Asia Pacific Journal of Mathematics, Vol. 1, No. 1 (2014), 44-66 ISSN 2357-2205

COUPLED COMMON FIXED POINT RESULTS IN ORDERED $S-$METRIC SPACES

ANIMESH GUPTA

Department of Applied Mathematics, Vidhyapeeth Institute of Science & Technology, Near SOS Balgram, Bhopal (M.P.) India

Formally in Department of Mathematics, Sagar Institute of Engineering, Technology and Research, Ratibad Bhopal (M.P.), India

ABSTRACT. The notion of coupled fixed point theorem introduced by Bhaskar and Lakxmikantham [16] and Sedghi et al [14] introduced the concept of S- metric space. Our aim of this article is to extend the concept of coupled fixed point in S- metric space and prove a coupled coincidence and common fixed point theorems for commuting mappings with mixed g-monotone property in partially ordered S-metric spaces. We also give some examples in support of our theorem.

2010 Mathematics Subject Classification. 45H10; 54H25.

Key words and phrases. Coupled coincidence point, Coupled fixed point, Mixed g- monotone, Mixed monotone.

1. INTRODUCTION

Fixed point theory is one of the most active field for the researchers. The first known result for fixed point theory in metric space was given by Banach [15] namely as Banach contraction principle. In last few decade this contraction principle was generalized and extend in many ways. Beside this, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [3] gave the concepts of $2-$ metric space further in 1992 Dhage [2] modified the concept of $2-$ metric space and introduced the concepts of $D-$ metric space but in 2005 Mustafa and Sims [4] pointed out that these attempts are not valid and introduced the concepts of $G-$ metric space and proved fixed point theorems in $G-$ metric space. Many authors proved different fixed point theorems in $G-$ metric space in different ways see in [13] and references theirin. Sedghi et al. [12] modified the concepts of $D-$ metric space and introduced the concepts of D^*- metric space also proved a common fixed point theorems in D^*- metric space.

Recently, Sedghi et al [14] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different form other spaces. In fact they gives following concepts of S- metric space.

Definition 1. *Let X be a nonempty set. An S- metric space on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ *if and only if* $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S- metric space.

Examples of such S - metric space are as follows,

Example 2. *Let $X = \mathbb{R}^n$ and $\| \, . \, \|$ a norm on X, then $S(x, y, z) = \| y + z - 2x \| + \| y - z \|$ is an S- metric on X.*

Example 3. *Let $X = \mathbb{R}^n$ and $\| \, . \, \|$ a norm on X, then $S(x, y, z) = \| x - z \| + \| y - z \|$ is an S- metric on X.*

Example 4. *Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S- metric on X.*

Lemma 5. *Let (X, S) be an S- metric space, then we have,*

$$S(x, x, y) = S(y, y, x)$$

.

Proof. By the third condition of S- metric, we have

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x)$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y)$$

which implies that

$$S(x, x, y) = S(y, y, x)$$

. □

Definition 6. *Let (X, S) be an S- metric space.*

(1) *A sequence $\{x_n\}$ in X is said to be converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $lim_{n \to \infty} x_n = x$.*

(2) *A sequence $\{x_n\}$ in X is said to be Cauchy sequence if and only if $S(x_n, x_m, x) \to 0$ as $n, m \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(x_n, x_m, x) < \epsilon$.*

Definition 7. *The S- metric space (X, S) is said to be complete if every Cauchy sequence is convergent.*

Every S- metric on X defines a metric d_S on X by

$$d_S(x, y) = S(x, x, y) + S(y, y, x) \quad \forall x, y \in X. \tag{1.1}$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X. Also, nonempty subset A in the S- metric space (X, S) is S- closed if $\bar{A} = A$.

Lemma 8. *Let (X, S) be a S- metric space and A is a nonempty subset of X. A is said S- closed iof for any sequence $\{x_n\}$ is A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

Beside this in present era the concept of mixed monotone operator were introduce by Gua and Lakshmikantham [21]. This technique are applied in various field of applied sciences as well as engineering to solve the problems. Its beautiful example was presented by Bhaskar and Lakxmikantham [16] as a coupled fixed point result in partial ordered metric spaces. After the publication of this work several coupled fixed point and coincidence point results have appeared in the recent literature. Some works noted in [18, 19, 20]. In [17] Lakshmikantham and Ciric, introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X and studied fixed point theorems in partially ordered metric spaces.

The aim of this paper is to prove a coupled coincidence and common fixed point theorems for commuting mappings with mixed g-monotone property in partially ordered S-metric spaces.

Definition 9. *Let (X,$\leq$) is a partially ordered set and $F : X \to X$. The mapping **F** is said to have the mixed monotone property if **F** is nondecreasing monotone in first argument and is a nonincreasing monotone in its second argument, that is, for any $x, y \in X$*

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y)$$

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \implies F(x, y_1) \geq F(x, y_2)$$

Definition 10. *Let (X,≤) is a partially ordered set and $F : X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g - monotone property if F is g - nondecreasing monotone in first argument and is a g - nonincreasing monotone in its second argument, that is, for any $x, y \in X$*

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \implies F(x_1, y) \leq F(x_2, y) \tag{1.2}$$

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \implies F(x, y_1) \geq F(x, y_2) \tag{1.3}$$

It is clear that Definition 10 reduced to 9 when g is the identity mapping.

Definition 11. *An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if*

$$F(x, y) = x \quad F(y, x) = y$$

Definition 12. *An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if*

$$F(x, y) = g(x) \quad F(y, x) = g(y)$$

It is easy to see that coupled coincidence point can be reduced to coupled fixed point on taking g be an identity mapping.

2. Main Results

Our first result is the following

Theorem 13. *Let $(X, S, \leq)$ be a partially ordered S- metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g- monotone property on X and there exists a $k \in \left[0, \frac{1}{2}\right)$*

$$S(F(x, y), F(u, v), F(w, z)) \leq k[S(gx, gu, gw) + S(gy, gv, gz)] \tag{2.1}$$

for all $x, y, z, u, v, w \in X$ for which $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$ where either $gu \neq gz$ or $gv \neq gw$. If there exists $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad and \quad gy_0 \geq F(y_0, x_0).$$

We assume the following hypotheses,

(i). $F : (X \times X) \subseteq g(X)$,

(ii). $g(X)$ *is* S*-complete,*

(iii). g *is* S*- continuous and commutes with* F.

Then F *and* g *have a coupled coincidence point. If* $gu = gz$ *and* $gv = gw$*, then* F *and* g *have common fixed point, that is, there exist* $x \in X$ *such that*

$$g(x) = F(x, x) = x.$$

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Since $F : X \times X \subseteq g(X)$, we can choose $gx_1, gy_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_0 = F(y_0, x_0)$. Again since $F : X \times X \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

(2.2) $$g(x_{n+1}) = F(x_n, y_n) \quad and \quad g(y_{n+1}) = F(y_n, x_n) \forall n \geq 0.$$

Next, we show that

(2.3) $$g(x_n) \leq g(x_{n+1}) \quad and \quad g(y_n) \geq g(y_{n+1}) \forall n \geq 0.$$

Since $g(x_0) \leq F(x_0, y_0) = g(x_1)$ and $g(y_0) \leq F(y_0, x_0) = g(y_1)$, therefore, (2.3) holds for $n = 0$. Next, suppose that (2.3) holds for some fixed $n \geq 0$, that is,

(2.4) $$g(x_n) \leq g(x_{n+1}) \quad and \quad g(y_n) \geq g(y_{n+1})$$

Since F is the mixed g-monotone property, from 2.4 and 1.2, we have

(2.5) $$F(x_n, y) \leq F(x_{n+1}, y) \quad and \quad F(y_{n+1}, x) \leq F(y_n, x)$$

for all $x, y \in X$ and from 2.4 and 1.3 we have

(2.6) $$F(y, x_n) \geq F(y, x_{n+1}) \quad and \quad F(x, y_{n+1}) \geq F(x, y_n)$$

for all $x, y \in X$. If we take $y = y_n$ and $x = x_n$ in 2.5, then we obtain

(2.7) $$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n) \quad and \quad F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1})$$

If we take $y = y_{n+1}$ and $x = x_{n+1}$ in 2.6 then

(2.8) $F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}) \quad and \quad g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n)$

Now, from 2.7 and 2.8, we have

$$g(x_{n+1}) \leq g(x_{n+2}) \quad and \quad g(y_{n+1}) \geq g(y_{n+2}). \tag{2.9}$$

Therefore, by the mathematical induction, we conclude that 2.3 holds for all $n \geq 0$. Continuing this process, one can easily verify that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq ..... \leq g(x_{n+1}) \leq ...$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq ..... \geq g(y_{n+1}) \geq ...$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then F and g have a coupled coincidence point. So we assume $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$, that is, we assume that either $g(x_{n+1}) = F(x_n, y_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n, x_n) \neq g(y_n)$.

Next, we claim that, for all $n \geq 0$,

$$S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \tag{2.10}$$

For $n = 1$, we have

$$\begin{aligned} S(gx_1, gx_1, gx_2) &= S(F(x_0, y_0), F(x_0, y_0), F(x_1, y_1)) \\ &\leq k[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ &\leq \frac{1}{2}(2k)^1[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned}$$

Thus 2.10 holds for $n = 1$. Therefore, we presume that 2.10 holds for $n > 0$. Since $g(x_{n+1}) \geq g(x_n)$ and $g(y_{n+1}) \leq g(y_n)$, from 2.1 and 2.2 we have

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)] \end{aligned} \tag{2.11}$$

and

$$
\begin{aligned}
S(gy_n, gy_n, gy_{n+1}) &= S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\
&\leq k[S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)]. \qquad (2.12)
\end{aligned}
$$

By adding 2.11 and 2.12, then we get

$$S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) \leq 2k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)] \qquad (2.13)$$

Continuing the process, we have for each $n \in N$,

$$
\begin{aligned}
S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) &\leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\
&\leq \frac{(2k)^n}{2(1-2k)}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]
\end{aligned}
$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty}[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] = 0.$$

That is

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$$

and

$$\lim_{n\to\infty} S(gy_n, gy_n, gy_{n+1}) = 0$$

Thus $\{gx_n\}$ and $\{gy_n\}$ are S- Cauchy sequences in $g(X)$.

Since $g(X)$ is S-complete, we get $\{gx_n\}$ and $\{gy_n\}$ are converges to some $x \in X$ and $y \in X$ respectively. Since g is S- continuous, we have $\{g(gx_n)\}$ and $\{f(gy_n)\}$ are converges to gx and gy respectively. that is

$$\lim_{n\to\infty} g(gx_n) = gx \qquad (2.14)$$

and

$$\lim_{n\to\infty} g(gy_n) = gy. \qquad (2.15)$$

Also from commutativity of F and g, we have

$$F((gx_n), g(y_n)) = gF(x_n, y_n) = g(gx_{n+1}) \qquad (2.16)$$

and

$$F((gy_n), g(x_n)) = gF(y_n, x_n) = g(gy_{n+1}) \tag{2.17}$$

Next we claim that (x, y) is a coupled coincidence point of F and g.

Now from the condition 2.1, we have

$$\begin{aligned} S(g(gx_{n+1}), g(gx_{n+1}), F(x, y)) &= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x, y)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)] \end{aligned}$$

Letting $n \to \infty$ and using the fact that S is continuous on its variables, we get that

$$S(gx, gx, F(x, y)) \leq k[S(gx, gx, gx) + S(gy, gy, gy)] = 0$$

Hence $gx = F(x, y)$. Similarly, we may show that $gy = F(y, x)$.

Finally, we claim that x is common fixed point of F and g.

Since (x, y) is a coupled coincidence point of the mappings F and g, we gave $gx = F(x, y)$ and $gy = F(y, x)$. Assume that $gx \neq gy$. Then by 2.1, we get

$$S(gx, gx, gy) = S(F(x, y), F(x, y), F(y, x)) \leq k[S(gx, gx, gy) + S(gy, gy, gx)] \tag{2.18}$$

also

$$S(gy, gy, gx) = S(F(y, x), F(y, x), F(x, y)) \leq k[S(gx, gx, gy) + S(gy, gy, gx)] \tag{2.19}$$

by adding 2.18 and 2.19 we have

$$S(gx, gx, gy) + S(gy, gy, gx) \leq 2k[S(gx, gx, gy) + S(gy, gy, gx)] \tag{2.20}$$

since $k < \frac{1}{2}$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) \leq [S(gx, gx, gy) + S(gy, gy, gx)] \tag{2.21}$$

Which contradiction. So $gx = gy$, and hence

$$F(x,y) = gx = gy = F(y,x)$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gy_n\}$ we have $\{gy_{n+1}\}$ is S- convergent to x. Thus

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(gx_{n+1}, gx_{n+1}, F(x,y)) \\ &= S(F(x_n, y_n), F(x_n, y_n), F(x,y)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)] \end{aligned}$$

Letting $n \to \infty$ and use the fact that S is continuous on its variables, we get

$$S(x,x,gx) \leq k[S(x,x,gx) + S(y,y,gy)]$$

Similarly, we may show that

$$S(y,y,gy) \leq k[S(x,x,gx) + S(y,y,gy)]$$

Thus

$$S(x,x,gx) + S(y,y,gy) \leq 2k[S(x,x,gx) + S(y,y,gy)].$$

Since $2k < 1$, the last inequality happens only if $S(x,x,gx) = 0$ and $S(y,y,gy) = 0$. Hence $x = gx$ and $y = gy$. Thus we get $gx = F(x,y) = x$. Thus F and g have a common fixed point. This completes the proof of the theorem. □

Our second result of this paper is following.

Theorem 14. *In the above theorem, in the place of condition (ii), if we assume the following conditions in the complete S- metric space X, namely*

(2.22) $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to x \in X,$ *then* $x_n \leq x, \ \ \forall n,$

and

(2.23) $\{y_n\} \subset X$ *is a nonincreasing sequence with* $y_n \to y \in X,$ *then* $y_n \geq y, \ \ \forall n.$

Then, we have the conclusion of Theorem 13, provided g is non decreasing.

Proof. Proceeding exactly as in Theorem 13, we have $\{gx_n\}$ and $\{gy_n\}$ are S- Cauchy sequences in X. Since (X,S) is a complete, there exists $(x,y) \in X \times X$ such that

$$\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = x \tag{2.24}$$

and

$$\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} g(y_n) = y \tag{2.25}$$

Therefore, from (iii) we arrive at 2.14, 2.15 2.16 and 2.17. Since $\{gx_n\}$ is a non decreasing sequence and $gx_n \to x$ and as $\{gy_n\}$ is non increasing sequence and $gy_n \to y$, by assumption 2.22 and 2.23 we have, $g(gx_n) \leq gx$ and $g(gy_n) \geq gy$ for all $n \geq 0$. If $g(gx_n) = gx$ and $g(gy_n) = gy$ for some n, then by construction $g(gx_{n+1}) = gx$ and $g(gy_{n+1}) = gy$ and (x,y) is coupled fixed point. So we assume either $g(gx_n) \neq gx$ or $g(gy_n) \neq gy$. Applying the contractive condition 2.1, we have

$$\begin{aligned}
S(F(x,y),F(x,y),gx) &\leq 2S(F(x,y),F(x,y),F(gx_n,gy_n)) + S(F(gx_n,gy_n),F(gx_n,gy_n),gx) \\
&= 2S(F(x,y),F(x,y),F(gx_n,gy_n)) + S(gF(x_n,y_n),gF(x_n,y_n),gx) \\
&\leq 2k[S(gx,gx,gx_n) + S(gy,gy,gy_n)] + S(g(gx_{n+1}),g(gx_{n+1}),gx)
\end{aligned}$$

Taking $n \to \infty$ in the above inequality we obtain $S(F(x,y),F(x,y),gx) = 0$, that is, $F(x,y) = g(x)$. Similarly we have that $F(y,x) = g(y)$. Remaining part of the proof follows from Theorem 13. Hence we have $g(x) = F(x,x) = x$. This completes the proof of the theorem.

□

Corollary 15. *Let $(X,S,\leq)$ be a partially ordered S- metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g- monotone property on X and there exists a $k \in \left[0, \frac{1}{2}\right)$*

$$S(F(x,y),F(x,y),F(w,z)) \leq k[S(gx,gx,gw) + S(gy,gy,gz)] \tag{2.26}$$

for all $x,y,z,w \in X$ for which $gx \geq gw$ and $gy \leq gz$. If there exists $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0,y_0) \quad and \quad gy_0 \geq F(y_0,x_0).$$

We assume the following hypotheses,

(i). *$F : (X \times X) \subseteq g(X)$,*

(ii). *$g(X)$ is S-complete,*

(iii). *g is S- continuous and commutes with F.*

Then F and g have a coupled coincidence point. If $gx = gz$ and $gy = gw$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x) = x.$$

Proof. It follows from Theorem 13 if we take $x = u$ and $y = v$. □

Corollary 16. *Let $(X, S, \leq)$ be a partially ordered S- metric space. Let $F : X \times X \to X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in \left[0, \frac{1}{2}\right)$*

$$S(F(x, y), F(u, v), F(w, z)) \leq k[S(x, u, w) + S(y, v, z)] \tag{2.27}$$

for all $x, y, z, u, v, w \in X$ for which $x \geq u \geq w$ and $y \leq v \leq z$ where either $u \neq z$ or $v \neq w$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad and \quad y_0 \geq F(y_0, x_0).$$

Then F has coupled fixed point in X. If $u = z$ and $v = w$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x).$$

Proof. If we define $g : X \to X$ be an identity mapping in Theorem 13 then result is follows. □

Corollary 17. *Let $(X, S, \leq)$ be a partially ordered S- metric space. Let $F : X \times X \to X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in \left[0, \frac{1}{2}\right)$*

$$S(F(x, y), F(x, y), F(w, z)) \leq k[S(x, x, w) + S(y, y, z)] \tag{2.28}$$

for all $x, y, z, w \in X$ for which $x \geq w$ and $y \leq z$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq F(x_0, y_0) \quad and \quad y_0 \geq F(y_0, x_0).$$

Then F has a coupled fixed point. If $x = z$ and $y = w$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x).$$

Proof. If we define $g : X \to X$ be an identity mapping in Corollary 16 then result is follows. □

Now we present some examples to illustrate our results given by Theorem 13 and Theorem 14.

Example 18. *Let $X = \mathbb{R}$ be ordered by the following relation*

$$x \leq y \iff x = y \quad or \quad (x, y \in [0,1] \ \textit{and} \ x \leq y).$$

Let a S- metric on X be defined by

$$S(x, y, z) = \mid x - z \mid + \mid y - z \mid .$$

Then (X, S) is a complete regular ordered S- metric space.
Let $g : X \to X$ and $F : X \times X \to X$ be defined by

$$g(x) = \begin{cases} \frac{1}{20}x & \textit{if } x < 0 \\ \frac{1}{2}(x) & \textit{if } x \in [0,1] \\ \frac{1}{20}x + \frac{9}{20} & \textit{if } x > 1, \end{cases} \qquad and \qquad F(x, y) = \tfrac{x+y}{20}$$

Take $k = \frac{1}{10}$. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied. Obviously, the mappings g and F have a unique common coupled fixed point $(0, 0)$.

Example 19. *Let $X = [0, 1]$, with the usual partial ordered $\leq$. Let a S- metric on X be defined by*

$$S(x, y, z) = \mid x - z \mid + \mid y - z \mid .$$

Then (X, S) is a complete regular ordered S- metric space.
Let $g : X \to X$ and $F : X \times X \to X$ be defined by

$$F(x, y) = \frac{x + y}{24} \quad g(x) = \tfrac{x}{3} \tag{2.29}$$

Take $k = \frac{1}{8} \in \left[0, \frac{1}{2}\right)$. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied. Obviously, the mappings g and F have a unique common coupled fixed point $(0, 0)$.

Example 20. *Let* $X = [0,1]$*, with the usual partial ordered* $\leq$*. Let a* S*- metric on* X *be defined by*

$$S(x,y,z) = \mid x-z \mid + \mid y-z \mid .$$

Then (X,S) *is a complete regular ordered* S*- metric space.*
Let $g : X \to X$ *and* $F : X \times X \to X$ *be defined by*

$$F(x,y) = \frac{1}{20}[\sin x + \sin y] \quad g(x) = \tfrac{x}{5} \tag{2.30}$$

for all $x, y \in X$*. Since* $\mid \sin x - \sin y \mid \leq \mid x-y \mid$ *holds for all* $x, y \in X$*. Then we have* $k = \frac{1}{4} \in \left[0, \frac{1}{2}\right)$ *. So all the conditions of Theorem 13 and Theorem 14 are satisfied. Then there exists a coupled fixed point of F. In this case* $(0,0)$ *is coupled fixed point of* F*.*

Furthermore we show that Theorem 13 is not true for following example,

Example 21. *Let* $X = [0,1]$*, with the usual partial ordered* $\leq$*. Let a* S*- metric on* X *be defined by*

$$S(x,y,z) = \mid x-z \mid + \mid y-z \mid .$$

Then (X,S) *is a complete regular ordered* S*- metric space.*
Let $g : X \to X$ *and* $F : X \times X \to X$ *be defined by*

$$F(x,y) = \frac{x+y}{6} \quad g(x) = \tfrac{x}{3} \tag{2.31}$$

Then there is no $k = \frac{1}{8} \in \left[0, \frac{1}{2}\right)$ *for which Theorem 13 is true Take* $k = \frac{1}{8} \in \left[0, \frac{1}{2}\right)$*. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied. Obviously, the mappings* g *and* F *have a unique common coupled fixed point* $(0,0)$*.*

Now in next section we give an another result for coupled fixed point which is generalization of our Theorem 13.

3. Generalization of Coupled Fixed Point Theorem

We begin this section with the following example,

Example 22. *Let* X *be the set* $[0,\infty)$ *and*

$$S(x,y,z) = \mid x-z \mid + \mid y-z \mid .$$

We set $g : X \to X$ *and* $F : X \times X \to X$ *be defined as* $g(x) = x^2$ *and*

$$F(x,y) = \begin{cases} \frac{x^2-y^2}{4} & \textit{if } x \geq y \\ 0 & \textit{if } x < y \end{cases}$$

Then for $k = \frac{1}{2} \notin \left[0, \frac{1}{2}\right)$ *we have coupled fixed point* $(0,0)$ *in* X.

Now we give first result of this section which as follows,

Theorem 23. *Let* $(X, S, \leq)$ *be a partially ordered* S*- metric space. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be mappings such that* F *has the mixed* g*- monotone property on* X *and there exists a* $k \in [0,1)$

$$S(F(x,y), F(u,v), F(w,z)) \leq \frac{k}{2}[S(gx, gu, gw) + S(gy, gv, gz)] \tag{3.1}$$

for all $x, y, z, u, v, w \in X$ *for which* $gx \geq gu \geq gw$ *and* $gy \leq gv \leq gz$ *where either* $gu \neq gz$ *or* $gv \neq gw$. *If there exists* $x_0, y_0 \in X$ *such that*

$$gx_0 \leq F(x_0, y_0) \quad and \quad gy_0 \geq F(y_0, x_0).$$

We assume the following hypotheses,

(i). $F : (X \times X) \subseteq g(X)$,

(ii). $g(X)$ *is* S*-complete,*

(iii). g *is* S*- continuous and commutes with* F.

Then F *and* g *have a coupled coincidence point. If* $gu = gz$ *and* $gv = gw$*, then* F *and* g *have common fixed point, that is, there exist* $x \in X$ *such that*

$$g(x) = F(x,x) = x.$$

Proof. Let $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$. Since $F : X \times X \subseteq g(X)$, we can choose $gx_1, gy_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_0 = F(y_0, x_0)$. Again since $F : X \times X \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \quad and \quad g(y_{n+1}) = F(y_n, x_n) \forall n \geq 0. \tag{3.2}$$

Next, we show that

$$g(x_n) \leq g(x_{n+1}) \quad and \quad g(y_n) \geq g(y_{n+1}) \forall n \geq 0. \tag{3.3}$$

Since $g(x_0) \leq F(x_0, y_0) = g(x_1)$ and $g(y_0) \leq F(y_0, x_0) = g(y_1)$, therefore, (3.3) holds for $n = 0$. Next, suppose that (3.3) holds for some fixed $n \geq 0$, that is,

(3.4) $$g(x_n) \leq g(x_{n+1}) \quad and \quad g(y_n) \geq g(y_{n+1})$$

Since F is the mixed g-monotone property, from 3.4 and 1.2, we have

(3.5) $$F(x_n, y) \leq F(x_{n+1}, y) \quad and \quad F(y_{n+1}, x) \leq F(y_n, x)$$

for all $x, y \in X$ and from 3.4 and 1.3 we have

(3.6) $$F(y, x_n) \geq F(y, x_{n+1}) \quad and \quad F(x, y_{n+1}) \geq F(x, y_n)$$

for all $x, y \in X$. If we take $y = y_n$ and $x = x_n$ in 3.5, then we obtain

(3.7) $$g(x_{n+1}) = F(x_n, y_n) \leq F(x_{n+1}, y_n) \quad and \quad F(y_{n+1}, x_n) \leq F(y_n, x_n) = g(y_{n+1})$$

If we take $y = y_{n+1}$ and $x = x_{n+1}$ in 3.6 then

(3.8) $$F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = g(y_{n+2}) \quad and \quad g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \geq F(x_{n+1}, y_n)$$

Now, from 2.7 and 3.8, we have

(3.9) $$g(x_{n+1}) \leq g(x_{n+2}) \quad and \quad g(y_{n+1}) \geq g(y_{n+2}).$$

Therefore, by the mathematical induction, we conclude that 3.3 holds for all $n \geq 0$. Continuing this process, one can easily verify that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq ..... \leq g(x_{n+1}) \leq ...$$

and

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq ..... \geq g(y_{n+1}) \geq ...$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then F and g have a coupled coincidence point. So we assume $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$, that is, we assume that either $g(x_{n+1}) = F(x_n, y_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n, x_n) \neq g(y_n)$.

Next, we claim that, for all $n \geq 0$,

$$S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{2}(2k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \tag{3.10}$$

For $n = 1$, we have

$$\begin{aligned} S(gx_1, gx_1, gx_2) &= S(F(x_0, y_0), F(x_0, y_0), F(x_1, y_1)) \\ &\leq \frac{k}{2}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)]. \end{aligned}$$

Thus 3.10 holds for $n = 1$. Therefore, we presume that 3.10 holds for $n > 0$. Since $g(x_{n+1}) \geq g(x_n)$ and $g(y_{n+1}) \leq g(y_n)$, from 3.1 and 3.2 we have

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \frac{k}{2}[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)] \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} S(gy_n, gy_n, gy_{n+1}) &= S(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq \frac{k}{2}[S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)]. \end{aligned} \tag{3.12}$$

By adding 3.11 and 3.12, then we get

$$S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) \leq 2k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)] \tag{3.13}$$

Continuing the process, we have for each $n \in N$,

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) &\leq k^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\ &\leq \frac{(k)^n}{(1-k)}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \end{aligned}$$

Letting $m, n \to \infty$, we have

$$\lim_{m,n\to\infty}[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] = 0.$$

That is

$$\lim_{m,n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0$$

and

$$\lim_{m,n\to\infty} S(gy_n, gy_n, gy_{n+1}) = 0$$

Thus $\{gx_n\}$ and $\{gy_n\}$ are S- Cauchy sequences in $g(X)$.

Since $g(X)$ is S-complete, we get $\{gx_n\}$ and $\{gy_n\}$ are converges to some $x \in X$ and $y \in X$ respectively. Since g is S- continuous, we have $\{g(gx_n)\}$ and $\{f(gy_n)\}$ are converges to gx and gy respectively. that is

(3.14) $$\lim_{n\to\infty} g(gx_n) = gx$$

and

(3.15) $$\lim_{n\to\infty} g(gy_n) = gy.$$

Also from commutativity of F and g, we have

(3.16) $$F((gx_n), g(y_n)) = gF(x_n, y_n) = g(gx_{n+1})$$

and

(3.17) $$F((gy_n), g(x_n)) = gF(y_n, x_n) = g(gy_{n+1})$$

Next we claim that (x, y) is a coupled coincidence point of F and g.

Now from the condition 3.1, we have

$$\begin{aligned} S(g(gx_{n+1}), g(gx_{n+1}), F(x,y)) &= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)) \\ &\leq \frac{k}{2}[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)] \end{aligned}$$

Letting $n \to \infty$ and using the fact that S is continuous on its variables, we get that

$$S(gx, gx, F(x,y)) \leq \frac{k}{2}[S(gx, gx, gx) + S(gy, gy, gy)] = 0$$

Hence $gx = F(x, y)$. Similarly, we may show that $gy = F(y, x)$.

Finally, we claim that x is common fixed point of F and g.

Since (x, y) is a coupled coincidence point of the mappings F and g, we gave $gx = F(x, y)$ and $gy = F(y, x)$. Assume that $gx \neq gy$. Then by 3.1, we get

$$S(gx, gx, gy) = S(F(x, y), F(x, y), F(y, x)) \leq \frac{k}{2}[S(gx, gx, gy) + S(gy, gy, gx)] \tag{3.18}$$

also

$$S(gy, gy, gx) = S(F(y, x), F(y, x), F(x, y)) \leq \frac{k}{2}[S(gx, gx, gy) + S(gy, gy, gx)] \tag{3.19}$$

by adding 3.18 and 3.19 we have

$$S(gx, gx, gy) + S(gy, gy, gx) \leq k[S(gx, gx, gy) + S(gy, gy, gx)] \tag{3.20}$$

since $k < 1$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) \leq [S(gx, gx, gy) + S(gy, gy, gx)] \tag{3.21}$$

Which contradiction. So $gx = gy$, and hence

$$F(x, y) = gx = gy = F(y, x)$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gy_n\}$ we have $\{gy_{n+1}\}$ is S- convergent to x. Thus

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(gx_{n+1}, gx_{n+1}, F(x, y)) \\ &= S(F(x_n, y_n), F(x_n, y_n), F(x, y)) \\ &\leq \frac{k}{2}[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)] \end{aligned}$$

Letting $n \to \infty$ and use the fact that S is continuous on its variables, we get

$$S(x, x, gx) \leq \frac{k}{2}[S(x, x, gx) + S(y, y, gy)]$$

Similarly, we may show that

$$S(y, y, gy) \leq \frac{k}{2}[S(x, x, gx) + S(y, y, gy)]$$

Thus

$$S(x,x,gx) + S(y,y,gy) \leq k[S(x,x,gx) + S(y,y,gy)].$$

Since $k < 1$, the last inequality happens only if $S(x,x,gx) = 0$ and $S(y,y,gy) = 0$. Hence $x = gx$ and $y = gy$. Thus we get $gx = F(x,y) = x$. Thus F and g have a common fixed point. This completes the proof of the theorem. □

The following example show that Theorem 23 is more general then Theorem 13.

Example 24. *Let $X = [0,1]$, with the usual partial ordered $\leq$. Let a S- metric on X be defined by*

$$S(x,y,z) = \mid x - z \mid + \mid y - z \mid .$$

Then (X,S) is a complete regular ordered S- metric space.
Let $g : X \to X$ and $F : X \times X \to X$ be defined by

$$F(x,y) = \frac{1}{4}[\sin x + \sin y] \quad g(x) = \tfrac{x}{2} \tag{3.22}$$

for all $x,y \in X$. Since $\mid \sin x - \sin y \mid \leq \mid x - y \mid$ holds for all $x,y \in X$. Then we have $k = \frac{1}{2} \in [0,1)$. So all the conditions of Theorem 23 and Theorem 26 are satisfied. Then there exists a coupled fixed point of F. In this case $(0,0)$ is coupled fixed point of F.

Example 25. *Let X be the set $[0,\infty)$ and*

$$S(x,y,z) = \mid x - z \mid + \mid y - z \mid .$$

We set $g : X \to X$ and $F : X \times X \to X$ be defined as $g(x) = x^2$ and

$$F(x,y) = \begin{cases} \frac{3x^2 - 3y^2 + 1}{10} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

Then for $k = \frac{3}{5} \in [0,1)$ we have coupled fixed point $(0,0)$ in X.

Theorem 26. *In the above Theorem 23, in the place of condition (ii), if we assume the following conditions in the complete S- metric space X, namely*

(3.23) *if* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \to x \in X$, *then* $x_n \leq x, \ \forall n,$

and

(3.24) *if* $\{y_n\} \subset X$ *is a nonincreasing sequence with* $y_n \to y \in X$, *then* $y_n \geq y, \ \forall n.$

Then, we have the conclusion of Theorem 23, provided g is non decreasing.

Proof. Proceeding exactly as in Theorem 23, we have $\{gx_n\}$ and $\{gy_n\}$ are S- Cauchy sequences in X. Since (X, S) is a complete, there exists $(x, y) \in X \times X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x \tag{3.25}$$

and

$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y \tag{3.26}$$

Therefore, from (iii) we arrive at 3.14, 3.15 3.16 and 3.17. Since $\{gx_n\}$ is a non decreasing sequence and $gx_n \to x$ and as $\{gy_n\}$ is non increasing sequence and $gy_n \to y$, by assumption 3.23 and 3.24 we have, $g(gx_n) \leq gx$ and $g(gy_n) \geq gy$ for all $n \geq 0$. If $g(gx_n) = gx$ and $g(gy_n) = gy$ for some n, then by construction $g(gx_{n+1}) = gx$ and $g(gy_{n+1}) = gy$ and (x, y) is coupled fixed point. So we assume either $g(gx_n) \neq gx$ or $g(gy_n) \neq gy$. Applying the contractive condition 3.1, we have

$$\begin{aligned} S(F(x,y), F(x,y), gx) &\leq 2S(F(x,y), F(x,y), F(gx_n, gy_n)) + S(F(gx_n, gy_n), F(gx_n, gy_n), gx) \\ &= 2S(F(x,y), F(x,y), F(gx_n, gy_n)) + S(gF(x_n, y_n), gF(x_n, y_n), gx) \\ &\leq k[S(gx, gx, gx_n) + S(gy, gy, gy_n)] + S(g(gx_{n+1}), g(gx_{n+1}), gx) \end{aligned}$$

Taking $n \to \infty$ in the above inequality we obtain $S(F(x,y), F(x,y), gx) = 0$, that is, $F(x,y) = g(x)$. Similarly we have that $F(y,x) = g(y)$. Remaining part of the proof follows from Theorem 23. Hence we have $g(x) = F(x,x) = x$. This completes the proof of the theorem.

□

Corollary 27. *Let $(X, S, \leq)$ be a partially ordered S- metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g- monotone property on X and there exists a $k \in [0,1)$*

$$S(F(x,y), F(x,y), F(w,z)) \leq \frac{k}{2}[S(gx, gx, gw) + S(gy, gy, gz)] \tag{3.27}$$

for all $x, y, z, w \in X$ for which $gx \geq gw$ and $gy \leq gz$. If there exists $x_0, y_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0) \quad \textit{and} \quad gy_0 \geq F(y_0, x_0).$$

We assume the following hypotheses,

(i). *$F:(X\times X)\subseteq g(X)$,*

(ii). *$g(X)$ is S-complete,*

(iii). *g is S- continuous and commutes with F.*

Then F and g have a coupled coincidence point. If $gx=gz$ and $gy=gw$, then F and g have common fixed point, that is, there exist $x\in X$ such that

$$g(x)=F(x,x)=x.$$

Proof. It follows from Theorem 23 if we take $x=u$ and $y=v$. □

Corollary 28. *Let $(X,S,\leq)$ be a partially ordered S- metric space. Let $F:X\times X\to X$ be mapping such that F has the mixed monotone property on X and there exists a $k\in[0,1)$*

$$S(F(x,y),F(u,v),F(w,z))\leq\frac{k}{2}[S(x,u,w)+S(y,v,z)] \tag{3.28}$$

for all $x,y,z,u,v,w\in X$ for which $x\geq u\geq w$ and $y\leq v\leq z$ where either $u\neq z$ or $v\neq w$. If there exists $x_0,y_0\in X$ such that

$$x_0\leq F(x_0,y_0)\quad and\quad y_0\geq F(y_0,x_0).$$

Then F has coupled fixed point in X. If $u=z$ and $v=w$, then F has fixed point, that is, there exist $x\in X$ such that

$$x=F(x,x).$$

Proof. If we define $g:X\to X$ be an identity mapping in Theorem 23 then result is follows. □

Corollary 29. *Let $(X,S,\leq)$ be a partially ordered S- metric space. Let $F:X\times X\to X$ be mapping such that F has the mixed monotone property on X and there exists a $k\in[0,1)$*

$$S(F(x,y),F(x,y),F(w,z))\leq\frac{k}{2}[S(x,x,w)+S(y,y,z)] \tag{3.29}$$

for all $x,y,z,w\in X$ for which $x\geq w$ and $y\leq z$. If there exists $x_0,y_0\in X$ such that

$$x_0\leq F(x_0,y_0)\quad and\quad y_0\geq F(y_0,x_0).$$

Then F has a coupled fixed point. If $x = z$ and $y = w$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x).$$

Proof. If we define $g : X \to X$ be an identity mapping in Corollary 28 then result is follows. □

Conclusion 30. *In this article one thing is observed that Theorem 13 implies Theorem 23 but converges may not be true. Example 24, 25 and 22 are in support of this fact. This fact is obvious because of the property of $\left[0, \frac{1}{2}\right) \subset [0, 1)$ but converges is not true.*

4. Acknowledgement

The authors thank the referees for their careful reading of the manuscript and for their suggestions.

References

1. R. Chugh, T. Kadian, A. Rani, B.E. Rhoades, Property P in G-metric spaces, Fixed Point Theory Appl. Vol. 2010, Article ID 401684.
2. B.C. Dhage, Generalized metric spaces mappings with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.
3. S. Gąğahler, 2-metrische Rąğaume und iher topoloische Struktur, Math. Nachr. 26 (1963), 115-148.
4. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289ÍC297.
5. Z. Mustafa, H. Obiedat, F. Awawdeh, Some common fixed point theorems for mapping on complete G-metric spaces, Fixed Point Theory Appl. Vol. 2008, Article ID 189870.
6. Z. Mustafa, A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory, Ph. D. Thesis, The University of Newcastle, Callaghan, Australia, 2005.
7. Z. Mustafa, B. Sims, Some results concerning D-metric spaces, Proc. Internat. Conf. Fixed Point Theory and Applications, pp. 189-198, Valencia, Spain, 2003.
8. S.V.R. Naidu , K.P.R. Rao, N. Srinivasa Rao, On the topology of D-metric spaces and the generation of D-metric spaces from metric spaces, Internat. J. Math. Math. Sci. 2004 (2004), No. 51, 2719-2740.
9. S.V.R. Naidu, K.P.R. Rao, N. Srinivasa Rao, On the concepts of balls in a D-metric space, Internat. J. Math. Math. Sci. 2005 (2005), 133-141.
10. S.V.R. Naidu, K.P.R. Rao, N. Srinivasa Rao, On convergent sequences and fixed point theorems in D-metric spaces, Internat. J. Math. Math. Sci. 2005 (2005), 1969-1988.
11. S. Sedghi, K.P.R. Rao, N. Shobe, Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces, Internat. J. Math. Math. Sci. 6 (2007), 225-237.
12. S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D^*-metric spaces, Fixed Point Theory Appl. Vol. 2007, Article ID 27906, 13 pages.
13. W. Shatanawi, Fixed point theory for contractive mappings satisfying ψ-maps in G-metric spaces, Fixed Point Theory Appl. Vol. 2010, Article ID 181650.

14. S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorem in S-metric spaces,Mat. Vesnik 64 (2012), 258-266.
15. S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales, Fund. Math. 3 (1922) 133-181.
16. T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Anal.* TMA 65, 1379–1393 (2006).
17. V. Lakshmikantham and L. Ciric, ąřCoupled fixed point theorems for nonlinear contractions in partially ordered metric spaces,ąś *Nonlinear Analysis: Theory, Methods and Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.
18. R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Analysis 87 (2008), 109–116.
19. I. Altun and H. Simsek, Some fixed point theorems on ordered metric spaces and application, Fixed Point Theory Appl. 2010(2010), Article ID 621492, 17 pages.
20. A. Amini-Harandi and H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, Nonlinear Analysis, 72 (2010), 2238–2242.
21. D. Guo and V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (1987), 623-632.