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LINEAR MAPS PRESERVING ESSENTIALLY NUMERICAL RANGE AND ESSENTIALLY UNITARY OPERATOR

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ABSTRACT. Let $\mathscr{H}$ be an infinite dimensional separable complex Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. We give the concrete forms of maps $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ that preserve the essential numerical range, the essential numerical radius in both direction or essential unitary operator in one directions.

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1. INTRODUCTION

Linear preserver problems is an active research area matrix theory, operator theory and Banach algebras, it has attracted the attention of many mathematicians in the last few decades [[5],[6] ,[7] ,[8] ,[12] ,[16] ,[18] ,[20] , [22], [23],[24] ,[26]]. By a linear preserver on an algebra $\mathscr{A}$ we mean a linear map, which roughly speaking preserves some properties or relations of elements in in $\mathscr{A}$. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. In this paper, we shall concentrate on the case when $\mathscr{A} = \mathscr{B}(\mathscr{H})$, the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. We should point out that a great deal of work has been devoted to the case when $\mathscr{H}$ is finite dimensional, that is, the case when $\mathscr{A}$ is a matrix algebra (see survey articles [[1] , [16] , [17]]), and that the first papers concerning this case date back to the previous century [[12]]. It seems that in the last few years the interest for the infinite dimensional situation grows.

Although the literature on linear preservers in the infinite dimensional case is far from being as vast as in the finite dimensional one.

This work is motivated by the result of M. Bendaoud , A. Bourhim and M. Sarih in [[3]], who characterized surjective maps from $\mathscr{B}(\mathscr{H})$ onto itself which preserve the essential spectral radius and M. Mbekhta in [[19]] who study linear maps preserving the set of Fredholm operators in $\mathscr{B}(\mathscr{H})$.

The goal of the present paper is to characterize linear maps from $\mathscr{B}(\mathscr{H})$ onto itself which preserve the essential numerical range, the essential numerical range radius or essential unitary operator .

2. Preliminaries

In this section we introduce some notation and terminology Let $\mathscr{A}$ be a complex normed algebra with unit. Denote by $\mathscr{A}^*$ the set of all bounded linear functional on $\mathscr{A}$. The algebraic numerical range of an element $a \in \mathscr{A}$ is defined by

$$V(a, \mathscr{A}) = \{\rho(a) : \rho \in \mathscr{A}^*, \rho(1) = 1 = ||\rho||\}.$$

Now Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. Denote by $\mathscr{K}(\mathscr{H})$ the ideal of all compact operators acting $\mathscr{H}$, and let π be π the quotient map from $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{C}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, note that π is a $*$-homomorphism. Denote further by $||.||_e$, the essential norm

$$||T||_e = inf\{||T + K|| : K \in \mathscr{K}(\mathscr{H})\} = ||\pi(T)||.$$

Definition 2.1. Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. The essential numerical range $V_e(T)$ of T is defined by

$$V_e(T) = V(\pi(T), \mathscr{C}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H}), ||.||_e).$$

We summarize the basic properties of the essential numerical range in the following theorem.

Theorem 2.2. *Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. Then :*

(1) $V_e(T) = 0$ *if and only if* $T \in \mathscr{K}(\mathscr{H})$;
(2) $V_e(T) = \bigcap V(T + K, \mathscr{B}(\mathscr{H})), K \in \mathscr{K}(\mathscr{H})$;
(3) $V_e(T) = \{\lambda 1\}$ *if and only if* $T = \lambda I + \mathscr{K}(\mathscr{H})$, *where* $\lambda \in \mathbb{C}$.

Definition 2.3. Let $T \in \mathscr{B}(\mathscr{H})$. The essential numerical radius of T, denoted $v_e(T)$ is

$$v_e(T) = \max\{|\lambda|, \lambda \in V_e(T)\}.$$

Proposition 2.4.

$$v_e(T) = v(\pi(T)), \quad T \in \mathscr{B}(\mathscr{H})$$

where $v(T)$ *the numerical radius, defined by* $v(T) = \max\{|\lambda|, \lambda \in V(T)\}$.

Proposition 2.5. *Let* $T \in \mathscr{B}(\mathscr{H})$*. Then* $v_e(T) \leq ||\pi(T)|| \leq 2v_e(T)$.

For these definitions and results on the essential numerical range and radius, see for instance [[2],[11]].

Definition 2.6. We say that an operator T is essentially unitary operator, if $\pi(T)$ is a unitary element in $\mathscr{C}(\mathscr{H})$.

Our references for essentially unitary operator, see [[15]]

A linear map ϕ from algebra $\mathscr{A}$ into an algebra $\mathscr{B}$ is called a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$ for every $x \in \mathscr{A}$.. A well known result of Herstein [[13],Theorem 3.1] this shows, that a Jordan homomorphism on prime algebra is either an homomorphism or an anti-homomorphism.

3. Linear maps preserving the essential numerical range

Let ϕ be a map on $\mathscr{B}(\mathscr{H})$. If for any $T \in \mathscr{B}(\mathscr{H})$ we have $V_e(\phi(T)) = V_e(T)$, then we say that ϕ preserves the essential numerical range in both directions.

Theorem 3.1. *Let* $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ *be a linear map map, which is surjective up to compact operators (ie;* $\mathscr{B}(\mathscr{H}) = range(\phi) + \mathscr{K}(\mathscr{H})$*). Then* ϕ *preserve the essential numerical range in both directions if and only if* $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$ *and the induced map* $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ *is either a continuous* $*$*-automorphism or a continuous* $*$*-anti-automorphism.*

Proof. It suffices to prove the only if part. Assume that ϕ is a surjective map up to compact operators and $V_e(\phi(T)) = V_e(T)$. We will complete the proof by three steps.

Step 1. ϕ is unital modulo compact operators.

Let $T \in \mathscr{K}(\mathscr{H})$ such that $\phi(I) = T + \mathscr{K}(\mathscr{H})$, since $V_e(\phi(I)) = V_e(I)$ so $V_e(T) = \{1\}$ and so $T = I + \mathscr{K}(\mathscr{H})$.

Step 2. The induced map ψ of ϕ preserve numerical range.

Pick $K \in \mathscr{K}(\mathscr{H})$, and let us prove that $\phi(K)$ is compact as well. We have $V_e(K) = 0$, or $V_e(\phi(K)) = V_e(K)$, consequently $V_e(\phi(K)) = 0$. Then by Theorem 2.2 $\phi(K)$ is a compact operator, therefore $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$.

Thus $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, and so ϕ induces a surjective linear map

$$\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H}),$$

defined by $\psi(\pi(T)) = (\pi \circ \phi)(T)$, Clearly, for every $T \in \mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} V(\psi(\pi(T))) &= V(\pi \circ \phi(T)) \\ &= V_e(\phi(T)) \\ &= V_e(T) \\ &= V(\pi(T)). \end{aligned}$$

Consequently, $V(\psi(T)) = V(T)$, for every $T \in \mathscr{C}(\mathscr{H})$.

Step 3. ψ is injective .

Let $T \in \mathscr{C}(\mathscr{H})$ such that $\psi(T) = 0$, since ψ preserve range numerical so $V(T) = 0$ and so $T = 0$, consequently ψ is injective.

Step 4. ψ is bounded .

We have $V_e(\phi(T)) = V_e(T)$ implies that $v_e(\phi(T)) = v_e(T)$, then by proposition 2.5, we have

$$\begin{aligned} ||\psi(T)||_e &= ||\pi(\phi(T))|| \\ &\leq 2v_e(\phi(T)) \\ &= 2v_e(T) \\ &\leq 2||\pi(T)|| \\ &\leq 2||T||_e \end{aligned}$$

This is implies, that ψ is bounded.

Now we get $V(\psi(T)) = V(T)$, since ψ is a bijective linear bounded map, then by (Pellegrni [[21], Theorem 2.2]) ψ and ψ^{-1} are state preserving, so $V(T) \subset \mathbb{R}^+$ if and only if $V(\psi(T)) \subset \mathbb{R}^+$, or $\mathscr{C}(\mathscr{H})$ is a $\mathbf{C}^*$-algebra then T is a positive operator if and only if $\psi(T)$ is a positive operator. This implies that ψ is bi-positive or $\psi(I) = I$ consequently, by K.yelinen [[25],theorem 1.1] ψ is C^*-isomorphism, i.e. $\psi(T^*) = \psi(T)^*$ for every $T \in \mathscr{C}(\mathscr{H})$, and $\psi(T^n) = \psi(T)^n$ for each self-adjoint operator $T \in \mathscr{C}(\mathscr{H})$, and each positive integer n. Since every operator $B \in \mathcal{C}(\mathscr{H})$ is written in the form $B = S + iT$ with S, T are self-adjoint. Then $S + T$ is self-adjoint operators, hence

$$\psi((S+T)^2) = (\psi(S) + \psi(T))^2,$$

so

$$\psi(ST + TS) = \psi(S)\psi(T) + \psi(T)\psi(S),$$

consequently

$$
\begin{aligned}
\psi(B^2) &= \psi((S+iT)^2) \\
&= \psi(S^2 - T^2 + i(ST+TS)) \\
&= \psi(S^2) - \psi(T^2) + i\psi(ST+TS) \\
&= \psi(S)^2 - \psi(T)^2 + i\psi(S)\psi(T) + \psi(T)\psi(S) \\
&= \psi(S+iT)^2 \\
&= \psi(B)^2.
\end{aligned}
$$

$\psi(B^2) = \psi(B)^2$, for all operator B in $\mathcal{C}(\mathscr{H})$. Consequently ψ is a Jordan automorphism , but it is known that a Jordan automorphism in prime algebra is an automorphism or an anti-automorphism, since $\mathcal{C}(\mathscr{H})$ is a prime algebra, this implies that ψ is an automorphism or an anti automorphism. As $\psi(T^*) = \psi(T)^*$ for every $T \in \mathscr{C}(\mathscr{H})$, we get ψ is an $*$-automorphism or an $*$-anti automorphism.

□

Theorem 3.2. *Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a linear map, which is surjective up to compact operators (ie; $\mathscr{B}(\mathscr{H}) = range(\phi) + \mathscr{K}(\mathscr{H})$). If $V_e(\phi(S)\phi(T)) = V_e(ST)$ for all $S, T \in \mathscr{B}(\mathscr{H})$. Then $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$ and the induced map $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is either a continuous $*$-automorphism or a continuous $*$-anti-automorphism multiplied by* ± 1.

Proof. We have $V_e(\phi(I)^2) = V_e(I) = \{1\}$ so $\phi(I)^2 = I + \mathscr{K}(\mathscr{H})$ and so $\phi(I)$ is invertible modulo compact operators. Now we consider a map ϕ_1 such that $\phi_1(T) = \phi(I)\phi(T)$ for all $T \in \mathscr{B}(\mathscr{H})$. Since ϕ is a linear map, which is surjective up to compact operators such that $V_e(\phi(S)\phi(T)) = V_e(ST)$ for all $S, T \in \mathscr{B}(\mathscr{H})$ and $\phi(I)$ is invertible modulo compact operators then ϕ_1 is is a linear map, which is surjective up to compact operators and $V_e(\phi_1(S)) = V_e(S)$ for all $S \in \mathscr{B}(\mathscr{H})$. As in the proof of Theorem 3.1 we have $\phi_1(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$ and the induced $\psi_1 : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is bounded, or $\phi(I)$ is invertible modulo compact operators, this implies that $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$ and the induced map $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is bounded. On the other hand, we have

$$\begin{aligned}
\pi(S)\pi(T) = 0 &\Leftrightarrow ST = 0 + \mathscr{K}(\mathscr{H}) \\
&\Leftrightarrow V_e(ST) = \{0\} \\
&\Leftrightarrow V_e(\phi(S)\phi(T)) = \{0\} \\
&\Leftrightarrow \phi(S)\phi(T) = 0 + \mathscr{K}(\mathscr{H}) \\
&\Leftrightarrow \pi(\phi(S))\pi(\phi(T)) = \pi(0) \\
&\Leftrightarrow \psi(\pi(S))\psi(\pi(T)) = \pi(0)
\end{aligned}$$

Therefore $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is a linear continuous map surjective such that $\psi(S)\psi(T) = 0$ if and only if $ST = 0$ for every $S, T \in \mathscr{C}(\mathscr{H})$. By (Wong [[27], Theorem 2.5]) $\psi(I)$ is a central operator and invertible in $\mathscr{C}(\mathscr{H})$, since the center of $\mathscr{C}(\mathscr{H})$ is trivial , i.e. , a scalar multiplied by I, or $\psi(I)^2 = I$ so $\psi(I) = \pm I$. Consequently $\phi(I) = \pm I + \mathscr{K}(\mathscr{H})$. If we have $\phi(I) = I + \mathscr{K}(\mathscr{H})$, by Theorem 3.1 the result follows, but if $\phi(I) = -I + \mathscr{K}(\mathscr{H})$, we may replace ϕ by $-\phi$ and we get the result. □

4. Linear map that preserve the essential numerical radius

Let ϕ be a map on $\mathscr{B}(\mathscr{H})$. If for any $T \in \mathscr{B}(\mathscr{H})$ we have $v_e(\phi(T)) = v_e(T)$, then we say that ϕ preserves the essential numerical range radius in both directions.

Theorem 4.1. *Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a linear map, which is surjective up to compact operators (ie; $\mathscr{B}(\mathscr{H}) = range(\phi) + \mathscr{K}(\mathscr{H})$). Then ϕ preserve the essential numerical radius in both directions if and only if $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$,and the induced map $\phi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is either a $*$-automorphism or a $*$-anti-automorphism multiplied by a scalar of modulus 1.*

Proof. It suffices to prove the only if part. Assume that ϕ preserves the essential numerical range radius. We first prove that $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, consider an operator $K \in \mathscr{K}(\mathscr{H})$, so $V_e(K) = 0$ and so $v_e(K) = 0$, or ϕ preserves the essential numerical range radius, this implies that $v_e(\phi(K)) = 0$, it follows that $V_e(\phi(K)) = 0$. From theorem 2.2 we get that $\phi(K)$ is a compact operator, so $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$. Thus $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$, and so ϕ induces a surjective linear map

$$\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H}),$$

defined by

$$\psi(\pi(T)) = (\pi \circ \phi)(T);\ T \in\in \mathscr{B}(\mathscr{H}).$$

Clearly, for every $T \in \mathscr{B}(\mathscr{H})$, we have

$$\begin{aligned} v(\psi(\pi(T))) &= v_e(\pi \circ \phi(T)) \\ &= v_e(\phi(T)) \\ &= v_e(T) \\ &= v(\pi(T)). \end{aligned}$$

Consequently $v(\psi(T)) = v(T)$ for every $T \in \mathscr{C}(\mathscr{H})$. As in the proof of Theorem 3.1 we prove that ψ is injective and continuous. Consequently ψ is a linear continuous bijective map that preserve the numerical radius in $\mathscr{C}(\mathscr{H})$. By J.T. Chan[[10], Theorem 3], ψ is a C^*-isomorphism multiplied by a fixed unitary operator in center of $\mathscr{C}(\mathscr{H})$, since by [[9]] the center $\mathscr{C}(\mathscr{H})$ is trivial , i.e. , the set of scalar operators, therefore ψ is C^*-isomorphism multiplied by a fixed scalar of module 1. Now as in the proof of Theorem 3.1 to obtain that ψ is a continuous $*$-automorphism multiplied by a fixed scalar of module 1, or ψ is continuous $*$-anti-automorphism multiplied by a fixed scalar of module 1.

□

5. LINEAR MAP PRESERVING ESSENTIAL UNITARY OPERATOR

Let ϕ be a map on $\mathscr{B}(\mathscr{H})$. If for any $A \in \mathscr{B}(\mathscr{H})$ we have $\phi(A)$, is essential unitary operator if A is essential operator, then we say that ϕ preserves the essential unitary operator in one direction.

Theorem 5.1. *Let $\phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$ be a linear map, unital up to compact operators (ie; $\phi(I) = I + \mathscr{K}(\mathscr{H})$) and surjective up to compact operators (ie; $\mathscr{B}(\mathscr{H}) = range(\phi) + \mathscr{K}(\mathscr{H})$) . Then ϕ preserve the essential unitary operator in one direction if and only if $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$,and the induced map $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is either a continuous $*$-automorphism or a continuous $*$-anti-automorphism .*

Proof. Step 1. $\phi(\mathscr{K}(\mathscr{H})) \subset \mathscr{K}(\mathscr{H})$.

Let $B \in \mathscr{K}(\mathscr{H})$ then there exist an operator $T \in \mathscr{K}(\mathscr{H})$ such that

$$\phi(I + B) = I + \phi(B) + T,$$

$$\phi(I - B) = I - \phi(B) + T,$$

and

$$\phi(I + iB) = I + i\phi(B) + T$$

or $\pm B+I$ and $iB+I$ are essential unitary operator, since ϕ preserve the essential unitary operator in one direction, hence $I+\pm\phi(B)+T$ and $I+i\phi(B)+T$ are essential unitary operators. We obtain

$$\phi(B)^*B+\phi(B)^*+\phi(B)\in\mathscr{K}(\mathscr{H}),$$

$$\phi(B)^*\phi(B)-\phi(B)^*-\phi(B)\in\mathscr{K}(\mathscr{H}),$$

and

$$\phi(B)^*\phi(B)-i\phi(B)^*+i\phi(B)\in\mathscr{K}(\mathscr{H}).$$

As $\mathscr{K}(\mathscr{H})$ is an ideal of $\mathscr{B}(\mathscr{H})$, those equalities implies that

$$\phi(B)^*\phi(B)\in\mathscr{K}(\mathscr{H}),$$

$$\phi(B)^*+\phi(B)\in\mathscr{K}(\mathscr{H})$$

and

$$\phi(B)^*-\phi(B)\in\mathscr{K}(\mathscr{H}),$$

hence $\phi(B)\in\mathscr{K}(\mathscr{H})$, Thus $\phi(\mathscr{K}(\mathscr{H}))\subset\mathscr{K}(\mathscr{H})$.

Step 2. The induced map ψ of ϕ is a linear map unital and surjective preserve unitary operator on one direction.

We have $\phi(\mathscr{K}(\mathscr{H}))\subset\mathscr{K}(\mathscr{H})$, so ϕ induces a surjective linear map

$$\psi:\mathscr{C}(\mathscr{H})\to\mathscr{C}(\mathscr{H}),$$

defined by $\psi(\pi(T))=(\pi\circ\phi)(T)$.

Now if $\pi(U)$ is a unitary operator in $\mathscr{C}(\mathscr{H})$, then U is essential unitary operator so

$$\phi(U)^*\phi(U)=\phi(U)\phi(U)^*=I+\mathscr{K}(\mathscr{H})$$

since π is a $*$-homomorphism, then

$$\pi(\phi(U)^*\phi(U))=\pi(\phi(U)\phi(U)^*)=\pi(I)$$

hence

$$\psi(\pi(U))^*\psi(\pi(U))=\psi(\pi(U))\psi(\pi(U))^*=\pi(I)$$

thus, we get that ψ preserve the unitary operator in one direction in $\mathscr{C}(\mathscr{H})$.

Step 3. ψ is a continuous map.

If T is self-adjoint operator in $\mathscr{C}(\mathscr{H})$, such that $||T||_e\leq 1$, then $U=T+i\sqrt{1-T^2}$ is a unitary operator and we have $T=\frac{1}{2}(U+U^*)$. In general, we can decompose T into its real and imaginary parts and then write each as a linear combination of two unitaries operators. Consequently for every operator T such that $||T||_e\leq 1$, we can write $T = \frac{1}{2}(U_1 + U_2 + U_3 + U_4)$ where $U_i, i = 1, \ldots, 4$ is a unitary operator. Therefore

$$\begin{aligned} ||\psi(T)||_e &= ||\psi(\frac{1}{2}(U_1 + U_2 + U_3 + U_4))||_e \\ &\leq \frac{1}{2}(||\psi(U_1)||_e + ||\psi(U_2)||_e + ||\psi(U_3)||_e + ||\psi(U_4)||_e) \\ &\leq 2, \end{aligned}$$

it follows that $||\psi(T)||_e \leq 2$ for every operator T such that $||T||_e \leq 1$, this implies that ψ is bounded.

Step 4. $\psi(\pi(I)) = \pi(I)$.

We have $\phi(I) = I + \mathscr{K}(\mathscr{H})$ so $\pi(\phi(I)) = \pi(I)$ and so $\psi(\pi(I)) = \pi(I)$.

Step 5. ψ is an $*$-automorphism or an $*$-anti-automorphism.

Pick a self-adjoint $S \in \mathscr{C}(\mathscr{H})$. Then e^{itS} is a unitary operator for every $t \in \mathbb{R}$. Or ψ is a unital linear continuous map that preserve unitary operator in one direction, then

$$\begin{aligned} I &= \psi(e^{itS})^*\psi(e^{itS}) = \psi(I + itS + \frac{(it)^2}{2!}S^2 + \ldots)^*\psi(I + itS + \frac{(it)^2}{2!}S^2 + \ldots) \\ &= I + it(\psi(S) - \psi(S)^*) + t^2 - \frac{\psi(S^2)^* + \psi(S^2)}{2} + \psi(S)^*\psi(S) \ldots . \end{aligned}$$

Hence

$$\psi(S) = \psi(S)^* \tag{5.1}$$

and

$$-\frac{\psi(S^2)^* + \psi(S^2)}{2} + \psi(S)^*\psi(S) = 0 \tag{5.2}$$

for S self-adjoint.

For any $B \in \mathscr{C}(\mathscr{H})$ we can write $B = S + iT$ with S, T is self-adjoint. By (5.1) and linearity of ψ we get $\psi(B^*) = \psi(B)^*$.

Now we will prove that ψ is a Jordan homomorphism.

We have $\psi(B^*) = \psi(B)^*$, then by (5.2) we get $\psi((S)^2) = (\psi(S))^2$, for every self-adjoint operator $S \in \mathscr{C}(\mathscr{H})$. So as in the proof of Theorem 3.1 to obtain that ψ is Jordan homomorphism , but it is known a Jordan homomorphism in prime algebra is an homomorphism or an anti-automorphism, then as $\mathscr{C}(\mathscr{H})$ is a prime algebra then we have that ψ is an homomorphism or an anti-homomorphism, or ψ is self-adjoin, consequently ψ is an $*$-homomorphism or $*$-anti-homomorphism. Now we will prove that ψ is injective. We see that $\psi : \mathscr{C}(\mathscr{H}) \to \mathscr{C}(\mathscr{H})$ is a surjective continuous homomorphism or anti-homomorphism, then $Ker(\psi)$ is a closed ideal in $\mathscr{C}(\mathscr{H})$, or $\mathscr{C}(\mathscr{H})$ is simple so the kernel of ψ is $Ker(\psi) = \{0\}$.

□

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