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A NOTE ON SYMMETRICAL FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR

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Abstract. In this paper using the concept of (j,k)-symmetric points and differential operator $D^{\alpha,\beta}_{\lambda,\gamma}$ we introduced new subclasses $\mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$ and $\mathcal{K}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$. The integral representation and some properties for these classes are obtained.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$. The class $\mathcal{P}$ designates the class of function of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \; (z \in \mathcal{U}),$$

which are convex and satisfy the condition $\Re\{p(z)\} > 0$. For f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function ω in $\mathcal{U}$ such that $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$, and we denote this by $f \prec g$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

The convolution or Hadamard product of two analytic functions $f, g \in \mathcal{A}$ where f is defined by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, is

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Definition 1.1. *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of the odd univalent functions.

The notion of (j, k)-symmetrical functions $(k = 2, 3, ... ; j = 0, 1, 2, ..., k - 1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.
The theory of (j, k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings [11].

Definition 1.2. *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, .., k - 1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j, k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \ z \in \mathcal{U}.$$

The family of all (j, k)-symmetrical functions is denoted be $\mathcal{S}^{(j,k)}$. $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are respectively the classes of even, odd and k-symmetric functions. We have the following decomposition theorem.

Theorem 1.3. *[11] For every mapping $f : \mathcal{U} \mapsto \mathbb{C}$, there exists exactly a unique sequence of (j, k)- symmetrical functions $f_{j,k}$,*

$$f(z) = \sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} f(\varepsilon^v z), \tag{1.2}$$

where $(f \in \mathcal{A};\ k = 1, 2, ...; j = 0, 1, 2, ..., k-1)$.

We denote by $\mathcal{S}^*,\ \mathcal{K},\ \mathcal{C},\ \mathcal{C}^*$ the familiar subclasses consisting of functions which, respectively, starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$.
Afaf A. Abubaker and Maslina Darus in [7] defined the following differential operator
.

$$D^{\alpha,\beta}_{\lambda,\gamma} f(z) = z + \sum_{n=2}^{\infty} n^{\beta}(C(\gamma, n)[1 + \lambda(n-1)])^{\alpha} a_n z^n, \tag{1.3}$$

where $\lambda \geq 0$, $\alpha, \beta\ and\ \gamma \in \mathbb{N}_0,\ C(\gamma, n) = \frac{(\gamma+1)_{(n-1)}}{(n-1)!}$, and $(x)_n$ is the Pochhammer symbol defined by.

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(n)} = \begin{cases} 1, & n = 0, \\ x(x+1)...(x+n-1), & n = 1, 2, 3, ..., \end{cases}.$$

Here $D^{\alpha,\beta}_{\lambda,\gamma}$ can be written, in terms of convolution as

$$\varphi(z) = \left[\frac{\lambda z}{(1-z)^2} - \frac{\lambda z}{1-z} + \frac{z}{(1-z)}\right] * \frac{z}{(1-z)^{\gamma+1}},\ z \in \mathcal{U},$$

$$D^{\alpha,\beta}_{\lambda,\gamma} f(z) = \underbrace{\varphi(z) * ... * \varphi(z)}_{\alpha-times} * \sum_{n=1}^{\infty} n^{\beta} z^n * f(z) = \underbrace{D_{\gamma} * ... * D_{\gamma}}_{\alpha-times} * D^{\alpha,\beta}_{\lambda} f(z).$$

Where $D_{\gamma} = z + \sum_{n=2}^{\infty} C(\gamma, n) z^n$ and $D^{\alpha,\beta}_{\lambda} = z + \sum_{n=2}^{\infty} n^{\beta}[1 + \lambda(n-1)]^{\alpha} z^n$.
Note that $D^{0,1}_{\lambda,\gamma} f(z) = D^{1,0}_{1,0} f(z) = zf'(z)$ and $D^{0,0}_{\lambda,\gamma} f(z) = f(z)$. When $\alpha = 0$ we get the $S\tilde{a}l\tilde{a}gean$ differential operator [4]. When $\lambda = 0, \alpha = 1$, we obtain the Ruscheweyh operator [14]. When $\beta = 0, \alpha = 1$ we obtain the Al-Shaqsi and Maslina Darus differential operator [6] and when $\gamma = \beta = 0$ we obtain the Al-Oboudi differential operator [5].
In the present paper, we introduce new subclasses of analytic functions with respect to (j, k)-symmetric points defined by differential operator.
Applying the operator $D^{\alpha,\beta}_{\lambda,\gamma}\ to\ f_{j,k}(z)$ we get

$$D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-vj} D^{\alpha,\beta}_{\lambda,\gamma} f(\varepsilon^v z),\ \varepsilon^k = 1, \tag{1.4}$$

where $k = 1, 2, ...; j = 0, 1, 2, ..., k-1$.

Definition 1.4. *Let* $\mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$ *denote the class of functions in* $\mathcal{A}$ *satisfying the condition*

$$\frac{z(D_{\lambda,\gamma}^{\alpha,\beta}f(z))'}{D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)} \prec \phi(z), \tag{1.5}$$

where $\phi \in \mathcal{P}$.

Definition 1.5. *Let* $\mathcal{K}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$ *denote the class of functions in* $\mathcal{A}$ *satisfying the condition*

$$\frac{(z(D_{\lambda,\gamma}^{\alpha,\beta}f(z))')'}{(D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z))'} \prec \phi(z), \tag{1.6}$$

where $\phi \in \mathcal{P}$.

In our work we need the following lemmas.

Lemma 1.6. *[12] Let* $c > -1$ *and lat* $I_c:\ \mathcal{A} \to \mathcal{A}$ *be the integral operator defined by* $F = I_c(f)$, *where*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt.$$

Let φ *be a convex function, with* $\varphi(0) = 1$ *and* $\Re\{\varphi(z)+c\} > 0$ *in* $\mathcal{U}$. *If* $f \in \mathcal{A}$ *and* $\frac{zf'(z)}{f(z)} \prec \varphi(z)$, *then* $\frac{zF'(z)}{F(z)} \prec q(z) \prec \varphi(z)$, *where* q *is univalent and satisfies the differential equation*

$$q(z) + \frac{zq'(z)}{q(z)+c} = \varphi(z).$$

Lemma 1.7. *[15]Let* b, v *be complex numbers. Let* φ *be convex univalent in* $\mathcal{U}$ *with* $\varphi(0) = 1$ *and* $\Re[b\varphi + v] > 0$, $z \in \mathcal{U}$ *and let* $q(z) \in \mathcal{A}$ *with* $q(0) = 1$ *and* $q(z) \prec \varphi(z)$. *If* $p \in \mathcal{P}$, *then*

$$p(z) + \frac{zp'(z)}{bq(z)+v} \prec \varphi(z) \Rightarrow p(z) \prec \varphi(z).$$

2. Main Results

Theorem 2.1. *Let* $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$. *Then*

$$D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z) = \exp\left\{\frac{1}{k}\sum_{v=0}^{k-1}\int_0^z \frac{\phi(w_j(\varepsilon^v t))}{t}dt\right\}, \tag{2.1}$$

where $D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)$ *is defined by* (1.4), $w(z)$ *is analytic in* $\mathcal{U}$ *and* $w(0) = 0,\ |w(z)| < 1$.

Proof. suppose that $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$, then

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma}f(z))'}{D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z)} \prec \phi(z),$$

or

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma}f(z))'}{D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z)} = \phi(w_j(z)), \tag{2.2}$$

where $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0,\ |w(z)| < 1$. Substituting z by $\varepsilon^v z$, in (2.2) respectively $(v = 0, 1, 2, ..., k-1)$, we have

$$\frac{\varepsilon^v z(D^{\alpha,\beta}_{\lambda,\gamma}f(\varepsilon^v z))'}{D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(\varepsilon^v z)} = \phi(w_j(\varepsilon^v z)),$$

or

$$\frac{\varepsilon^{v-vj} z(D^{\alpha,\beta}_{\lambda,\gamma}f(\varepsilon^v z))'}{D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z)} = \phi(w_j(\varepsilon^v z)). \tag{2.3}$$

By summing up (2.3) we get

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z))'}{D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z)} = \frac{1}{k}\sum_{v=0}^{k-1}\phi(w_j(\varepsilon^v z)). \tag{2.4}$$

Integrating equality (2.4) we have

$$\log D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z) = \frac{1}{k}\sum_{v=0}^{k-1}\int_0^z \frac{\phi(w_j(\varepsilon^v t))}{t}dt. \tag{2.5}$$

By (2.5) we can obtain equality (2.1). □

Theorem 2.2. *Let* $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$ *. Then*

$$D^{\alpha,\beta}_{\lambda,\gamma}f(z) = \int_0^z \left[\frac{1}{t}\exp\left\{\frac{1}{k}\sum_{v=0}^{k-1}\int_0^t \frac{\phi(w_j(\varepsilon^v y))}{y}dy\right\}\right]dt, \tag{2.6}$$

where $w(z)$ *is analytic in* $\mathcal{U}$ *and* $w(0) = 0,\ |w(z)| < 1$.

Proof. suppose that $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$, then

$$z(D^{\alpha,\beta}_{\lambda,\gamma}f(z))' = D^{\alpha,\beta}_{\lambda,\gamma}f_{j,k}(z).\phi(z).$$

By Theorem 2.1, we have

$$(D^{\alpha,\beta}_{\lambda,\gamma}f(z))' = \frac{1}{z}\exp\left\{\frac{1}{k}\sum_{v=0}^{k-1}\int_0^z \frac{\phi(w_j(\varepsilon^v t))}{t}dt\right\}. \tag{2.7}$$

Integrating (2.7), we get (2.6). □

Corollary 2.3. *Let* $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$. *Then*

$$D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z) \ \in \mathcal{S}^*(\phi)$$

Proof. From (2.4) in Theorem 2.1, we have

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z))'}{D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z)} = \frac{1}{k}\sum_{v=0}^{k-1} \phi(w_j(\varepsilon^v z)) = \phi(\zeta). \tag{2.8}$$

For $\zeta \in \mathcal{U}$, since $\phi(\mathcal{U})$ is convex. This mean $D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z) \ \in \mathcal{S}^*(\phi)$. □

Theorem 2.4. *Let* $f \in \mathcal{A},\ \phi \in \mathcal{P}$, *then*

$$f \in \mathcal{K}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi) \ \Leftrightarrow \ zf' \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi).$$

Proof. suppose that $f \in \mathcal{K}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$, then

$$\frac{(z(D^{\alpha,\beta}_{\lambda,\gamma} f(z))')'}{D^{\alpha,\beta}_{\lambda,\gamma} f'_{j,k}(z)} \prec \phi(z),$$

set

$$g(z) = z + \sum_{n=2}^{\infty} n^{\beta}(C(\gamma,n)[1+\lambda(n-1)]^{\alpha})z^n,$$

then

$$D^{\alpha,\beta}_{\lambda,\gamma} f(z) = g * f, \ z(D^{\alpha,\beta}_{\lambda,\gamma} f(z))' = D^{\alpha,\beta}_{\lambda,\gamma} zf'(z),$$

so

$$\frac{(z(D^{\alpha,\beta}_{\lambda,\gamma} f(z))')'}{(D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z))'} = \frac{z(D^{\alpha,\beta}_{\lambda,\gamma} zf'(z))'}{D^{\alpha,\beta}_{\lambda,\gamma} zf'_{j,k}(z)}.$$

Thus $f \in \mathcal{K}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$ if and only if $zf' \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$. □

Theorem 2.5. *Let* $\phi \in \mathcal{P}, \beta \in \mathbb{N}_0$. *Then,*

$$\mathcal{S}^{(j,k)}(\alpha,\beta+1,\gamma,\lambda,\phi) \subset \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi). \tag{2.9}$$

Proof. suppose that $f \in \mathcal{S}^{(j,k)}(\alpha,\beta,\gamma,\lambda,\phi)$, then

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma} f(z))'}{D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z)} \prec \phi(z).$$

Now let

$$p(z) = \frac{z(D^{\alpha,\beta}_{\lambda,\gamma} f(z))'}{D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z)},$$

where $p(z)$ is analytic function and by using the equation

$$z(D_{\lambda,\gamma}^{\alpha,\beta}f(z))' = D_{\lambda,\gamma}^{\alpha,\beta+1}f(z), \tag{2.10}$$

we get

$$p(z) = \frac{D_{\lambda,\gamma}^{\alpha,\beta+1}f(z)}{D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)}.$$

By differentiating, we have

$$z(D_{\lambda,\gamma}^{\alpha,\beta+1}f(z))' = zp'(z)D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z) + zp(z)(D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z))'. \tag{2.11}$$

Hence

$$\frac{z(D_{\lambda,\gamma}^{\alpha,\beta+1}f(z))'}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)} = \frac{D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)}zp'(z) + \frac{z(D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z))'}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)}p(z). \tag{2.12}$$

Applying (2.10) for the function $f_{j,k}$ we obtain

$$\frac{z(D_{\lambda,\gamma}^{\alpha,\beta+1}f(z))'}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)} = \frac{D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)}zp'(z) + p(z). \tag{2.13}$$

Let $q(z) = \frac{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)}{D_{\lambda,\gamma}^{\alpha,\beta}f_{j,k}(z)}$ in (2.13) we get

$$\frac{z(D_{\lambda,\gamma}^{\alpha,\beta+1}f(z))'}{D_{\lambda,\gamma}^{\alpha,\beta+1}f_{j,k}(z)} = \frac{zp'(z)}{q(z)} + p(z),$$

by (2.9) we have

$$\frac{zp'(z)}{q(z)} + p(z) \prec \phi(z).$$

We can see $q(z) \prec \phi(z)$, hence applying Lemma 1.7 we compleat the proof .

□

Corollary 2.6. *Let $\phi \in \mathcal{P}, \beta \in \mathbb{N}_0$. Then,*

$$\mathcal{K}^{(j,k)}(\alpha, \beta+1, \gamma, \lambda, \phi) \subset \mathcal{K}^{(j,k)}(\alpha, \beta, \gamma, \lambda, \phi).$$

Theorem 2.7. *Let $\phi \in \mathcal{P}, \lambda > 0$ with $\Re\{\phi(z) + (\frac{1}{\lambda}) - 1\} > 0$. If $f \in \mathcal{S}^{(j,k)}(\alpha, \beta, \gamma, \lambda, \phi)$. Then,*

$$\frac{z(D_{\lambda,\gamma}^{\alpha-1,\beta}(D_\gamma f_{j,k}(z)))'}{D_{\lambda,\gamma}^{\alpha-1,\beta}(D_\gamma f_{j,k}(z))} \prec q(z) \prec \phi(z), \tag{2.14}$$

where $D^{\alpha-1,\beta}_{\lambda,\gamma}(D_\gamma f_{j,k}(z)) = D^{\alpha-1,\beta}_{\lambda,\gamma} * D_\gamma f_{j,k}(z)$ *and* q *is the univalent solution of the differential equation*

$$q(z) + \frac{zq'(z)}{q(z) + (1 \setminus \lambda) - 1} = \phi(z) \tag{2.15}$$

Proof. Let $f \in \mathcal{S}^{(j,k)}(\alpha, \beta, \gamma, \lambda, \phi)$. Then by Corollary 2.3 we have

$$\frac{z(D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z))'}{D^{\alpha,\beta}_{\lambda,\gamma} f_{j,k}(z)} \prec \phi(z), \tag{2.16}$$

from the definition of $D^{\beta,\alpha}_{\lambda,\gamma}$, we get

$$D^{\beta,\alpha}_{\lambda,\gamma} f_{j,k}(z) = (1-\lambda)(D^{\beta,\alpha-1}_{\lambda,\gamma} * D_\gamma f_{j,k}(z)) + \lambda z (D^{\beta,\alpha-1}_{\lambda,\gamma} * D_\gamma f_{j,k}(z))',$$

which implies that

$$D^{\beta,\alpha-1}_{\lambda,\gamma} * D_\gamma f_{j,k}(z) = \frac{1}{\lambda z^{(1\setminus\lambda)-1}} \int_0^z t^{(1\setminus\lambda)-2} D^{\beta,\alpha}_{\lambda,\gamma} f_{j,k}(t) dt. \tag{2.17}$$

Using (2.16), (2.17) and Lemma 1.6 can be applied to get (2.14). Where $c = (1 \setminus \lambda) - 1 > -1$ and $\Re\{\phi\} > 0$ with $\Re\{\phi(z) + (\frac{1}{\lambda}) - 1\} > 0$ and $q(z)$ satisfies (2.15). Thus complete the proof. □

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