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ECCENTRIC DIGRAPHS OF TREES

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ABSTRACT. In a digraph G, the eccentricity $e(u)$ of a vertex u is the maximum distance from u to any other vertex in G. A vertex v is an eccentric vertex of u if the distance from u to v equals $e(u)$. An eccentric digraph $ED(G)$ of a graph G has vertices, same as that of G but has directed edges that correspond to the relation that v is an eccentric vertex of u. In other words, there are directed edges in $ED(G)$ from a vertex u to those vertices v which are farthest from u. In this paper we have examined the problem of finding the structure of the second iterated eccentric digraph of a tree and obtained a solution.

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1. Introduction

The field of graph theory is rich in its theoretical development as well as in finding application in different areas. One of the well-investigated notions in graph theory is the distance between vertices in a graph, especially when a graph is used in modeling real-world problems. This notion of distance has given rise to other concepts on graphs. Based on the related concept of eccentricity of a vertex which is the maximum distance from a vertex to any other vertex in a graph, the idea of an eccentric digraph of a graph was introduced by Buckley [3] and this was extended to the eccentric digraph of a digraph by Boland and Miller [2]. Several authors [2, 5, 6, 7] have investigated the problem of finding eccentric digraphs of graphs and digraphs. The problem of finding the structure of the second iteration of the eccentric digraph of a tree, is one among a list of problems proposed by Boland et al [1]. In this paper we have examined this problem and obtained a solution. Also we have obtained the domination numbers of the eccentric digraph and the second iterated eccentric digraph of a tree as well as a diameter maximal graph with odd diameter.

2. Preliminaries

We recall needed notions. For unexplained notions and notations on graphs and digraphs, we refer to [4, 8].

A directed graph or a digraph $G = (V, E)$ consists of a finite nonempty set $V = V(G)$ of objects called vertices or points and a set $E = E(G)$ of ordered pairs of vertices called directed edges or arcs; that is, $E(G)$ represents a binary relation defined on $V(G)$. If the set E consists of unordered pairs of vertices, called edges, then G is an undirected graph. A tree is a connected undirected graph with no cycles. The order $|G|$ of G is the cardinality $|V(G)|$ of $V(G)$ i.e. $|G| = |V(G)|$. The digraph G is called a complete symmetric digraph or a symmetric clique if G contains the arc (u, v) as well as the arc (v, u) for every pair of vertices u, v in G and it is denoted by K_p^*, if G contains p vertices. In the case of undirected graph, a complete graph on p vertices, in which every vertex is adjacent to every other vertex, is denoted by K_p and its complement which has the same p vertices but has no edges is denoted by $\overline{K_p}$. A maximal strongly connected subgraph of a digraph G is called a strong component of G.

For two vertex disjoint digraphs G_1 and G_2, $G_1 \oplus G_2$ is the digraph obtained by joining an arc from each vertex of G_1 to every vertex of G_2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the union of G_1 and G_2 is defined as the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ while the graph $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$. For three or more graphs $G_1, G_2, G_3, \cdots, G_n$, the sequential join [4] $G_1 + G_2 + G_3 + \cdots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$. Also, for a set $S \subseteq V$, the subgraph induced by the vertices in S, is denoted by $G[S]$ so that this subgraph has all the elements of S as its vertices and has all the arcs (or edges) among the vertices of S that are present in the digraph (or the graph) G with vertex set V. Two graphs (or digraphs) G and H are said to be isomorphic if there exists a bijection f from $V(G)$ to $V(H)$ such that (u, v) is an edge (or arc) in G if and only if $(f(u), f(v))$ is an edge (or arc) in H and the isomorphism is denoted by $G \cong H$.

If (u, v) is an arc of the digraph G, then u is said to be adjacent to v and v is adjacent from u. The set of vertices which are adjacent from a given vertex v in a digraph G is denoted by $N^+(v)$ and the set of vertices adjacent to v, is denoted by $N^-(v)$. The cardinality of $N^+(v)$ is the out-degree of v, that is denoted by $deg^+(v)$ and the cardinality of $N^-(v)$ is the in-degree of v, which is denoted by $deg^-(v)$. The set of mutually non-adjacent vertices of G is called a stable set. The number of arcs in a shortest directed path from u to v is the distance from u to v, denoted by $d(u, v)$. If there is no directed path from u to v in G, then we define $d(u, v) = \infty$. In the case of a undirected graph, the distance $d(u, v)$ is the number of edges in a shortest path from u to v. The eccentricity $e(u)$ of a vertex u in a digraph (or graph) G is the maximum distance from u to any other vertex in G. Vertex v is an eccentric vertex of u if $d(u, v) = e(u)$. The set of all eccentric vertices of u is denoted by $E(u)$. The radius of G is the minimum eccentricity of the vertices in G; the diameter is the maximum eccentricity of the vertices in G. A vertex whose eccentricity equals the radius, is called a central vertex of G. A vertex whose eccentricity equals the diameter, is called a peripheral vertex of G. The set of all central vertices of G is called the center of G, denoted by $C(G)$. The set of all peripheral vertices is denoted by $P(G)$ and it is called as periphery of G. The set of all eccentric vertices of G is denoted by $EP(G)$.

A graph G is said to be a diameter maximal graph (also called an upper diameter critical graph), if $diam(G + e) < diam(G)$ for every $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G. In an undirected graph G, a $u - v$ path is called an eccentric path if either u is an eccentric vertex of v or v is an eccentric vertex of u, whereas in the case of a digraph, a $u - v$ directed path is called an eccentric path if v is an eccentric vertex of u. A set $S \subseteq V$ is called a dominating set of a digraph $G = (V, E)$ if each vertex of $V(G)$ is adjacent from at least one vertex of S. The cardinality of a minimal dominating set is called the domination number of G, denoted by $\gamma(G)$. A graph (or digraph) G is called a unique eccentric point graph (u.e.p.) [9] if every vertex of G has only one eccentric vertex. We note that if a vertex in a digraph G has out-degree zero, then this vertex has all the other vertices of the given digraph as its eccentric vertices.

The eccentric digraph of a digraph G, denoted $ED(G)$, is the digraph on the same vertex set as G, but with an arc from vertex u to vertex v in $ED(G)$ if and only if v is an eccentric vertex of u. The eccentric digraph of a (undirected) graph G is defined in a similar manner. As an illustration, a tree T is shown in Fig. 1. The eccentric digraph $ED(T)$ of the tree T in Fig. 1 and the second iterated eccentric digraph $ED^2(T) = ED(ED(T))$ of the tree T are shown in Figs. 2 and 3 respectively. For every digraph G, there exist smallest positive integer numbers $p > 0$ and $t \geq 0$, such that

$ED^{p+t}(G) = ED^t(G)$. The integers p and t are called the period of G, denoted by $p(G)$ and the tail of G, denoted by $t(G)$. If $t = 0$, then G is called periodic.

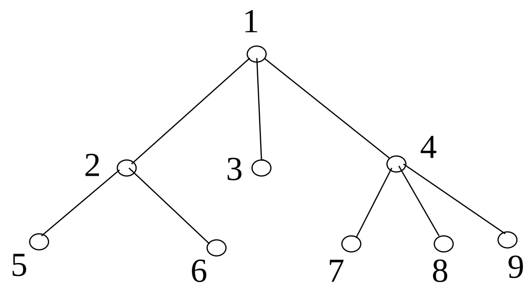

FIGURE 1. A Tree T

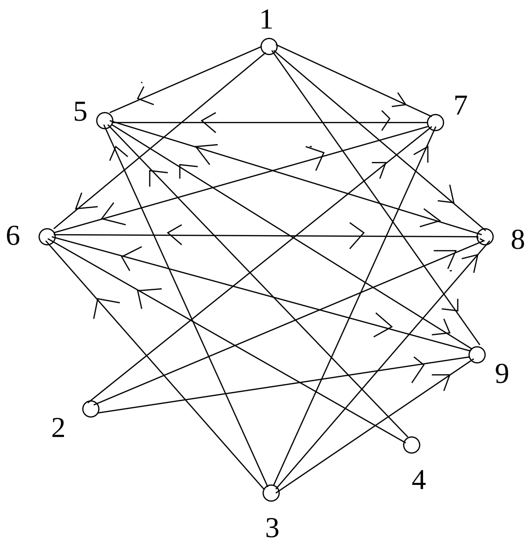

FIGURE 2. Eccentric digraph $ED(T)$ of the Tree T in Fig.1

3. Trees and their Eccentric Digraphs

We first consider the digraph $\overline{K^*_{n-l}} \oplus K^*_l$, and examine the structure of the eccentric digraph of G.

Theorem 3.1. *If the digraph* $\overline{K^*_{n-l}} \oplus K^*_l = G = (V, E)$ *with* $|V| = n$, *and some* $l, 1 \leq l < n$, *then*
(i) *there exists a digraph* H *such that* $ED(H) = G$
(ii) $ED(G) = \overline{K^*_l} \oplus K^*_{n-l}$ *and* G *is periodic with period* 2.

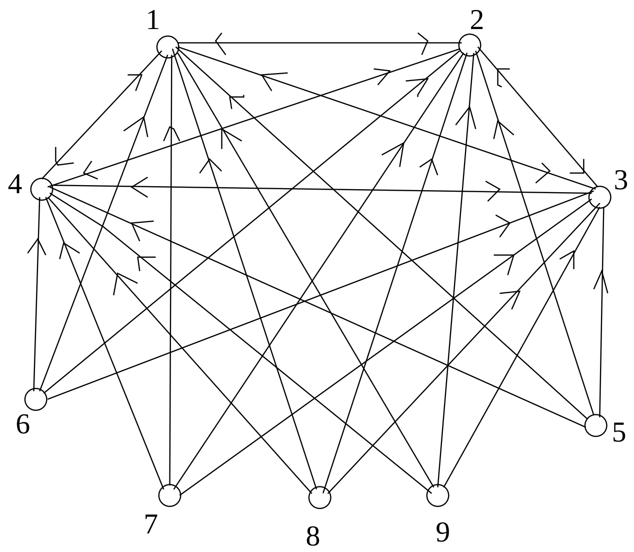

FIGURE 3. Second iterated eccentric digraph $ED^2(T)$ of the tree T in Fig.1

Proof. Let $S = \{u_1, u_2, u_3, \cdots, u_l\} \subseteq V$ and $V - S = \{v_1, v_2, v_3, \cdots, v_{n-l}\}$ such that $G[S] = K_l^*$ and $V - S$ is a stable set. Now to prove statement (i), consider a strong digraph H_1 with vertex set $V(H_1) = V - S$. We construct a digraph $H = (V, F)$ where

$$F = E(H_1) \cup \{(u_i, v_1)/1 \leq i \leq l\}.$$

Then $G[V - S]$ is a strong component of H and S is a stable set of H. Moreover, no vertex of S is reached in H. This implies that in H every vertex of S is an eccentric vertex of every other vertex of V and no vertex of $V - S$ can be an eccentric vertex. Hence in $ED(H)$ every vertex of S is adjacent from every other vertex of V and no vertex of $V - S$ is adjacent from any vertex of V. Thus $ED(H) = \overline{K_{n-l}^*} \oplus K_l^* = G$.

Now we prove statement (ii). It is clear that no vertex in $V - S$ is reached in G from any vertex of V. This implies that all the vertices of $V - S$ are eccentric vertices in G. It follows that the eccentric digraph $ED(G)$ has the same vertex set V and there is an edge from every vertex of V to every vertex of $V - S$ and $deg^-(v) = 0$ for all $v \in S$. It follows that in $ED(G)$, S is a stable set and $V - S$ is a symmetric clique. That is

$$ED(G) = \overline{K_l^*} \oplus K_{n-l}^*.$$

Now in $ED^2(G)$, every vertex of $V - S$ is of in-degree zero and there is an edge from every vertex of V to every vertex of S. Hence

$$ED^2(G) = \overline{K_{n-l}^*} \oplus K_l^* = G.$$

Thus G is periodic with period two. □

Corollary 3.2. Let G be a digraph on $2n$ vertices such that $G = \overline{K^*_{2n-l}} \oplus K^*_l$, for some $l, 1 \leq l < 2n$. Then $ED(G) \cong G$, when $l = n$.

Proof. Let G be a digraph on $2n$ vertices such that $G = \overline{K^*_{2n-l}} \oplus K^*_l$, for some l. Then by Theorem 3.1, $ED(G) = \overline{K^*_l} \oplus K^*_{2n-l}$. Furthermore if $l = n$, then $G = \overline{K^*_n} \oplus K^*_n$ and $ED(G) = \overline{K^*_n} \oplus K^*_n$. Thus $ED(G) \cong G$.

□

Now we consider trees T in order to study the structure of the second iterated eccentric digraph $ED^2(T)$. First we show in the following Lemma that the periphery of T is exactly the set of all eccentric vertices of T.

Lemma 3.3. *If T is a tree then $P(T) = EP(T)$.*

Proof. Let T be a tree and suppose that $P(T) \neq EP(T)$. Then there is a vertex v in $EP(T)$ such that $v \notin P(T)$. Since $deg(u) = 1$, for all u in $P(T)$, $deg(v) > 1$. Let v_1 and v_2 be the neighbors of v. Since $v \in EP(T)$, there is a vertex $w \in V(T)$ such that $v \in E(w)$. Furthermore, a $w - v$ path passes through either v_1 or v_2 but not both. If $w - v$ path passes through v_1, then $d(w, v_2) > d(w, v)$ which is not true. This leads to $deg(v) = 1$ and hence $v \in P(T)$. Therefore $P(T) = EP(T)$. □

Theorem 3.4. *If T is a tree on n vertices with $|P(T)| = l$, then*

$$ED^2(T) = \overline{K^*_l} + K^*_{n-l}, n > l + 1.$$

Proof. Let T be a tree on n vertices with $|P(T)| = l$. Let

$$P(T) = \{u_1, u_2, u_3, \cdots, u_l\}$$

and

$$V(T) - P(T) = \{v_1, v_2, v_3, \cdots, v_{n-l}\}.$$

Then by Lemma 3.3, $|P(T)| = |EP(T)| = l$. Also, any two vertices of $P(T)$ are either eccentric vertices of each other or have a common neighbor. This implies that in $ED(T)$, the set of all vertices u_i induces a strong digraph and the other vertices $v_i's$ cannot be reached and hence each and every vertex $v_i (1 \leq i \leq n - l)$ is eccentric vertex to every other vertex in $ED(T)$. Consequently in $ED^2(T)$ the set of all vertices v_i induces a complete symmetric digraph and there are edges from every u_i to every v_i. Hence $ED^2(T) = \overline{K^*_l} + K^*_{n-l}$, $n > l + 1$. □

Corollary 3.5. Let T be a tree on n vertices with $|P(T)| = l$. Then the period $p(T) = 2$ and the tail $t(T) = 2$.

Proof. By Theorem 3.4, $ED^2(T) = \overline{K^*_{n-r}} \oplus K^*_r$. Now by Theorem 3.1, $ED^2(T)$ is periodic with period 2. Thus the period and tail of T are each equal to two. □

The following Theorem gives tight bounds on the number of edges of the eccentric digraph $ED(T)$ of a tree T.

Theorem 3.6. *If T is a tree on n vertices with radius r then*

$$2 \leq |E(ED(T)| \leq l(n-r)$$

where $|P(T)| = l$.

Proof. Let $T = (V, E)$ be a tree on n vertices with radius r and $|P(T)| = l$. Let $V = V_1 \cup V_2 \cup V_3 \cup V_4$ where V_1 is the set of all central vertices, $V_2 = P(T)$, $V_3 = \{u \in V - (V_1 \cup V_2)/u$ is an internal vertex of some eccentric path$\}$ and $V_4 = V - (V_1 \cup V_2 \cup V_3)$. Now by Lemma 3.3, $|EP(T)| = |P(T)| = l$. This implies that, $|V_1| \leq 2$, $|V_2| = l$, $|V_3| \leq l(r-1)$ and $|V_4| \leq n - 1 - lr$. Each of the vertices of V_2 as well as of V_3 has at most $l-1$ eccentric vertices and every vertex of V_4 has at most l eccentric vertices whereas V_1 has exactly l eccentric vertices.
It is clear that

$$\begin{aligned}
|E(ED(T)| &= \sum_{v \in V} |E(v)| \\
&= \sum_{v \in V_1} |E(v)| + \sum_{v \in V_2} |E(v)| + \sum_{v \in V_3} |E(v)| + \sum_{v \in V_4} |E(v)| \\
&\leq l + l(l-1) + l(r-1)(l-1) + (n-1-lr)l \\
&= l(n-r)
\end{aligned}$$

That is $|E(ED(T)| \leq l(n-r)$. Since a tree has at least 2 vertices, $|E(ED(T)| \geq 2$. Furthermore the lower bound is attained when $T = K_2$ and the upper bound is attained when the degree of the unique central vertex is l. □

Theorem 3.7. *If T is a tree on n vertices with $|P(T)| = l$, then the domination number $\gamma(ED(T)) = n - l$ and $\gamma(ED^2(T)) = l$.*

Proof. Let T be a tree on n vertices with $|P(T)| = l$. Then by Lemma 3.3, $|EP(T)| = l$. Also it is clear that in $ED(T)$, $n - l$ vertices are of in-degree zero. Hence $\gamma(ED(T)) = n - l$. Furthermore by Theorem 3.4, $ED^2(T) = \overline{K^*_l} \oplus K^*_{n-l}$. This implies that $\gamma(ED^2(T)) = l$. □

4. Diameter Maximal Graphs and Their Eccentric Digraphs

We now consider a diameter maximal graph G of odd diameter and obtain properties of the eccentric graph of G.

Lemma 4.1. [4] *Let G be a diameter maximal graph.*
$i)$ *If G is connected then it has a unique pair of eccentric peripheral vertices and for some $d-1$ positive integers $a_i, 1 \leq i \leq d-1$, G has the form*

$$K_1 + K_{a_1} + K_{a_2} + \cdots + K_{a_{d-1}} + K_1$$

$ii)$ *If G is disconnected then $G = K_m \cup K_n$.*
$iii)$ *If G has an odd diameter then G is a $u.e.p$ graph.*

Theorem 4.2. *The eccentric digraph $ED(G)$ of a diameter maximal graph G of odd diameter is a K_2^* with one end having m independent in-neighbors and other end having n independent in-neighbors.*

Proof. Let G be a diameter maximal graph with odd diameter d. Then by Lemma 4.1 there exists $d-1$ positive integers $a_1, a_2, a_3, \cdots, a_{d-1}$ such that

$$G = K_1(=\{u\}) + K_{a_1} + K_{a_2} + \cdots + K_{a_{d-1}} + K_1(=\{v\}).$$

Also $P(G) = EP(G)$ with $|P(G)| = 2$ and G is a u.e.p graph. Let $P(G) = \{u, v\}$. Then it is clear that for all $w \in K_{a_i}, 1 \leq i \leq (d-1)/2, E(w) = \{v\}$ and for all $w \in K_{a_i}, (d-1)/2 \leq i \leq (d-1), E(w) = \{u\}$. This implies that in $ED(G)$, all the vertices $w \in K_{a_i}, 1 \leq i \leq (d-1)/2$ are adjacent to v and all the vertices $w \in K_{a_i}, (d-1)/2 \leq i \leq (d-1)$ are adjacent to u. Also u and v are adjacent to each other in $ED(G)$. Thus, $ED(G)$ is a K_2^* with one end having m independent in-neighbors and the other end having n independent in-neighbors. □

Theorem 4.3. *If G is a diameter maximal graph with odd diameter, then G has period two and tail two.*

Proof. If G is a diameter maximal graph with odd diameter. Then by Lemma 4.1, $|P(G)| = 2$. Let $P(G) = \{u, v\}$. Now by Theorem 4.2, $ED(G)$ is a K_2^* with one end u having m independent in-neighbors and the other end v having n independent in-neighbors. Let

$$N^-_{ED(G)}(u) = \{u_1, u_2, u_3, \cdots, u_m\}$$

and

$$N^-_{ED(G)}(v) = \{v_1, v_2, v_3, \cdots, v_n\}.$$

Since $deg^-_{ED(G)}(w) = 0$, for all $w \in N^-_{ED(G)}(u) \cup N^-_{ED(G)}(v)$, we have

$$EP(ED(G)) = N^-_{ED(G)}(u) \cup N^-_{ED(G)}(v) = \{u_1, u_2, u_3, \cdots, u_m, v_1, v_2, v_3, \cdots, v_n\}.$$

It can be seen that in $ED(G)$, $E(u) = E(v) = EP(ED(G))$ and $E(w) = N^-_{ED(G)}(v)$ or $E(w) = N^-_{ED(G)}(u)$ according as $w \in N^-_{ED(G)}(u)$ or $w \in N^-_{ED(G)}(v)$. This implies that $ED^2(G) = \overline{K_2^*} \oplus K^*_{m+n}$. Then by theorem 3.1, $ED^4(G) = ED^2(G)$. Therefore G has period two and tail two. □

Theorem 4.4. *If G is a diameter maximal graph with odd diameter then*
(i) $\gamma(ED(G)) = m + n$ *and* $(ii)\gamma(ED^2(G)) = 2$.

Proof. Let G be a diameter maximal graph with odd diameter. In order to prove statement (i), we note that by theorem 4.2, the eccentric digraph $ED(G)$ is a K_2^* with one end having m independent in-neighbors and other end having n independent in-neighbors. This implies that $\gamma(ED(G)) = m + n$.
Now to prove statement (ii), we have, as in the proof of Theorem 4.3, $ED^2(G) = \overline{K_2^*} \oplus K^*_{m+n}$. Hence $\gamma(ED^2(G)) = 2$. □

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