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FIXED POINT AND COMMON FIXED POINT THEOREMS IN CONE BALL-METRIC SPACES

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Abstract. In this paper, we define a new cone ball-metric and get fixed points and common fixed points for the Meir-Keeler type functions in cone ball-metric spaces.
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1. Introduction and preliminaries

Huang and Zhang [4] have introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. In 2006, Mustafa and Sims [6] introduced a more appropriate generalization of metric spaces, G-metric spaces. Recently, Beg et al. [2] introduced the notion of generalized cone metric s-paces, and proved some fixed point results for mappings satisfying certain contractive conditions. In [3] Chen and Tsai introduce the notion of the cone ball-metric $\mathcal{B}$. In this paper, we proves fixed point and common fixed results in cone ball-metric.

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers, and we recall some definitions of the cone metric spaces and some of the properties [4], as follow:

Definition 1.1. [4] Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is nonempty, closed, and $P \neq \{\theta\}$,
(ii) $a, b \in \mathbb{R}^+$, $x, y \in P \Rightarrow ax + by \in P$,
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

For given a cone $P \subset E$, we can define a partial ordering with respect to P by $x \preccurlyeq y$ or $x \succcurlyeq y$ if and only if $y - x \in P$ for all $x, y \in E$. The real Banach space E equipped with the partial ordered induced by P is denoted by $(E, \preccurlyeq)$. We shall write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \preccurlyeq x \preccurlyeq y \quad \Rightarrow \quad \|x\| \leq K\|y\|.$$

The least positive number K satisfying above is called the normal constant of P.

The cone P is called regular if every non-decreasing sequence which is bounded from above is convergent, that is, if $\{x_n\}$ is a sequence such that

$$x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq \cdots \preccurlyeq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every non-increasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Moreover, P is called stronger minihedral if every subset of E which is bounded above has a supremum [1].

In the following we always suppose that E is a real Banach space with a stronger minihedral regular cone P and $intP \neq \phi$, and $\preccurlyeq$ is a partial ordering with respect to P.

Metric spaces are playing an important role in mathematics and the applied sciences. In 2003, Mustafa and Sims [6] introduced a more appropriate and robust notion of a generalized metric space as follows.

Definition 1.2. [6] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following axioms:

(1) $G(x, y, z) = 0$ if and only if $x = y = z$;
(2) $G(x, x, y) > 0$, for all $x \neq y$;
(3) $G(x, y, z) \geq G(x, x, y)$, for all $x, y, z \in X$;
(4) $G(x, y, z) = G(x, z, y) = G(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$, for all $x, y, z, w \in X$.

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

This research subject is interesting and widespread. But is too abstract makes the human difficulty with to understand. So Chen and Tsai [3] introduce the concept of cone ball-metric spaces and prove fixed point results on such spaces for functions satisfying the contractions involving the Meir-Keeler type functions.

Infact Chen and Tsai [3] introduced the following notion of the cone ball-metric $\mathcal{B}$.

Definition 1.3. Let (X, d) be a cone metric space, $\mathcal{B} : X \times X \times X \to E$, $x, y, z \in X$ and we denote

$$\delta(B) = \sup\{d(a, b) : a, b \in B\},$$

and

$$\mathcal{B}(x, y, z) = \delta(B),$$

where $B = \cap\{F \subset X | F \text{ is a closed ball and } \{x, y, z\} \subset F\}$. Then we call $\mathcal{B}$ a ball-metric with respect to the cone metric d, and $(X, \mathcal{B})$ a cone ball-metric space. It is clear that $\mathcal{B}(x, x, y) = d(x, y)$.

Remark 1.4. It is clear that the cone ball-metric $\mathcal{B}$ has the following properties:

(1) $\mathcal{B}(x, y, z) = \theta$ if and only if $x = y = z$;
(2) $\mathcal{B}(x, x, y) \succ \theta$, for all $x \neq y$;
(3) $\mathcal{B}(x, x, y) \preccurlyeq \mathcal{B}(x, y, z)$, for all $x, y, z \in X$;
(4) $\mathcal{B}(x, y, z) = \mathcal{B}(x, z, y) = \mathcal{B}(z, y, x) = \cdots$ (symmetric in all three variables);
(5) $\mathcal{B}(x, y, z) \preccurlyeq \mathcal{B}(x, w, w) + \mathcal{B}(w, y, z)$, for all $x, y, z, w \in X$;
(6) $\mathcal{B}(x, y, z) \preccurlyeq \mathcal{B}(x, w, w) + \mathcal{B}(y, w, w) + \mathcal{B}(z, w, w)$, for all $x, y, z, w \in X$.

Definition 1.5. Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(1) Cauchy sequence if for every $\varepsilon \in E$ with $\theta \ll \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m, l > n_0$, $\mathcal{B}(x_n, x_m, x_l) \ll \varepsilon$.
(2) Convergent sequence if for every $\varepsilon \in E$ with $\theta \ll \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $\mathcal{B}(x_n, x_m, x) \ll \varepsilon$ for some $x \in X$. Here x is called the limit of the sequence $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.6. Let $(X, \mathcal{B})$ be a cone ball-metric space. Then X is said to be complete if every Cauchy sequence is convergent in X.

Proposition 1.7. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent:*

(i) *$\{x_n\}$ converges to x;*
(ii) *$\mathcal{B}(x_n, x_n, x) \to \theta$ as $n \to \infty$;*
(iii) *$\mathcal{B}(x_n, x, x) \to \theta$ as $n \to \infty$;*
(iv) *$\mathcal{B}(x_n, x_m, x) \to \theta$ as $n, m \to \infty$.*

Proposition 1.8. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}$ be a sequence in X, $x, y \in X$. If $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.*

Proof. Let $\varepsilon \in E$ with $\theta \ll \varepsilon$ be given. Since $x_n \to x$ and $x_n \to y$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$,

$$\mathcal{B}(x_n, x_m, x) \ll \frac{\varepsilon}{3} \text{ and } \mathcal{B}(x_n, x_m, y) \ll \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned} \mathcal{B}(x, x, y) &\preccurlyeq \mathcal{B}(x, x_n, x_n) + \mathcal{B}(x_n, x, y) \\ &= \mathcal{B}(x, x_n, x_n) + \mathcal{B}(y, x_n, x) \\ &\preccurlyeq \mathcal{B}(x, x_n, x_n) + \mathcal{B}(y, x_m, x_m) + \mathcal{B}(x_m, x_n, x) \\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, $\mathcal{B}(x, x, y) \ll \frac{\varepsilon}{\alpha}$ for all $\alpha \geq 1$, and so $\frac{\varepsilon}{\alpha} - \mathcal{B}(x, x, y) \in P$ for all $\alpha \geq 1$. Since $\frac{\varepsilon}{\alpha} \to \theta$ as $\alpha \to \infty$ and P is closed, we have that $-\mathcal{B}(x, x, y) \in P$. This implies that $\mathcal{B}(x, x, y) = \theta$, since $\mathcal{B}(x, x, y) \in P$. So $x = y$. □

Proposition 1.9. *Let $(X, \mathcal{B})$ be a cone ball-metric space and $\{x_n\}, \{y_m\}, \{z_l\}$ be three sequences in X. If $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, then $\mathcal{B}(x_n, y_m, z_l) \to \mathcal{B}(x, y, z)$ as $n \to \infty$.*

Proof. Let $\varepsilon \in E$ with $\theta \ll \varepsilon$ be given. Since $x_n \to x$, $y_m \to y$, $z_l \to z$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m, l > n_0$,

$$\mathcal{B}(x_n, x, x) \ll \frac{\varepsilon}{3}, \ \mathcal{B}(y_m, y, y) \ll \frac{\varepsilon}{3}, \ \mathcal{B}(z_l, z, z) \ll \frac{\varepsilon}{3},$$

Therefore,

$$\begin{aligned} \mathcal{B}(x_n, y_m, z_l) &\preccurlyeq \mathcal{B}(x_n, x, x) + \mathcal{B}(x, y_m, z_l) \\ &\preccurlyeq \mathcal{B}(x_n, x, x) + \mathcal{B}(y_m, y, y) + \mathcal{B}(y, x, z_l) \\ &\preccurlyeq \mathcal{B}(x_n, x, x) + \mathcal{B}(y_m, y, y) + \mathcal{B}(z_l, z, z) + \mathcal{B}(z, x, y) \\ &\ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \mathcal{B}(x, y, z), \end{aligned}$$

that is,

$$\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z) \ll \varepsilon.$$

Similarly,

$$\mathcal{B}(x, y, z) - \mathcal{B}(x_n, y_m, z_l) \ll \varepsilon.$$

Therefore, for all $\alpha \geq 1$, we have

$$\mathcal{B}(x_n, y_m, z_l) - \mathcal{B}(x, y, z) \ll \frac{\varepsilon}{\alpha},$$

and

$$\mathcal{B}(x,y,z) - \mathcal{B}(x_n,y_m,z_l) \ll \frac{\varepsilon}{\alpha}.$$

These imply that

$$\frac{\varepsilon}{\alpha} - \mathcal{B}(x_n,y_m,z_l) + \mathcal{B}(x,y,z) \in P,$$

$$\frac{\varepsilon}{\alpha} + \mathcal{B}(x_n,y_m,z_l) - \mathcal{B}(x,y,z) \in P.$$

Since P is closed and $\frac{\varepsilon}{\alpha} \to \theta$ as $\alpha \to \infty$, we have that

$$\lim_{n,m,l\to\infty} [-\mathcal{B}(x_n,y_m,z_l) + \mathcal{B}(x,y,z)] \in P,$$

$$\lim_{n,m,l\to\infty} [\mathcal{B}(x_n,y_m,z_l) - \mathcal{B}(x,y,z)] \in P.$$

These show that

$$\lim_{n,m,l\to\infty} \mathcal{B}(x_n,y_m,z_l) = \mathcal{B}(x,y,z).$$

So we complete the proof. □

2. Main results

In the section, we first recall the notion of the Meir-Keeler type function. A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler type function (see [5]), if for each $\eta \in \mathbb{R}^+$, there exists $\delta > 0$ such that for $t \in \mathbb{R}^+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new weaker Meir-Keeler type function in a cone ball-metric space $(X, \mathcal{B})$, as follows:

Definition 2.1. Let $(X, \mathcal{B})$ be a cone ball-metric space with cone P, and let $\psi : intP \cup \{\theta\} \to intP \cup \{\theta\}$. Then the function ψ is called a weaker Meir-Keeler type function in X, if for each η, $\theta \ll \eta$, there exists δ, $\theta \ll \delta$ such that for $x,y,z \in X$ with $\eta \preccurlyeq \mathcal{B}(x,y,z) \ll \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\mathcal{B}(x,y,z)) \ll \eta$.

Further, we let the function $\psi : intP \cup \{\theta\} \to intP \cup \{\theta\}$ satisfying the following conditions:

(i) ψ be a weaker Meir-Keeler type function;
(ii) for each $t \in intP$, we have $\theta \ll \psi(t) \ll t$ and $\psi(\theta) = \theta$;
(iii) for $t_n \in intP \cup \{\theta\}$, if $\lim_{n\to\infty} t_n = \gamma \gg \theta$, then $\lim_{n\to\infty} \psi(t_n) \ll \gamma$;
(iv) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is non-increasing.

Then we call this mapping a ψ-function.

We now state our main common fixed point result for the weaker Meir-Keeler type function in a cone ball-metric space $(X, \mathcal{B})$, as follows:

Theorem 2.2. *Let $(X,\mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and f,g be two self-mapplings of X such that $fX\subset gX$. Suppose that there exists a ψ-function such that*

$$\mathcal{B}(fx,fy,fz)\preccurlyeq\psi(L(x,y,z)), \tag{2.1}$$

where

$$L(x,y,z) \;=\; \max\{\mathcal{B}(gx,gy,gz),\mathcal{B}(gx,fx,fx),\mathcal{B}(gy,fy,fy),\mathcal{B}(gz,fz,fz)\}.$$

If gX is closed, then f and g have a coincidence point in X.

Moreover, if f and g commute at their coincidence points, then f and g have a unique common fixed point in X

Proof. Given $x_0\in X$. Since $fX\subset gX$, we can choose $x_1\in X$ such that $gx_1=fx_0$. Continuing this process, we define the sequence $\{x_n\}$ in X recursively as follows:

$$fx_n=gx_{n+1}\ \text{ for each }\ n\in\mathbb{N}\cup\{0\}.$$

In what follows we will suppose that $fx_{n+1}\neq fx_n$ for all $n\in\mathbb{N}$, since if $fx_{n+1}=fx_n$ for some n, then $fx_{n+1}=gx_{n+1}$, that is , f,g have a coincidence point x_{n+1}, and so we complete the proof.

By (2.1), we have

$$\mathcal{B}(fx_n,fx_{n+1},fx_{n+1})\preccurlyeq\psi(L(x_n,x_{n+1},x_{n+1})),$$

where

$$\begin{aligned}L(x_n,x_{n+1},x_{n+1}) &= \max\{\mathcal{B}(gx_n,gx_{n+1},gx_{n+1}),\mathcal{B}(gx_n,fx_n,fx_n),\\ &\quad \mathcal{B}(gx_{n+1},fx_{n+1},fx_{n+1}),\mathcal{B}(gx_{n+1},fx_{n+1},fx_{n+1})\}\\ &= \max\{\mathcal{B}(fx_{n-1},fx_n,fx_n),\mathcal{B}(fx_{n-1},fx_n,fx_n),\\ &\quad \mathcal{B}(fx_n,fx_{n+1},fx_{n+1}),\mathcal{B}(fx_n,fx_{n+1},fx_{n+1})\}\\ &= \max\{\mathcal{B}(fx_{n-1},fx_n,fx_n),\mathcal{B}(fx_n,fx_{n+1},fx_{n+1})\}.\end{aligned}$$

Therefore, by the condition (ii) of ψ, we conclude that for each $n\in\mathbb{N}$,

$$\mathcal{B}(fx_n,fx_{n+1},fx_{n+1}) \;\ll\; \mathcal{B}(fx_{n-1},fx_n,fx_n),$$

and

$$\begin{aligned}\mathcal{B}(fx_n,fx_{n+1},fx_{n+1}) &\preccurlyeq\psi(\mathcal{B}(fx_{n-1},fx_n,fx_n))\\ &\preccurlyeq\cdots\\ &\preccurlyeq\psi^n(\mathcal{B}(fx_0,fx_1,fx_1)).\end{aligned}$$

Since $\{\psi^n(\mathcal{B}(fx_0, fx_1, fx_1))\}_{n\in\mathbb{N}}$ is non-incresing, it must converge to some η, $\theta \preccurlyeq \eta$. We claim that $\eta = \theta$. On the contrary, assume that $\theta \ll \eta$. Then by the definition of the ψ-function, there exists δ, $\theta \ll \delta$ such that for $\theta \ll \mathcal{B}(fx_0, fx_1, fx_1)$ with $\eta \preccurlyeq \mathcal{B}(fx_0, fx_1, fx_1) \ll \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\mathcal{B}(fx_0, fx_1, fx_1)) \ll \eta$. Since $\lim_{n\to\infty} \psi^n(\mathcal{B}(fx_0, fx_1, fx_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \preccurlyeq \psi^m \mathcal{B}(fx_0, fx_1, fx_1) \ll \delta + \eta$, for all $m \geq m_0$. Thus, we conclude that $\psi^{m_0+n_0}(\mathcal{B}(fx_0, fx_1, fx_1)) \ll \eta$. So we get a contradiction. So $\lim_{n\to\infty} \psi^n(\mathcal{B}(fx_0, fx_1, fx_1)) = \theta$, and so we have

$$\lim_{n\to\infty} \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) = \theta.$$

Next, we claim that the sequence $\{fx_n\}$ is a Cauchy sequence. Suppose that $\{fx_n\}$ is not a Cauchy sequence. Then there exists $\gamma \in E$ with $\theta \ll \gamma$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd,
(2) $\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \succcurlyeq \gamma$, and
(3) m_k is the smallest even number such that the conditions (1), (2) hold.

Since $\lim_{n\to\infty} \mathcal{B}(fx_n, fx_{n+1}, fx_{n+1}) = \theta$ and by (2), (3), we have that

$$\begin{aligned}
\gamma &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \\
&\preccurlyeq \mathcal{B}(fx_{n_k}, f_{m_k-1}, f_{m_k-1}) + \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}) \\
&\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k-2}, fx_{m_k-2}) + \mathcal{B}(fx_{m_k-2}, fx_{m_k-1}, fx_{m_k-1}) \\
&\quad + \mathcal{B}(fx_{m_k-1}, f_{m_k}, f_{m_k}) \\
&\preccurlyeq \gamma + \mathcal{B}(fx_{m_k-2}, fx_{m_k-1}, fx_{m_k-1}) + \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}).
\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) = \gamma.$$

Since

$$\begin{aligned}
\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) &\preccurlyeq \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}) + \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \\
&\quad + \mathcal{B}(fx_{m_k}, fx_{m_k-1}, fx_{m_k-1}).
\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) \preccurlyeq \gamma. \tag{2.2}$$

On the other hand,

$$\begin{aligned}\gamma &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \\ &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{n_k-1}, fx_{n_k-1}) + \mathcal{B}(fx_{n_k-1}, fx_{m_k}, fx_{m_k}) \\ &\preccurlyeq \mathcal{B}(fx_{n_k}, fx_{n_k-1}, fx_{n_k-1}) + \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) \\ &\quad +\mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}).\end{aligned}$$

Taking $\lim_{k\to\infty}$, we also deduce

$$\gamma \preccurlyeq \lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}). \tag{2.3}$$

By (2.2) and (2.3), we get

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) = \gamma.$$

And, by (2.1), we have that

$$\mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \preccurlyeq \psi(L(x_{n_k}, x_{m_k}, x_{m_k}))$$

where

$$\begin{aligned}L(x_{n_k}, x_{m_k}, x_{m_k}) &= \max\{\mathcal{B}(gx_{n_k}, gx_{m_k}, gx_{m_k}), \mathcal{B}(gx_{n_k}, fx_{n_k}, fx_{n_k}), \\ &\quad \mathcal{B}(gx_{m_k}, fx_{m_k}, fx_{m_k}), \mathcal{B}(gx_{m_k}, fx_{m_k}, fx_{m_k})\} \\ &= \max\{\mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}), \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}), \\ &\quad \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}), \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k})\}.\end{aligned}$$

(I) If

$$L(x_{n_k}, x_{m_k}, x_{m_k}) = \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}),$$

then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{m_k-1}, fx_{m_k-1}) = \gamma,$$

and

$$\gamma \preccurlyeq \lim_{k\to\infty} \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \ll \gamma,$$

a contradiction.

(II) If

$$L(x_{n_k}, x_{m_k}, x_{m_k}) = \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}),$$

or

$$L(x_{n_k}, x_{m_k}, x_{m_k}) = \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}),$$

then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \mathcal{B}(fx_{n_k-1}, fx_{n_k}, fx_{n_k}) = \theta,$$

$$\lim_{k\to\infty} \mathcal{B}(fx_{m_k-1}, fx_{m_k}, fx_{m_k}) = \theta,$$

and

$$\gamma \preccurlyeq \lim_{k\to\infty} \mathcal{B}(fx_{n_k}, fx_{m_k}, fx_{m_k}) \preccurlyeq \theta,$$

a contradiction.

Follow (I) and (II), we get the sequence $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exist $\nu, \mu \in X$ such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = g(\mu) = \nu.$$

We shall show that μ is a coincidence point of f and g, that is, we claim that

$$\mathcal{B}(g\mu, f\mu, f\mu) = \theta.$$

If not, assume that $\mathcal{B}(g\mu, f\mu, f\mu) \neq \theta$, then by (2.1), we have

$$\begin{aligned} \mathcal{B}(g\mu, f\mu, f\mu) &\preccurlyeq \mathcal{B}(g\mu, fx_n, fx_n) + \mathcal{B}(fx_n, f\mu, f\mu) \\ &\preccurlyeq \mathcal{B}(g\mu, fx_n, fx_n) + \psi(L(x_n, \mu, \mu)), \end{aligned}$$

where

$$L(x_n, \mu, \mu) \in \{\mathcal{B}(gx_n, g\mu, g\mu), \mathcal{B}(gx_n, fx_n, fx_n), \mathcal{B}(g\mu, f\mu, f\mu), \mathcal{B}(g\mu, f\mu, f\mu)\}.$$

(III) If

$$L(x_n, \mu, \mu) = \mathcal{B}(gx_n, g\mu, g\mu),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty} \mathcal{B}(gx_n, g\mu, g\mu) = \mathcal{B}(g\mu, g\mu, g\mu) = \theta,$$

and

$$\begin{aligned} \mathcal{B}(g\mu, f\mu, f\mu) &= \lim_{n\to\infty} \mathcal{B}(g\mu, fx_n, fx_n) + \lim_{n\to\infty} \psi(\mathcal{B}(gx_n, g\mu, g\mu)) \\ &\preccurlyeq \theta, \end{aligned}$$

a contradiction.

(IV) If

$$L(x_n, \mu, \mu) = \mathcal{B}(gx_n, fx_n, fx_n),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty} \mathcal{B}(gx_n, fx_n, fx_n) = \mathcal{B}(g\mu, g\mu, g\mu) = \theta,$$

and

$$\mathcal{B}(g\mu, f\mu, f\mu) = \lim_{n\to\infty} \mathcal{B}(g\mu, fx_n, fx_n) + \lim_{n\to\infty} \psi(\mathcal{B}(gx_n, fx_n, fx_n) \preccurlyeq \theta,$$

a contradiction.

(V) If

$$L(x_n, \mu, \mu) = \mathcal{B}(g\mu, f\mu, f\mu),$$

then

$$\mathcal{B}(g\mu, f\mu, f\mu) = \psi(\mathcal{B}(g\mu, f\mu, f\mu)) \ll \mathcal{B}(g\mu, f\mu, f\mu),$$

a contradiction.

Follow (III)-(V), we obtain that $\mathcal{B}(g\mu, f\mu, f\mu) = \theta$, that is, $g\mu = f\mu = \nu$, and so μ is a coincidence point of f and g.

Suppose that f and g commute at μ. Then

$$f\nu = fg\mu = gf\mu = g\nu.$$

Later, we claim that $\mathcal{B}(f\mu, f\nu, f\nu) = \theta$. By (2.1), we have

$$\mathcal{B}(f\mu, f\nu, f\nu) \preccurlyeq \psi(L(\mu, \nu, \nu)),$$

where

$$\begin{aligned} L(x, y, z) &= \max\{\mathcal{B}(g\mu, g\nu, g\nu), \mathcal{B}(g\mu, f\mu, f\mu), \mathcal{B}(g\nu, f\nu, f\nu), \mathcal{B}(g\nu, f\nu, f\nu)\} \\ &= \max\{\mathcal{B}(f\mu, f\nu, f\nu), \mathcal{B}(f\mu, f\mu, f\mu), \mathcal{B}(f\nu, f\nu, f\nu), \mathcal{B}(f\nu, f\nu, f\nu)\} \\ &= \max\{\mathcal{B}(f\mu, f\nu, f\nu), \theta\}. \end{aligned}$$

Therefore, if

$$\mathcal{B}(f\mu, f\nu, f\nu) \preccurlyeq \psi(\mathcal{B}(f\mu, f\nu, f\nu)) \ll \mathcal{B}(f\mu, f\nu, f\nu),$$

then we get a contradition, which implies that $\mathcal{B}(f\mu, f\nu, f\nu) = \theta$, $\mathcal{B}(\nu, f\nu, f\nu) = \theta$, that is, $\nu = f\nu = g\nu$. So ν is a common fixed point of f and g.

Let $\overline{\nu}$ be another common fixed point of f and g. By (2.1),

$$\mathcal{B}(\overline{\nu}, \nu, \nu) = \mathcal{B}(f\overline{\nu}, f\nu, f\nu) \preccurlyeq \psi(L(\overline{\nu}, \nu, \nu)),$$

where

$$\begin{aligned} L(x, y, z) &= \max\{\mathcal{B}(g\overline{\nu}, g\nu, g\nu), \mathcal{B}(g\overline{\nu}, f\overline{\nu}, f\overline{\nu}), \mathcal{B}(g\nu, f\nu, f\nu), \mathcal{B}(g\nu, f\nu, f\nu)\} \\ &= \max\{\mathcal{B}(f\overline{\nu}, f\nu, f\nu), \mathcal{B}(f\overline{\nu}, f\overline{\nu}, f\overline{\nu}), \mathcal{B}(f\nu, f\nu, f\nu), \mathcal{B}(f\nu, f\nu, f\nu)\} \\ &= \max\{\mathcal{B}(f\overline{\nu}, f\nu, f\nu), \theta\} \\ &= \{\mathcal{B}(\overline{\nu}, \nu, \nu), \theta\}. \end{aligned}$$

Therefore, we also conclude that $\mathcal{B}(\overline{\nu},\nu,\nu)=\theta$, that is $\overline{\nu}=\nu$. So we show that ν is the unique common fixed point of g and f. □

Next, we state the following fixed point results for the weaker Meir-Keeler type functions in ball-metric spaces.

Theorem 2.3. *Let $(X,\mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and $f:X\to X$. Suppose that there exists a ψ-function such that*

$$\mathcal{B}(fx,fy,fz)\preccurlyeq\psi(\mathcal{L}(x,y,z)) \text{ for all } x,y,z\in X, \tag{2.4}$$

where

$$\mathcal{L}(x,y,z)=\max\{\mathcal{B}(x,y,z),\mathcal{B}(x,fx,fx),\mathcal{B}(y,fy,fy),\mathcal{B}(fx,y,z)\}$$

Then f has a unique fixed point (say μ) in X and f is continuous at μ.

Proof. Given $x_0\in X$. Define the sequence $\{x_n\}$ in X recursively as follows:

$$fx_{n-1}=x_n \text{ for each } n\in\mathbb{N}.$$

In what follows we will suppose that $x_{n+1}\neq x_n$ for all $n\in\mathbb{N}$, since if $x_{n+1}=x_n$ for some n, then $x_{n+1}=fx_n=x_n$, and so we complete the proof.

By (2.4), we deduce

$$\begin{aligned}\mathcal{B}(x_n,x_{n+1},x_{n+1}) &= \mathcal{B}(fx_{n-1},fx_n,fx_n)\\ &\preccurlyeq \psi(\mathcal{L}(x_{n-1},x_n,x_n)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(x_{n-1},x_n,x_n) &= \max\{\mathcal{B}(x_{n-1},x_n,x_n),\mathcal{B}(x_{n-1},fx_{n-1},fx_{n-1}),\\ &\quad \mathcal{B}(x_n,fx_n,fx_n),\mathcal{B}(fx_{n-1},x_n,x_n)\}\\ &= \max\{\mathcal{B}(x_{n-1},x_n,x_n),\mathcal{B}(x_{n-1},x_n,x_n),\\ &\quad \mathcal{B}(x_n,x_{n+1},x_{n+1}),\mathcal{B}(x_n,x_n,x_n)\}\\ &= \max\{\mathcal{B}(x_{n-1},x_n,x_n),\mathcal{B}(x_n,x_{n+1},x_{n+1})\}.\end{aligned}$$

If

$$\mathcal{L}(x_{n-1},x_n,x_n)=\mathcal{B}(x_n,x_{n+1},x_{n+1}),$$

then

$$\begin{aligned}\mathcal{B}(x_n,x_{n+1},x_{n+1}) &\preccurlyeq \psi(\mathcal{L}(x_{n-1},x_n,x_n))\\ &\ll \mathcal{B}(x_n,x_{n+1},x_{n+1}),\end{aligned}$$

a contradiction. So we deduce that

$$\begin{aligned} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq \psi(\mathcal{L}(x_{n-1}, x_n, x_n)) \\ &\ll \mathcal{B}(x_{n-1}, x_n, x_n), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq \psi(\mathcal{B}(x_{n-1}, x_n, x_n)) \\ &\preccurlyeq \psi^2(\mathcal{B}(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\preccurlyeq \cdots\cdots \\ &\preccurlyeq \psi^n(\mathcal{B}(x_0, x_1, x_1)). \end{aligned}$$

Since $\{\psi^n(\mathbb{B}(x_0, x_1, x_1))\}_{n\in\mathbb{N}}$ is non-incresing, it must converge to some η, $\eta \succcurlyeq \theta$. We claim that $\eta = \theta$. On the contrary, assume that $\eta \gg \theta$. Then by the definition of the ψ-function, there exists $\delta \gg \theta$ such that for $x_0, x_1 \in X$ with $\eta \preccurlyeq \mathcal{B}(x_0, x_1, x_1) \ll \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\mathcal{B}(fx_0, fx_1, fx_1)) \ll \eta$. Since $\lim_{n\to\infty} \psi^n(\mathcal{B}(x_0, x_1, x_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \preccurlyeq \psi^m \mathcal{B}(x_0, x_1, x_1) \ll \delta + \eta$, for all $m \geq m_0$. Thus, we get $\psi^{m_0+n_0}(\mathcal{B}(x_0, x_1, x_1)) \ll \eta$, and we get a contradiction. So $\lim_{n\to\infty} \psi^n(\mathcal{B}(x_0, x_1, x_1)) = \theta$, and so we have $\lim_{n\to\infty} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) = \theta$.

For $m, n \in \mathbb{N}$ with $m > n > \kappa_0$, we claim that the following result holds:

$$\mathcal{B}(x_n, x_m, x_m) \prec \varepsilon \text{ for all } m > n > \kappa_0. \tag{2.5}$$

Let $\varepsilon \in E$ with $\varepsilon \gg 0$ be given. Since $\lim_{n\to\infty} \varphi^n(\mathcal{B}(x_0, x_1, x_1)) = \theta$ and $\psi(\varepsilon) \ll \varepsilon$, there exists $\kappa_0 \in \mathbb{N}$ such that

$$\psi^n(\mathcal{B}(x_0, x_1, x_1)) \ll \varepsilon - \psi(\varepsilon) \text{ for all } n \geq \kappa_0,$$

that is,

$$\mathcal{B}(x_n, x_{n+1}, x_{n+1}) \ll \varepsilon - \psi(\varepsilon) \text{ for all } n \geq \kappa_0. \tag{2.6}$$

We prove (2.5) by induction on m. Assume that the inequality (2.5) holds for $m = k$. Then by (2.6), we have that for $m = k + 1$,

$$\begin{aligned} \mathcal{B}(x_n, x_{k+1}, x_{k+1}) &\preccurlyeq \mathcal{B}(x_n, x_{n+1}, x_{n+1}) + \mathcal{B}(x_{n+1}, x_{k+1}, x_{k+1}) \\ &\ll \mathcal{B}(x_n, x_{n+1}, x_{n+1}) + \psi(\mathcal{B}(x_n, x_k, x_k) \\ &\ll \varepsilon - \psi(\varepsilon) + \psi(\varepsilon) \\ &= \varepsilon. \end{aligned}$$

Thus, we conclude that $\mathcal{B}(x_n, x_m, x_m) \ll \varepsilon$ for all $m > n > \kappa_0$. So $\{x_n\}$ is a Cauchy sequence in X. Since $(X, \mathcal{B})$ is a complete cone ball-metric space, there exists $\mu \in X$ such that $\lim_{n\to\infty} x_n = \mu$, that is, $\mathcal{B}(x_n, x_n, \mu) \to \theta$.

For $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{B}(\mu, \mu, f\mu) &\preccurlyeq \mathcal{B}(\mu, \mu, x_n) + \mathcal{B}(x_n, x_n, f\mu)\\ &\preccurlyeq \mathcal{B}(\mu, \mu, x_n) + \mathcal{B}(fx_{n-1}, fx_{n-1}, f\mu)\\ &\preccurlyeq \mathcal{B}(\mu, \mu, x_n) + \psi(\mathcal{L}(x_{n-1}, x_{n-1}, \mu)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(x_{n-1}, x_{n-1}, \mu) &= \max\{\mathcal{B}(x_{n-1}, x_{n-1}, \mu), \mathcal{B}(x_{n-1}, fx_{n-1}, fx_{n-1}),\\ &\quad \mathcal{B}(x_{n-1}, fx_{n-1}, fx_{n-1}), \mathcal{B}(fx_{n-1}, x_{n-1}, \mu)\}\\ &= \max\{\mathcal{B}(x_{n-1}, x_{n-1}, \mu), \mathcal{B}(x_{n-1}, x_n, x_n),\\ &\quad \mathcal{B}(x_{n-1}, x_n, x_n), \mathcal{B}(x_n, x_{n-1}, \mu)\}.\end{aligned}$$

(I) If

$$\mathcal{L}(x_{n-1}, x_{n-1}, \mu) = \mathcal{B}(x_{n-1}, x_{n-1}, \mu),$$

then

$$\mathcal{B}(\mu, \mu, f\mu) \ll \mathcal{B}(\mu, \mu, x_n) + \mathcal{B}(x_{n-1}, x_{n-1}, \mu).$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu, \mu, f\mu) = \theta$, and so $\mu = f\mu$.

(II) If

$$\mathcal{L}(x_{n-1}, x_{n-1}, \mu) = \mathcal{B}(x_{n-1}, x_n, x_n),$$

then

$$\mathcal{B}(\mu, \mu, f\mu) \ll \mathcal{B}(\mu, \mu, x_n) + \mathcal{B}(x_{n-1}, x_n, x_n).$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu, \mu, f\mu) = \theta$, and so $\mu = f\mu$.

(III) If

$$\mathcal{L}(x_{n-1}, x_{n-1}, \mu) = \mathcal{B}(x_n, x_{n-1}, \mu) \preccurlyeq \mathcal{B}(x_n, x_{n-1}, x_{n-1}) + \mathcal{B}(x_{n-1}, x_{n-1}, \mu),$$

then

$$\mathcal{B}(\mu, \mu, T\mu) \ll \mathcal{B}(\mu, \mu, x_n) + \mathcal{B}(x_n, x_{n-1}, x_{n-1}) + \mathcal{B}(x_{n-1}, x_{n-1}, \mu).$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu, \mu, f\mu) = \theta$, and so $\mu = f\mu$.

Follow (I), (II) and (III), we have that μ is a fixed point of f.

Let ν be another fixed point of f with $\mu \neq \nu$. Then

$$\mathcal{B}(\mu, \nu, \nu) = \mathcal{B}(f\mu, f\nu, f\nu) \preccurlyeq \psi(\mathcal{L}(\mu, \nu, \nu)),$$

where

$$\begin{aligned}\mathcal{L}(\mu,\nu,\nu) &= \max\{\mathcal{B}(\mu,\nu,\nu),\mathcal{B}(\mu,f\mu,f\mu),\mathcal{B}(\nu,f\nu,f\nu),\mathcal{B}(f\mu,\nu,\nu)\}\\ &= \max\{\mathcal{B}(\mu,\nu,\nu),\mathcal{B}(\mu,\mu,\mu),\mathcal{B}(\nu,\nu,\nu),\mathcal{B}(\mu,\nu,\nu)\}\\ &= \max\{\mathcal{B}(\mu,\nu,\nu)\},\theta\}.\end{aligned}$$

Therefore, if $\mathcal{B}(\mu,\nu,\nu) \ll \mathcal{B}(\mu,\nu,\nu)$, then we get a contradiction. So $\mu = \nu$, and we show that μ is a unique fixed point of f.

To show that f is continuous at μ. Let $\{y_n\}$ be any sequence in X such that $\{y_n\}$ convergent to μ. Then

$$\begin{aligned}\mathcal{B}(\mu,\mu,fy_n) &= \mathcal{B}(f\mu,f\mu,fy_n)\\ &\preccurlyeq \varphi(\mathcal{L}(\mu,\mu,y_n)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(\mu,\mu,y_n) &= \max\{\mathcal{B}(\mu,\mu,y_n),\mathcal{B}(\mu,f\mu,f\mu),\mathcal{B}(\mu,f\mu,f\mu),\mathcal{B}(f\mu,\mu,y_n)\}\\ &= \max\{\mathcal{B}(\mu,\mu,y_n),\theta\}.\end{aligned}$$

Thus

$$\mathcal{B}(\mu,\mu,fy_n) \ll \mathcal{B}(\mu,\mu,y_n).$$

Letting $n \to \infty$. Then we deduce that $\{fy_n\}$ is convergent to $f\mu = \mu$. Hence f is continuous at μ. □

By Theorem 2.3, we immediate get the following corollary.

Corollary 2.4. *Let $(X,\mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and $f : X \to X$. Suppose that there exists a ψ-function such that*

$$\mathcal{B}(fx,fy,fz) \preccurlyeq \psi(\mathcal{B}(x,y,z) \qquad (x,y,z \in X).$$

Then f has a unique fixed point (say μ) in X and f is continuous at μ.

In the sequel, we introduce the stronger Meir-Keeler cone-type function $\phi : intP \cup \{\theta\} \to [0,1)$ in cone ball-metric spaces, and prove the fixed point theorem for this type of function.

Definition 2.5. Let $(X,\mathcal{B})$ be a cone ball-metric space with cone P, and let

$$\phi : intP \cup \{\theta\} \to [0,1).$$

Then the function ϕ is called a stronger Meir-Keeler type function, if for each $\eta \in P$ with $\eta \gg \theta$, there exists $\delta \gg \theta$ such that for $x,y,z \in X$ with $\eta \preccurlyeq \mathcal{B}(x,y,z) \ll \delta+\eta$, there exists $\gamma_\eta \in [0,1)$ such that $\phi(\mathcal{B}(x,y,z)) < \gamma_\eta$.

Theorem 2.6. *Let $(X, \mathcal{B})$ be a complete cone ball-metric space, P be a regular cone in E and $f : X \to X$. Suppose that there exists a stronger Meir-Keeler type function $\phi : intP \cup \{0\} \to [0, 1)$ such that*

$$\mathcal{B}(fx, fy, fz) \preccurlyeq \phi(\mathcal{L}(x, y, z)) \cdot \mathcal{L}(x, y, z) \text{ for all } x, y, z \in X, \tag{2.7}$$

where

$$\mathcal{L}(x, y, z) = \max\{\mathcal{B}(x, y, z), \mathcal{B}(x, fx, fx), \mathcal{B}(y, fy, fy), \mathcal{B}(fx, y, z)\}$$

Then f has a unique fixed point (say μ) in X and f is continuous at μ.

Proof. Given $x_0 \in X$. Define the sequence $\{x_n\}$ in X recursively as follows:

$$fx_{n-1} = x_n \text{ for each } n \in \mathbb{N}.$$

In what follows, we will suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, since if $x_{n+1} = x_n$ for some n, then $x_{n+1} = fx_n = x_n$, and so we complete the proof.

By (2.7), we deduce

$$\begin{aligned} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) &= \mathcal{B}(fx_{n-1}, fx_n, fx_n) \\ &\preccurlyeq \phi(\mathcal{L}(x_{n-1}, x_n, x_n)) \cdot \mathcal{L}(x_{n-1}, x_n, x_n) \\ &\ll \gamma_\eta \cdot \mathcal{L}(x_{n-1}, x_n, x_n), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}(x_{n-1}, x_n, x_n) &= \max\{\mathcal{B}(x_{n-1}, x_n, x_n), \mathcal{B}(x_{n-1}, fx_{n-1}, fx_{n-1}), \\ &\quad \mathcal{B}(x_n, fx_n, fx_n), \mathcal{B}(fx_{n-1}, x_n, x_n)\} \\ &= \max\{\mathcal{B}(x_{n-1}, x_n, x_n), \mathcal{B}(x_{n-1}, x_n, x_n), \\ &\quad \mathcal{B}(x_n, x_{n+1}, x_{n+1}), \mathcal{B}(x_n, x_n, x_n)\} \\ &= \max\{\mathcal{B}(x_{n-1}, x_n, x_n), \mathcal{B}(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

If

$$\mathcal{L}(x_{n-1}, x_n, x_n) = \mathcal{B}(x_n, x_{n+1}, x_{n+1}),$$

then

$$\mathcal{B}(x_n, x_{n+1}, x_{n+1}) \ll \gamma_\eta \cdot \mathcal{B}(x_n, x_{n+1}, x_{n+1}),$$

a contradiction. So we deduce that

$$\begin{aligned} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) &\preccurlyeq \phi(\mathcal{L}(x_{n-1}, x_n, x_n)) \\ &\ll \gamma_\eta \cdot \mathcal{B}(x_{n-1}, x_n, x_n). \end{aligned}$$

Then the sequence $\{\mathcal{B}(x_n, x_{n+1}, x_{n+1})\}$ is decreasing and bounded below. Let

$$\lim_{n\to\infty} \mathcal{B}(x_n, x_{n+1}, x_{n+1}) = \eta \succcurlyeq \theta.$$

Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta \gg \theta$ such that for all $n > \kappa_0$

$$\eta \preccurlyeq \mathcal{B}(x_n, x_{n+1}, x_{n+1}) \ll \eta + \delta.$$

For each $n \in \mathbb{N}$, since $\phi : intP \cup \{\theta\} \to [0, 1)$ is a stronger Meir-Keeler type function, for these η and δ we have that for $x_{\kappa_0+n}, x_{\kappa_0+n+1} \in X$ with $\eta \preccurlyeq \mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+n+1}, x_{\kappa_0+n+1}) \ll \delta + \eta$, there exists $\gamma_\eta \in [0, 1)$ such that $\phi(\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+n+1}, x_{\kappa_0+n+1})) < \gamma_\eta$. Thus, by (2.7), we can deduce

$$\begin{aligned}\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+n+1}, x_{\kappa_0+n+1}) &= \phi(\mathcal{B}(x_{\kappa_0+n-1}, x_{\kappa_0+n}, x_{\kappa_0+n})) \cdot \mathcal{B}(x_{\kappa_0+n-1}, x_{\kappa_0+n}, x_{\kappa_0+n}) \\ &\ll \gamma_\eta \cdot \mathcal{B}(x_{\kappa_0+n-1}, x_{\kappa_0+n}, x_{\kappa_0+n}),\end{aligned}$$

and it follows that for each $n \in \mathbb{N}$,

$$\begin{aligned}\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+n+1}, x_{\kappa_0+n+1}) &\ll \gamma_\eta \cdot \mathcal{B}(x_{\kappa_0+n-1}, x_{\kappa_0+n}, x_{\kappa_0+n}) \\ &\ll \cdots \\ &\ll \gamma_\eta^n \cdot \mathcal{B}(x_{\kappa_0}, x_{\kappa_0+1}, x_{\kappa_0+1}).\end{aligned}$$

So

$$\lim_{n\to\infty} \mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+n+1}, x_{\kappa_0+n+1}) = \theta, \text{ since } \gamma_\eta < 1.$$

We next claim that $\lim_{m,n\to\infty} \mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+m}) = \theta$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+m}) &\preccurlyeq \sum_{i=n}^{m-1} \mathcal{B}(x_{\kappa_0+i}, x_{\kappa_0+i+1}, x_{\kappa_0+i+1}) \\ &\ll \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} \mathcal{B}(x_{\kappa_0+1}, x_{\kappa_0+2}, x_{\kappa_0+2}),\end{aligned}$$

and hence $\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+m}) \to \theta$ as $m, n \to \infty$, since $0 < \gamma_\eta < 1$.

By the properties of the cone ball-metric, we obtain

$$\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+l}) \preccurlyeq \mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+m}) + \mathcal{B}(x_{\kappa_0+m}, x_{\kappa_0+m}, x_{\kappa_0+l}),$$

taking limit as $m, n, l \to \infty$, we get $\mathcal{B}(x_{\kappa_0+n}, x_{\kappa_0+m}, x_{\kappa_0+l}) \to \theta$. So $\{x_n\}$ is a Cauchy sequence. Since $(X, \mathcal{B})$ is a complete cone ball-metric space, there exists $\mu \in X$ such that $\lim_{n\to\infty} x_n = \mu$, that is, $\mathcal{B}(x_n, x_n, \mu) \to \theta$.

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{B}(\mu,\mu,f\mu) &\preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \mathcal{B}(x_n,x_n,f\mu) \\
&\preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \mathcal{B}(fx_{n-1},fx_{n-1},f\mu) \\
&\preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \phi(\mathcal{L}(x_{n-1},x_{n-1},\mu)) \cdot \mathcal{L}(x_{n-1},x_{n-1},\mu) \\
&\preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \gamma_\eta \cdot \mathcal{L}(x_{n-1},x_{n-1},\mu),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}(x_{n-1},x_{n-1},\mu) &= \max\{\mathcal{B}(x_{n-1},x_{n-1},\mu), \mathcal{B}(x_{n-1},fx_{n-1},fx_{n-1}), \\
&\qquad \mathcal{B}(x_{n-1},Tx_{n-1},fx_{n-1}), \mathcal{B}(fx_{n-1},x_{n-1},\mu)\} \\
&= \max\{\mathcal{B}(x_{n-1},x_{n-1},\mu), \mathcal{B}(x_{n-1},x_n,x_n), \\
&\qquad \mathcal{B}(x_{n-1},x_n,x_n), \mathcal{B}(x_n,x_{n-1},\mu)\}.
\end{aligned}$$

(I) If

$$\mathcal{L}(x_{n-1},x_{n-1},\mu) = \mathcal{B}(x_{n-1},x_{n-1},\mu),$$

then

$$\mathcal{B}(\mu,\mu,f\mu) \preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \gamma_\eta \cdot \mathcal{B}(x_{n-1},x_{n-1},\mu).$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu,\mu,f\mu) = \theta$, and so $\mu = f\mu$.

(II) If

$$\mathcal{L}(x_{n-1},x_{n-1},\mu) = \mathcal{B}(x_{n-1},x_n,x_n),$$

then

$$\mathcal{B}(\mu,\mu,f\mu) \preccurlyeq \mathcal{B}(\mu,\mu,x_n) + \gamma_\eta \cdot \mathcal{B}(x_{n-1},x_n,x_n).$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu,\mu,T\mu) = \theta$, and so $\mu = f\mu$.

(III) If

$$\mathcal{L}(x_{n-1},x_{n-1},\mu) = \mathcal{B}(x_n,x_{n-1},\mu) \preccurlyeq \mathcal{B}(x_n,x_{n-1},x_{n-1}) + \mathcal{B}(x_{n-1},x_{n-1},\mu),$$

then

$$\mathcal{B}(\mu,\mu,f\mu) \leq \mathcal{B}(\mu,\mu,x_n) + \gamma_\eta \cdot [\mathcal{B}(x_n,x_{n-1},x_{n-1}) + \mathcal{B}(x_{n-1},x_{n-1},\mu)].$$

Letting $n \to \infty$, we conclude that $\mathcal{B}(\mu,\mu,f\mu) = \theta$, and so $\mu = f\mu$.

Follow (I), (II) and (III), we have that μ is a fixed point of f.

Let ν be another fixed point of f with $\mu \neq \nu$. Then

$$\begin{aligned}\mathcal{B}(\mu,\nu,\nu) &= \mathcal{B}(f\mu,f\nu,f\nu)\\ &\preccurlyeq \psi(\mathcal{L}(\mu,\nu,\nu))\cdot\mathcal{L}(\mu,\nu,\nu)\\ &\ll \gamma_\eta\cdot\mathcal{L}(\mu,\nu,\nu),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(\mu,\nu,\nu) &= \max\{\mathcal{B}(\mu,\nu,\nu),\mathcal{B}(\mu,f\mu,f\mu),\mathcal{B}(\nu,f\nu,f\nu),\mathcal{B}(T\mu,\nu,\nu)\}\\ &= \max\{\mathcal{B}(\mu,\nu,\nu),\mathcal{B}(\mu,\mu,\mu),\mathcal{B}(\nu,\nu,\nu),\mathcal{B}(\mu,\nu,\nu)\}\\ &= \max\{\mathcal{B}(\mu,\nu,\nu),\theta\}.\end{aligned}$$

Thus if $\mathcal{B}(\mu,\nu,\nu) \ll \gamma_\eta\cdot\mathcal{B}(\mu,\nu,\nu)$, then we get a contradiction. So $\mu=\nu$, and we show that μ is a unique fixed point of T.

To show that f is continuous at μ. Let $\{y_n\}$ be any sequence in X such taht $\{y_n\}$ convergent to μ. Then

$$\begin{aligned}\mathcal{B}(\mu,\mu,fy_n) &= \mathcal{B}(f\mu,f\mu,fy_n)\\ &\preccurlyeq \psi(\mathcal{L}(\mu,\mu,y_n))\cdot\mathcal{L}(\mu,\mu,y_n)\\ &\ll \gamma_\eta\cdot\mathcal{L}(\mu,\mu,y_n),\end{aligned}$$

where

$$\mathcal{L}(\mu,\mu,y_n) = \max\{\mathcal{B}(\mu,\mu,y_n),\mathcal{B}(\mu,T\mu,T\mu),\mathcal{B}(\mu,f\mu,f\mu),\mathcal{B}(f\mu,\mu,y_n)\}.$$

Thus

$$\mathcal{B}(\mu,\mu,Ty_n) \ll \gamma_\eta\cdot\mathcal{B}(\mu,\mu,y_n).$$

Letting $n\to\infty$. Then we deduce that $\{fy_n\}$ is convergent to $f\mu=\mu$. Hence f is continuous at μ. □

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