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ESTIMATE OF SECOND HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In the present investigation an upper bound of second Hankel determinant $|a_2a_4 - a_3^2|$ for the functions belonging to the class $M_s(\alpha; A, B)$ is studied. The results due to various authors follow as special cases.

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1. Introduction

By A, we denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the unit disc $E = \{z :| z |< 1\}$.

Let U is the class of bounded analytic functions $w(z)$ in the unit disc E and of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n,\ z \in E,$$

which satisfy the conditions $w(0) = 0, |w(z)| < 1$.

Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc E, then $f(z)$ is said to be subordinate to $F(z)$ if there exists a function $w(z) \in U$ such that $f(z) = F(w(z))$ and we write as $f(z) \prec F(z)$.

$M_s(\alpha; A, B)$ represents the class of functions $f(z)$ in A which satisfy the condition

$$\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1-\alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1, \, z \in E. \tag{1.2}$$

The following observations are obvious:

(i) $M_s(\alpha; 1, -1) \equiv M_s(\alpha)$, the class introduced by Selvaraj and Vasanthi [12].

(ii) $M_s(0; 1, -1) \equiv S_s^*$, the class of starlike functions with respect to symmetric points introduced by Sakaguchi [11].

(iii) $M_s(1; 1, -1) \equiv K_s$, the class of convex functions with respect to symmetric points introduced by Das and Singh [1].

(iv) $M_s(0; A, B) \equiv S_s^*(A, B)$, the subclass of starlike functions with respect to symmetric points introduced and studied by Goel and Mehrok [2].

(v) $M_s(1; A, B) \equiv K_s(A, B)$, the subclass of convex functions with respect to symmetric points.

In 1976, Noonan and Thomas [9] stated the qth Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as second Hankel determinant $H_2(2)$ and obtain an upper bound to the functional $H_2(2)$ for $f(z) \in M_s(\alpha; A, B)$.

Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Earlier Janteng et al. [3, 4, 5], Mehrok and Singh [8], Singh [13, 14] obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

2. Main Results

Let P be the family of all functions p analytic in E for which $Re(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \tag{2.1}$$

for $z \in E$.

Lemma 2.1. If $p \in P$, then $|p_k| \leq 2(k = 1, 2, 3, ...)$.
This result is due to Pommerenke [10].

Lemma 2.2. If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some x and z satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [6, 7]

Theorem 2.1. If $f \in M_s(\alpha; A, B)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(A - B)^2}{4(1 + 2\alpha)^2}. \tag{2.2}$$

Proof. As $f \in M_s(\alpha; A, B)$, so there exists a Schwarz function $w(z) \in U$ such that

$$\frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1 - \alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} = \phi(w(z)) \tag{2.3}$$

where

(2.4)
$$\phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + ... = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \tag{2.5}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $h(z)$ by

$$h(z) = \frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1 - \alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{2.6}$$

In view of the equations (2.3), (2.5) and (2.6), we have

$$h(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1z+c_2z^2+c_3z^3+...}{2+c_1z+c_2z^2+c_3z^3+...}\right)$$
$$= \phi\left(\frac{1}{2}c_1z + \frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3...\right)$$
$$= 1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2$$
$$+\left[\frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}\right]z^3 + ...$$

Thus

(2.7)
$$b_1 = \frac{B_1c_1}{2}; b_2 = \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}; b_3 = \frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}.$$

Using (2.4) and (2.6) in (2.7), we obtain

(2.8)
$$a_2 = \frac{(A-B)c_1}{4(1+\alpha)},$$

(2.9)
$$a_3 = \frac{(A-B)}{8(1+2\alpha)}[2c_2 - (B+1)c_1^2]$$

and

(2.10)
$$a_4 = \frac{(A-B)}{64(1+3\alpha)}[8c_3 + 2(A-5B-4)c_1c_2 + (B+1)(3B-A+2)c_1^3].$$

Using (2.8), (2.9) and (2.10), it yields

(2.11)
$$a_2a_4 - a_3^2 = \frac{(A-B)^2}{C(\alpha)}[2Lc_1(4c_3) + Mc_1^2(2c_2) - Nc_1^4 - 4R(4c_2^2)]$$

where $C(\alpha) = 256(1+3\alpha)(1+\alpha)(1+2\alpha)^2$,
$L = (1+2\alpha)^2$,
$M = (1+2\alpha)^2A + [8(1+3\alpha)(1+\alpha) - 5(1+2\alpha)^2]B + [8(1+3\alpha)(1+\alpha) - 4(1+2\alpha)^2]$,
$N = (B+1)[(1+2\alpha)^2A + [4(1+3\alpha)(1+\alpha) - 3(1+2\alpha)^2]B + [4(1+3\alpha)(1+\alpha) - 2(1+2\alpha)^2]]$
and
$R = (1+3\alpha)(1+\alpha)$.

Using Lemma 2.1 and Lemma 2.2 in (2.11), we obtain

$$|a_2a_4-a_3^2| = \frac{(A-B)^2}{C(\alpha)}\left|\begin{array}{r} -[(1+2\alpha)^2AB+[4(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2]B^2]c_1^4 \\ +[(1+2\alpha)^2A+[8(1+\alpha)(1+3\alpha)-5(1+2\alpha)^2]B]c_1^2(4-c_1^2)x \\ -2[8(1+\alpha)(1+3\alpha)-[2(1+\alpha)(1+3\alpha)-(1+2\alpha)^2]c_1^2](4-c_1^2)x^2 \\ +4(1+\alpha)(1+3\alpha)c_1(4-c_1^2)(1-|x|^2)z \end{array}\right|.$$

Assume that $c_1 = c$ and $c \in [0,2]$, using triangular inequality and $|z| \leq 1$, we have

$$|a_2a_4-a_3^2| \leq \frac{(A-B)^2}{C(\alpha)}\left[\begin{array}{r} [2(4-c^2)(8(1+\alpha)(1+3\alpha)-(2(1+\alpha)(1+3\alpha)-(1+2\alpha)^2)c^2) \\ -4(1+\alpha)(1+3\alpha)c(4-c^2)]\delta^2 \\ +|(1+2\alpha)^2A+[8(1+\alpha)(1+3\alpha)-5(1+2\alpha)^2]B|(4-c^2)c^2\delta \\ +|(1+2\alpha)^2AB+[4(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2]B^2|c^4 \\ +4(1+\alpha)(1+3\alpha)c(4-c^2) \end{array}\right].$$

Therefore

$$|a_2a_4-a_3^2| \leq \frac{(A-B)^2}{C(\alpha)}F(\delta),$$

where $\delta = |x| \leq 1$ and

$$\begin{aligned} F(\delta) = {} & [2(4-c^2)(8(1+\alpha)(1+3\alpha)-(2(1+\alpha)(1+3\alpha)-(1+2\alpha)^2)c^2) \\ & -4(1+\alpha)(1+3\alpha)c(4-c^2)]\delta^2 \\ & +|(1+2\alpha)^2A+[8(1+\alpha)(1+3\alpha)-5(1+2\alpha)^2]B|(4-c^2)c^2\delta \\ & +|(1+2\alpha)^2AB+[4(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2]B^2|c^4 \\ & +4(1+\alpha)(1+3\alpha)c(4-c^2) \end{aligned}$$

is an increasing function.
Therefore Max$F(\delta) = F(1)$.
Consequently

$$|a_2a_4-a_3^2| \leq \frac{(A-B)^2}{C(\alpha)}G(c), \qquad (2.12)$$

where $G(c) = F(1)$.
So

$$G(c) = S(\alpha)c^4 + T(\alpha)c^2 + 64(1+3\alpha)(1+\alpha)$$

where

$$S(\alpha)=\left[\begin{array}{r}|(1+2\alpha)^2AB+[4(1+\alpha)(1+3\alpha)-3(1+2\alpha)^2]B^2| \\ -|(1+2\alpha)^2A+[8(1+\alpha)(1+3\alpha)-5(1+2\alpha)^2]B| \\ +2[2(1+\alpha)(1+3\alpha)-(1+2\alpha)^2]\end{array}\right]$$

and

$$T(\alpha)=\left[\begin{array}{r}4|(1+2\alpha)^2A+[8(1+\alpha)(1+3\alpha)-5(1+2\alpha)^2]B| \\ -8[4(1+\alpha)(1+3\alpha)-(1+2\alpha)^2]\end{array}\right].$$

Now

$$G^{'}(c)=4S(\alpha)c^3+2T(\alpha)c$$

and

$$G^{''}(c)=12S(\alpha)c^2+2T(\alpha).$$

$G^{'}(c)=0$ gives

$$c[2S(\alpha)c^2+T(\alpha)]=0.$$

$G^{''}(c)$ is negative at $c=0$.
So Max$G(c)=G(1)$.
Hence from (2.12), we obtain (2.2).
The result is sharp for $c_1=0$, $c_2=2$ and $c_3=0$.

For $A=1, B=-1$, Theorem 2.1 gives the following result due to Singh [13].

Corollary 2.1. If $f(z)\in M_s(\alpha)$, then

$$|a_2a_4-a_3^2|\leq\frac{1}{(1+2\alpha)^2}.$$

For $\alpha=0, A=1, B=-1$, Theorem 2.1 gives the following result due to Janteng et al. [5].

Corollary 2.2. If $f(z)\in S_s^*$, then

$$|a_2a_4-a_3^2|\leq 1.$$

For $\alpha=1, A=1, B=-1$, Theorem 2.1 gives the following result due to Janteng et al. [5].

Corollary 2.3. If $f(z)\in K_s$, then

$$|a_2a_4-a_3^2|\leq\frac{1}{9}.$$

For $\alpha = 0$, Theorem 2.1 gives the following result.

Corollary 2.4. If $f(z) \in S_s^*(A, B)$, then

$$|a_2a_4 - a_3^2| \leq \frac{(A-B)^2}{4}.$$

For $\alpha = 1$, Theorem 2.1 gives the following result.

Corollary 2.5. If $f(z) \in K_s(A, B)$, then

$$|a_2a_4 - a_3^2| \leq \frac{(A-B)^2}{36}.$$

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