1. [4] *Let X be a linear space of dimension greater than 1 over $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers. Suppose $\|.,.\|$ be a nonnegative real-valued*

function on $X \times X$ satisfying the following conditions:
(i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent vectors.*
(ii) $\|x, y\| = \|y, x\|$ *for all $x, y \in X$.*
(iii) $\|\lambda x, y\| = |\lambda|\|x, y\|$ *for all $\lambda \in \mathbb{R}$ and $x, y \in X$.*
(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all $x, y, z \in X$.*
Then $\|., .\|$ is called a 2-norm on X and $(X, \|., .\|)$ is called a linear 2-normed space.

Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X, p_b(x) = \|x, b\|$ is a seminorm and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X.

Definition 2.2. [6], [7] *Let X and Y be linear spaces, D be a nonempty subset of $X \times Y$ such that for every $x \in X$ and $y \in Y$, the sets*

$$D_x = \{y \in Y : (x, y) \in D\} \; ; \; D_y = \{x \in X : (x, y) \in D\}$$

are linear subspaces of the spaces Y and X, respectively. A function $\|., .\| : D \to [0, \infty)$ is called a generalized 2-norm on D if it satisfies the following conditions:
$(N1)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ *for all $(x, y) \in D$ and every scalar α.*
$(N2)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for all $(x, y), (x, z) \in D$.*
$(N3)\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all $(x, z), (y, z) \in D$.*
Then $(D, \|., .\|)$ is called a 2-normed set. In particular, if $D = X \times Y$,$(X \times Y, \|., .\|)$ is called a generalized 2-normed space. Moreover, if $X = Y$, then generalized 2-normed space is denoted by $(X, \|., .\|)$.

Definition 2.3. [6] *Let X be a real linear space. Denote by $\mathcal{X}$ a non empty subset $X \times X$ with the property $\mathcal{X} = \mathcal{X}^{-1}$ and such that the set $\mathcal{X}^y = \{x \in \mathcal{X}; (x; y) \in \mathcal{X}\}$ is a linear subspace of X, for all $y \in X$. A function $\|., .\| : \mathcal{X} \to [0, \infty)$ satisfying the following conditions:*
$(S1)\|x, y\| = \|y, x\|$*for all $(x, y) \in \mathcal{X}$,*
$(S2)\|\alpha x, y\| = |\alpha|\|x, y\| = \|x, \alpha y\|$ *for any real number α and all $(x, y) \in \mathcal{X}$,*
$(S3)\|x, y + z\| \leq \|x, y\| + \|x, z\|$ *for all $x, y, z \in X$ such that $(x, y), (x, z) \in \mathcal{X}$,*
will be called a generalized symmetric 2-norm on $\mathcal{X}$. The set $\mathcal{X}$ is called a symmetric 2-normed set. In particular, if $\mathcal{X} = X \times X$, the function $\|., .\|$ will be called a generalized symmetric 2-norm on X and the pair $(X; \|., .\|)$ a generalized symmetric 2-normed space.

Every 2-normed space is a generalized 2-normed space but the converse is not true. The following example is a generalized 2-normed space that is not a 2-normed space.

Example 2.4. [11] *Let X be a real linear space having two norms $\|.\|_1$ and $\|.\|_2$. Then $(X, \|.,.\|)$ is a generalized 2-normed space with the generalized 2-norm defined by*

$$\|x, y\| = \|x\|_1.\|y\|_2 \; ; x, y \in X.$$

Specially if $\|.\|_1 = \|.\|_2$, our generalized 2-normed space will be a generalized symmetric 2-normed space.
For further examples see [6]-[11].

3. 2-BEST COAPPROXIMATION IN QUOTIENT GENERALIZED 2-NORMED SPACE

In the following theorem a generalized 2-norm on the space $\frac{X}{G_1} \times \frac{Y}{G_2}$ is defined.

Theorem 3.1. [1] *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed linear space, and G_1 and G_2 be subspaces of X and Y respectively. Define*

$$\|.,.\| : \frac{X}{G_1} \times \frac{Y}{G_2} \longrightarrow [0, +\infty)$$
$$\|x + G_1, y + G_2\| = \inf_{(g_1,g_2)\in G_1\times G_2} \|x + g_1, y + g_2\|$$

for every $x \in X$ and $y \in Y$. Then $\|.,.\|$ is a generalized 2-norm on $\frac{X}{G_1} \times \frac{Y}{G_2}$.

In [1], the authors have been shown that $\|.,.\|$ is a generalized 2-norm that it is not necessary a 2-norm.

Definition 3.2. [16] *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, G_1 be a subspace of X and G_2 be a subspace of Y. Then, $G_1 \times G_2$ is called 2-proximinal if for every $(x, y) \in X \times Y$ there exists $(g_0, g'_0) \in G_1 \times G_2$ such that*

$$\|x - g_0, y - g'_0\| = \inf\{\|x - g_1, y - g_2\| : (g_1, g_2) \in G_1 \times G_2\}.$$

In this case, (g_0, g'_0) is called 2-best approximation of (x, y) in $G_1 \times G_2$ and the set of all 2-best approximations of (x, y) in $G_1 \times G_2$ is denoted by $P^2_{G_1\times G_2}(x, y)$.

Now, we define the concept of 2-best coapproximation in generalized 2-normed s-paces as follows.

Definition 3.3. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed space, G_1 be a subspace of X and G_2 be a subspace of Y. Then, $G_1 \times G_2$ is called 2-coproximinal if for every $(x, y) \in X \times Y$ there exists $(g_0, g'_0) \in G_1 \times G_2$ such that*

$$\|g_0 - g_1, g'_0 - g_2\| \leq \|x - g_1, y - g_2\| \; \forall (g_1, g_2) \in G_1 \times G_2.$$

In this case, (g_0, g'_0) is called 2-best coapproximation of (x, y) in $G_1 \times G_2$ and the set of all 2-best coapproximations of (x, y) in $G_1 \times G_2$ is denoted by $R^2_{G_1\times G_2}(x, y)$. $G_1 \times G_2$ is called 2-cochebyshev if $R^2_{G_1\times G_2}(x, y)$ is exactly singleton for each $(x, y) \in X \times Y$.

Theorem 3.4. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed linear space, G_1, K_1 and G_2, K_2 are subspaces of X and Y respectively such that $K_1 \subset G_1$ and $K_2 \subset G_2$. If (g_1, g_2) is a 2-best coapproximation to (x, y) in $G_1 \times G_2$, then $(g_1 + K_1, g_2 + K_2)$ is a 2-best coapproximation to $(x + K_1, y + K_2)$ in $\frac{G_1}{K_1} \times \frac{G_2}{K_2}$.*

Proof. If $(g_0 + K_1, g_0' + K_2)$ is not a 2-best coapproximation to $(x + K_1, y + K_2)$ in $\frac{G_1}{K_1} \times \frac{G_2}{K_2}$, then there exists $(g_1', g_2') \in G_1 \times G_2$ such that

$$\|x - g_1' + K_1, y - g_2' + K_2\| < \|g_0 - g_1 + K_1, g_0' - g_2 + K_2\|.$$

Hence there exists $(k_1, k_2) \in K_1 \times K_2$ such that

$$\|x - g_1' + k_1, y - g_2' + k_2\| < \|g_0 - g_1 + K_1, g_0' - g_2 + K_2\|.$$

Therefore since

$$\|g_0 - g_1 + K_1, g_0' - g_2 + K_2\| < \|(g_0 - k_1) - g_1, (g_0' - k_2) - g_2\|,$$

we have

$$\|x - (g_1' - k_1), y - (g_2' - k_2)\| < \|(g_0 - k_1) - g_1, (g_0' k_2) - g_2\|,$$

and this is a contradiction. Hence $(g_0 + K_1, g_0' + K_2)$ is a 2-best coapproximation for $(x + K_1, y + K_2)$ in $\frac{G_1}{K_1} \times \frac{G_2}{K_2}$. □

Corollary 3.5. *Let $(X \times Y, \|.,.\|)$ be a generalized 2-normed linear space, G_1, K_1 and G_2, K_2 are subspaces of X and Y respectively such that $K_1 \subset G_1$ and $K_2 \subset G_2$. If $G_1 \times G_2$ is 2-coproximinal in $X \times Y$, then $\frac{G_1}{K_1} \times \frac{G_2}{K_2}$ is 2-coproximial in $\frac{X}{K_1} \times \frac{Y}{K_2}$.*

Theorem 3.6. *Let $K_1 \times K_2$ be a 2-proximinal closed subspace of a generalized 2-normed space $X \times Y$. If $(g_0 + K_1, g_0' + K_2) \in R^2_{\frac{G_1}{K_1} \times \frac{G_2}{K_2}}(x + K_1, y + K_2)$, then there exists $(k_0, k_0') \in K_1 \times K_2$ such that $(g_0 + k_0, g_0' + k_0') \in R^2_{G_1 \times G_2}(x, y)$.*

Proof. Let $(g_0 + K_1, g_0' + K_2) \in R^2_{\frac{G_1}{K_1} \times \frac{G_2}{K_2}}(x + K_1, y + K_2)$, where $(g_0, g_0') \in G_1 \times G_2$. Then for every $(g_1, g_2) \in G_1 \times G_2$,

$$\|(g_1 - K_1) - (g_0 - K_1), (g_2 - K_2) - (g_0' - K_2)\| \leq \|(x - K_1) - (g_1 - K_1), (y - K_2) - (g_2 - K_2)\|,$$

or

$$\|(g_1 - g_0) + K_1, (g_2 - g_0') + K_2\| \leq \|(x - g_1) + K_1, (y - g_2) + K_2\|.$$

By 2-proximinality of $K_1 \times K_2$ in $G_1 \times G_2$, there exists $(k_0, k_0') \in K_1 \times K_2$ such that

$$\|(g_1 - g_0) - k_0, (g_2 - g_0') - k_0'\| = \|(g_1 - g_0) + K_1, (g_2 - g_0') + K_2\|.$$

Now we have

$$\begin{aligned}\|g_1-(g_0+k_0),g_2-(g_0'+k_0')\| &= \|(g_1-g_0)-k_0,(g_2-g_0')-k_0')\| \\ &= \|(g_1-g_0)+K_1,(g_2-g_0')+K_2)\| \\ &\le \|(x-g_1)+K_1,(y-g_2)+K_2)\| \\ &\le \|x-g_1,y-g_2\|\end{aligned}$$

for every $(g_1,g_2)\in G_1\times G_2$. Therefore $(g_0+k_0,g_0'+k_0')\in R^2_{G_1\times G_2}(x,y)$. □

Corollary 3.7. *Let $K_1\times K_2$ be a 2-proximinal closed subspace of a generalized 2-normed space $X\times Y$. If $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-coproximinal in $\frac{X}{K_1}\times\frac{Y}{K_2}$, then $G_1\times G_2$ is 2-coproximinal in $X\times Y$.*

Theorem 3.8. *Let $K_1\times K_2$ be a 2-proximinal closed subspace of a generalized 2-normed space $X\times Y$. If $G_1\times G_2$ is 2-cochebyshev in $X\times Y$, then $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is also 2-cochebyshev in $\frac{X}{K_1}\times\frac{Y}{K_2}$.*

Proof. By corollary(3.5), $\frac{G_1}{K_1}\times\frac{G_2}{K_2}$ is 2-coproximinal in $\frac{X}{K_1}\times\frac{Y}{K_2}$. Let $(x+K_1,y+K_2)\in(\frac{X}{K_1}\times\frac{Y}{K_2})\setminus(\frac{G_1}{K_1}\times\frac{G_2}{K_2})$ be arbitrary and $(g_1+K_1,g_2+K_2),(g_3+K_1,g_4+K_2)\in R^2_{G_1\times G_2}(x+K_1,y+K_2)$. By theorem (3.6), there exist $(k_1,k_2),(k_3,k_4)\in R^2_{G_1\times G_2}(x,y)$. Since $G_1\times G_2$ is 2-cochebyshev $(g_1+k_1,g_1+k_1)=(g_3+k_3,g_4+k_4)$ and then $(g_1+K_1,g_2+K_2)=(g_3+K_1,g_4+K_2)$. □

Theorem 3.9. *Let $G_1\times G_2$ and $K_1\times K_2$ be subspaces of a generalized 2-normed linear space $X\times Y$ and $K_1\times K_2\subset G_1\times G_2$. If $G_1\times G_2$ is 2-coproximinal in $X\times Y$, then*

$$\pi(R^2_{G_1\times G_2}(x,y))\subseteq R^2_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1,y+G_2).$$

Furthermore if $K_1\times K_2$ is 2-proximinal in $X\times Y$, then

$$\pi(R_{G_1\times G_2}(x,y))=R^2_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1,y+G_2),$$

where $\pi:X\times Y\to\frac{X}{K_1}\times\frac{Y}{K_2}$ which is defined by $\pi(x,y)=(x+K_1,y+K_2)$, is the canonical map.

Proof. By theorem (3.4), it is clear that $\pi(R^2_{G_1\times G_2}(x,y))\subseteq R^2_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1,y+G_2)$. Now let $K_1\times K_2$ be 2-proximinal in $X\times Y$. By theorem (3.6), if $(g_1+K_1,g_2+K_2)\in R^2_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1,y+G_2)$, then there exists $(k_1,k_2)\in K_1\times K_2$ such that $(g_1+k_1,g_2+k_2)\in R_{G_1\times G_2}(x,y)$. Therefore

$$(g_1+K_1,g_2+K_2)=(g_1+k_1+K_1,g_2+k_2+K_2)=\pi(g_1+k_1,g_2+k_2)\in\pi(R^2_{G_1\times G_2}(x,y)).$$

Hence $R_{\frac{G_1}{K_1}\times\frac{G_2}{K_2}}(x+G_1,y+G_2)\subseteq\pi(R^2_{G_1\times G_2}(x,y))$. □

Theorem 3.10. *For a linear subspace $G_1 \times G_2$ of a generalized 2-normed linear space $X \times Y$, the following statements are equivalent:*
1) $G_1 \times G_2$ is 2-coproximinal.
2) We have

$$\begin{aligned} X \times Y &= (G_1 \times G_2) + (R^2_{G_1\times G_2})^{-1}(0,0) \\ &= \{(g_1 + x, g_2 + y) | (g_1, g_2) \in G_1 \times G_2, (x,y) \in (R^2_{G_1\times G_2})^{-1}(0,0)\}. \end{aligned}$$

3) $G_1 \times G_2$ is closed and for the canonical mapping $\pi : X \times Y \longrightarrow \frac{X}{G_1} \times \frac{Y}{G_2}$ we have

$$\pi\big((R^2_{G_1\times G_2})^{-1}(0,0)\big) = \frac{X}{G_1} \times \frac{Y}{G_2}$$

Proof. (1) $\Longleftrightarrow$ (2) Let $(x,y) \in X \times Y$. Since $G_1 \times G_2$ is 2-coproximinal, there exits $(g_1, g_2) \in R^2_{G_1\times G_2}(x,y))$, and so $(x - g_1, y - g_2) \in (R^2_{G_1\times G_2})^{-1}(0,0))$. Since

$$(x,y) = (g_1 + (x - g_1), g_2 + (y - g_2)) \in (G_1 \times G_2) + R^{-1}_{G_1\times G_2}(0,0),$$

we get $X \times Y \subseteq (G_1 \times G_2) + R^2_{G_1\times G_2})^{-1}(0,0) \subseteq X \times Y$ and so $X \times Y = (G_1 \times G_2) + R^2_{G_1\times G_2})^{-1}(0,0)$. Conversely, let $(x,y) \in X \times Y = (G_1 \times G_2) + (R^2_{G_1\times G_2})^{-1}(0,0))$. Then $(x,y) = (g_1 + x_1, g_2 + y_1)$, where, $(g_1, g_2) \in G_1 \times G_2$ and $(x_1, y_1) \in R^{-1}_{G_1\times G_2}(0,0)$. Hence $(0,0) \in R^2_{G_1\times G_2}(x_1, y_1) = R^2_{G_1\times G_2}(x - g_1, y - g_2)$. Therefore $(g_1, g_2) \in R^2_{G_1\times G_2}(x,y)$, so $G_1 \times G_2$ is 2-coproximinal.
(1) $\Longleftrightarrow$ (3) First we show that $G_1 \times G_2$ is closed. Let $(p_1, p_2) \in \overline{G_1 \times G_2} \setminus G_1 \times G_2$ and $(g_1, g_2) \in R^2_{G_1\times G_2}(p_1, p_2)$. Then there exits a sequence $(g_n, g'_n) \in G_1 \times G_2$ such that $(g_n, g'_n) \to (p_1, p_2)$ and $\|g_1 - g_n, g_2 - g'_n\| \leq \|p_1 - g_n, p_2 - g'_n\|$ for all n. Now if $n \to \infty$ implies that $(g_n, g'_n) \to (g_1, g_2)$ and so $(p_1, p_2) = (g_1, g_2) \in G_1 \times G_2$. Hence $G_1 \times G_2$ is closed. Now suppose $(x + G_1, y + G_2) \in \frac{X}{G_1} \times \frac{Y}{G_2}$ and $(g_1, g_2) \in P_{G_1\times G_2}(x,y)$. Then $(x - g_1, y - g_2) \in (R^2_{G_1\times G_2})^{-1}(0,0))$ and $\pi(x - g_1, y - g_2) = (x + G_1, y + G_2)$. Conversely, if we have (3) and $(x,y) \in X \times Y$. Then $(x + G_1, y + G_2) \in \frac{X}{G_1} \times \frac{Y}{G_2} = \pi(R^2_{G_1\times G_2})^{-1}(0,0))$. So $(x + G_1, y + G_2) = \pi(x_1, y_1)$, where $(x_1, y_1) \in (R^2_{G_1\times G_2})^{-1}(0,0))$. Hence $(x + G_1, y + G_2) = (x_1 + G_1, y_2 + G_2)$ where $(0,0) \in R^2_{G_1\times G_2}(x,y))$. So $(x - x_1, y - y_1) = (g_1, g_2) \in G_1 \times G_2$ and $(0,0) \in R^2_{G_1\times G_2}(x - g_1, y - g_2))$. Therefore $(g_1, g_2) \in R^2_{G_1\times G_2}(x,y))$, and hence $G_1 \times G_2$ is 2-coproximinal. Which this complete the proof. □

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