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FIXED POINT OF FUZZY MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, we work out on two fixed point theorems for fuzzy mappings on Hilbert spaces. The proof rely on the parallelogram law in Hilbert spaces.
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1. Introduction

Heilpern [2] introduced the concept of fuzzy mappings as a mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space and proved a fixed point theorem for fuzzy mappings. Various authors extended and generalized Heilpernąŕs result [1], [3], [4], [5], and [6]. In the present paper, we some proved fixed point theorems of fuzzy mappings as introduced by Heilpern applied to Hilbert spaces.

2. Preliminaries

In the following discussions we mainly follow the definitions and notations due to Heilpern [2]. Let H be a Hilbert space and $F(H)$ be collection of all fuzzy sets in H. Let $A \in F(H)$ and $\alpha \in [0,1]$ The α -level set of A, denoted by A_α is defined as

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0,1]$$
$$A_0 = \overline{\{}x : A(x)\, 0\},$$

where $\bar{B}$ stands for a closure of a set B.

Definition 2.1. A fuzzy subset A of H is said to be an approximate quantity if and only if its α-level set is a non-fuzzy compact convex subset of H for each $\alpha \in [0,1]$ and

$\sup_{x\in H} A(x) = 1$.
From the collection $F(H)$, the subcollection of all approximate quantities is denoted by $W(H)$.

Definition 2.2. Let $A, B \in W(H)$ and $\alpha \in [0,1]$, then

(1) $P_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} \|x-y\|$;
(2) $D_\alpha(A,B) = dist(A_\alpha, B_\alpha)$, where ąřdistąś denotes the Hausdorff metric between A_α and B_α ;
(3) $D(A,B) = \sup_\alpha D_\alpha(A,B)$ and
(4) $P(A,B) = \sup_\alpha P_\alpha(A,B)$.

It is to be noted that for any 'α', P_α is a non-decreasing as well as continuous function.

Definition 2.3. Let $A, B \in W(H)$. An approximate quantity A is said to be more accurate than B, denoted by $A \subset B$, iff $A(x) < B(x)$ for each $x \in H$. The relation $\subset$ induces a partial ordering on $W(H)$.

Definition 2.4. A mapping F from the set H onto $W(H)$ is said to be a fuzzy mapping. Any $x \in H$ is called fixed point of a mapping $F : H \to W(H)$ if

$$\{x\} \subset Fx,$$

where $\{x\}$ is the fuzzy set with a membership function equal to the characteristic function of the crisp set $\{x\}$.

We shall use the following lemmas due to Heilpern [2].

Lemma 2.5. *Let $x \in H$, $A \in W(H)$, then $\{x\} \subset A$ if and only if $P_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.*

Lemma 2.6. $P_\alpha(x,A) \le \|x-y\| + P_\alpha(y,A)$ *for any $x, y \in H$.*

Lemma 2.7. *If $\{x_0\} \subset A$, then $P_\alpha(x_0, A) \le D_\alpha(A,B)$ for each $B \in W(H)$.*

3. Main Results

n this section we prove common fixed point theorems for fuzzy mappings.

Theorem 3.1. *Let H be a Hilbert space, F and G are fuzzy mappings from H into $W(H)$ satisfying*

$$D^2(Fx,Gy) \le a_1\|x-y\| + a_2 P_\alpha^2(x,Fx) + a_3 P_\alpha^2(y,Gy) + a_4\left\{\frac{P_\alpha^2(x,Gy) + P_\alpha^2(y,Fx)}{1 + P_\alpha^2(y,Fx)P_\alpha^2(x,Gy)}\right\} \tag{3.1}$$

for all $x, y \in H$ and for all $\alpha \in [0,1]$ and a_1, a_2, a_3, a_4 are nonnegative numbers satisfying

$$a_1 + a_2 + a_3 + a_4 < 1. \tag{3.2}$$

Then there exist a point z in H such that $\{z\} \subset Fz \cap Gz$.

Proof. Let $x_0 \in H$. We construct the sequence $\{x_n\}$ as follows.

$$\{x_1\} \subset Fx_0, \quad \{x_2\} \subset Gx_1, \quad \cdots \{x_{2n+1}\} \subset Fx_2n, \quad \{x_{2n+2}\} \subset Gx_{2n+1},$$

and

$$\|x_i - x_{i+1}\| \leq D(Fx_{i-1}, Gx_i), \quad i = 1, 2, 3 \cdots.$$

Now

$$\begin{aligned}
\|x_{2n} - x_{2n+1}\|^2 &\leq D^2(Fx_{2n-1}, Gx_{2n}) \\
&\leq a_1\|x_{2n} - x_{2n-1}\|^2 + a_2 P_\alpha^2(x_{2n-1}, Fx_{2n-1}) + a_3 P_\alpha^2(x_{2n}, Gx_{2n}) \\
&\quad + a_4 \Big\{ \frac{P_\alpha^2(x_{2n-1}, Gx_{2n}) + P_\alpha^2(x_{2n}, Fx_{2n-1})}{1 + P_\alpha^2(x_{2n-1}, Gx_{2n}) P_\alpha^2(x_{2n}, Fx_{2n-1})} \Big\} \\
&\leq a_1\|x_{2n} - x_{2n-1}\|^2 + a_2\|x_{2n-1} - x_{2n}\|^2 + a_3\|x_{2n} - x_{2n+1}\|^2 \\
&\quad + a_4 \Big\{ \frac{\|x_{2n} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n+1}\|^2}{1 + \|x_{2n} - x_{2n}\|^2 \|x_{2n-1} - x_{2n+1}\|^2} \Big\} \\
&\leq a_1\|x_{2n} - x_{2n-1}\|^2 + a_2\|x_{2n-1} - x_{2n}\|^2 + a_3\|x_{2n} - x_{2n+1}\|^2 \\
&\quad + a_4\{2\|x_{2n-1} - x_{2n}\|^2 + 2\|x_{2n} - x_{2n+1}\|^2\}
\end{aligned}$$

(By using Parallelogram Law $\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$)

which gives

$$\|x_{2n} - x_{2n+1}\|^2 \leq k\|x_{2n} - x_{2n-1}\|^2,$$

where

$$0 < k = \frac{a_1 + a_2 + 2a_4}{1 - a_3 - 2a_4} < 1$$

Again,

$$\begin{aligned}\|x_{2n-1}-x_{2n}\|^2 &\leq D^2(Fx_{2n-2},Gx_{2n-1})\\ &\leq a_1\|x_{2n-2}-x_{2n-1}\|^2+a_2P_\alpha^2(x_{2n-2},Fx_{2n-2})+a_3P_\alpha^2(x_{2n-1},Gx_{2n-1})\\ &\quad+a_4\Big\{\frac{P_\alpha^2(x_{2n-2},Gx_{2n-1})+P_\alpha^2(x_{2n-1},Fx_{2n-2})}{1+P_\alpha^2(x_{2n-2},Gx_{2n-1})P_\alpha^2(x_{2n-1},Fx_{2n-2})}\Big\}\\ &\leq a_1\|x_{2n-2}-x_{2n-1}\|^2+a_2\|x_{2n-2}-x_{2n-1}\|^2+a_3\|x_{2n-1}-x_{2n}\|^2\\ &\quad+a_4\Big\{\frac{\|x_{2n-2}-x_{2n}\|^2+\|x_{2n-1}-x_{2n-1}\|^2}{1+\|x_{2n-2}-x_{2n}\|^2+\|x_{2n-1}-x_{2n-1}\|^2}\Big\}\\ &\leq a_1\|x_{2n-2}-x_{2n-1}\|^2+a_2\|x_{2n-2}-x_{2n-1}\|^2+a_3\|x_{2n-1}-x_{2n}\|^2\\ &\quad+a_4\{2\|x_{2n-2}-x_{2n-1}\|^2+2\|x_{2n-1}-x_{2n}\|^2\}\end{aligned}$$

(By using Parallelogram Law $\|x+y\|^2\leq 2\|x\|^2+2\|y\|^2$)

which gives

$$\|x_{2n}-x_{2n+1}\|^2\leq k\|x_{2n}-x_{2n-1}\|^2,$$

where

$$0<k=\frac{a_1+a_2+2a_4}{1-a_3-2a_4}<1$$

In general, it follows that

$$\|x_{n+1}-x_n\|^2\leq k\|x_n-x_{n-1}\|^2,\quad(0<k<1).$$

Hence $\{x_n\}$ is a Cauchy sequence in H and therefore it converges to a limit in H. We assume $\lim_{n\to\infty}x_n=z$.
Again, using Lemma (2.7) and for all $\alpha\in[0,1]$,we have

$$\begin{aligned}P_\alpha^2(x_{2n+2},Fz) &\leq D_\alpha^2(Gx_{2n+1},Fz)\\ &\leq a_1\|z-x_{2n+1}\|^2+a_2P_\alpha^2(z,Fz)+a_3P_\alpha^2(x_{2n+1},Gx_{2n+1})\\ &\quad+a_4\Big\{\frac{P_\alpha^2(z,Gx_{2n+1})+P_\alpha^2(x_{2n+1},Fz)}{1+P_\alpha^2(z,Gx_{2n+1})P_\alpha^2(x_{2n+1},Fz)}\\ &\leq a_1\|z-x_{2n+1}\|^2+a_2P_\alpha^2(z,Fz)+a_3\|x_{2n+1},x_{2n+2}\|^2\\ &\quad+a_4\Big\{\frac{P_\alpha^2(z,x_{2n+2})+P_\alpha^2(x_{2n+1},Fz)}{1+P_\alpha^2(z,x_{2n+2})P_\alpha^2(x_{2n+1},Fz)}\Big\}\end{aligned}$$

Making $n\to\infty$ and using the fact that P_α is continuous,

$$P_\alpha^2(z,Fz)\leq(a_2+a_4)P_\alpha^2(z,Fz).$$

As $(a_2+a_4)<1$, it follows that $P_\alpha^2(z,Fz)=0$, hence by Lemma (2.5), $\{z\}\subset Fz$.
Similarly, $\{z\}\subset Gz$. Hence, $\{z\}\subset Fz\cap Gz$. □

Theorem 3.2. *Let H be a Hilbert space, F and G are fuzzy mappings from H into $W(H)$ satisfying*

$$D^2(Fx, Gy) \leq q \max\Big\{\|x-y\|^2, P_\alpha^2(x, Fx), P_\alpha^2(y, Gy), \frac{P_\alpha^2(x, Gy) + P_\alpha^2(y, Fx)}{1 + P_\alpha^2(x, Gy)P_\alpha^2(y, Fx)}\Big\} \tag{3.3}$$

for all $x, y \in H$ and for all $\alpha \in [0,1]$ and $0 < q < \frac{1}{4}$. Then there exist a point $z \in H$ such that $\{z\} \subset Fz \cap Gz$.

Proof. Let $x_0 \in H$. We construct the sequence $\{x_n\}$ as in Theorem (3.1) and correspondingly

$$\begin{aligned}
\|x_{2n} - x_{2n+1}\|^2 &\leq D^2(Fx_{2n}, Gx_{2n-1}) \\
&\leq q \max\Big\{\|x_{2n} - x_{2n-1}\|^2, P_\alpha^2(x_{2n}, Fx_{2n}), P_\alpha^2(x_{2n-1}, Gx_{2n-1}), \\
&\quad \frac{P_\alpha^2(x_{2n}, Gx_{2n-1}) + P_\alpha^2(x_{2n-1}, Fx_{2n})}{1 + P_\alpha^2(x_{2n}, Gx_{2n-1})P_\alpha^2(x_{2n-1}, Fx_{2n})}\Big\} \\
&\leq q \max\Big\{\|x_{2n} - x_{2n-1}\|^2, \|x_{2n} - x_{2n+1}\|^2, \|x_{2n-1} - x_{2n}\|^2, \\
&\quad \frac{\|x_{2n} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n+1}\|^2}{1 + \|x_{2n} - x_{2n}\|^2\|x_{2n-1} - x_{2n+1}\|^2}\Big\} \\
&\leq q \max\{\|x_{2n} - x_{2n-1}\|^2, \|x_{2n} - x_{2n+1}\|^2, \\
&\quad 2[\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2]\} \\
&\leq q\{2[\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2]\}
\end{aligned}$$

which yields

$$\|x_{2n} - x_{2n+1}\|^2 \leq \frac{2q}{1-2q}\|x_{2n} - x_{2n-1}\|^2. \tag{3.4}$$

Again

$$\begin{aligned}
\|x_{2n}-x_{2n-1}\|^2 &\le D^2(Fx_{2n-2}, Gx_{2n-1})\\
&\le q\max\Big\{\|x_{2n-2}-x_{2n-1}\|^2, P_\alpha^2(x_{2n-2}, Fx_{2n-2}), P_\alpha^2(x_{2n-1}, Gx_{2n-1}),\\
&\quad \frac{P_\alpha^2(x_{2n-2}, Gx_{2n-1}) + P_\alpha^2(x_{2n-1}, Fx_{2n-2})}{1+P_\alpha^2(x_{2n-2}, Gx_{2n-1})P_\alpha^2(x_{2n-1}, Fx_{2n-2})}\Big\}\\
&\le q\max\Big\{\|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-1}-x_{2n}\|^2,\\
&\quad \frac{\|x_{2n-2}-x_{2n}\|^2 + \|x_{2n-1}-x_{2n-1}\|^2}{1+\|x_{2n-2}-x_{2n}\|^2\|x_{2n-1}-x_{2n-1}\|^2}\Big\}\\
&\le q\max\{\|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-1}-x_{2n}\|^2,\\
&\quad 2[\|x_{2n-2}-x_{2n-1}\|^2 + \|x_{2n-1}-x_{2n}\|^2]\}\\
&\le q\{2[\|x_{2n-2}-x_{2n-1}\|^2 + \|x_{2n-1}-x_{2n}\|^2]\}
\end{aligned}$$

which yields

$$\|x_{2n}-x_{2n-1}\|^2 \le \frac{2q}{1-2q}\|x_{2n-2}-x_{2n-1}\|^2. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\|x_{n+1}-x_n\|^2 \le k\|x_n-x_{n-1}\|^2,$$

where $0 < k = \frac{2q}{1-2q} < 1$.
Hence $\{x_n\}$ is a Cauchy sequence in H and therefore it converges to a limit in H. We assume

$$\lim_{n\to\infty} x_n = z.$$

Again using Lemma (2.7), we have

$$\begin{aligned}
P_\alpha^2(x_{2n+2}, Fz) &\le D_\alpha^2(Gx_{2n+1}, Fz)\\
&\le D^2(Gx_{2n+1}, Fz)\\
&\le q\max\Big\{\|z-x_{2n+1}\|^2, P_\alpha^2(z, Fz), P_\alpha^2(x_{2n+1}, Gx_{2n+1}),\\
&\quad \frac{P_\alpha^2(z, Gx_{2n+1}) + P_\alpha^2(x_{2n+1}, Fz)}{1+P_\alpha^2(z, Gx_{2n+1})P_\alpha^2(x_{2n+1}, Fz)}\Big\}\\
&\le q\max\Big\{\|z-x_{2n+1}\|^2, P_\alpha^2(z, Fz), \|x_{2n+1}-x_{2n+2}\|^2\\
&\quad \frac{\|z-x_{2n+2}\|^2 + P_\alpha^2(x_{2n+1}, Fz)}{1+\|z-x_{2n+2}\|^2 P_\alpha^2(x_{2n+1}, Fz)}\Big\}.
\end{aligned}$$

Making $n\to\infty$ and using the fact that P_α is continuous,

$$P_\alpha^2(z, Fz) \le qP_\alpha^2(z, Fz).$$

As $0 < q < \frac{1}{4}$, it follows that $P_\alpha^2(z, Fz) = 0$, hence by Lemma (2.5), $\{z\} \subset Fz$. Similarly, $\{z\} \subset Gz$. Hence $\{z\} \subset Fz \cap Gz$. □

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