ON GENERALIZED ABSOLUTE MATRIX SUMMABILITY

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Abstract. In this paper a general theorem on \( |A, p_n; \delta|_k \) summability factors, which generalizes a theorem of Bor [3] on \( |\vec{N}, p_n|_k \) summability factors, has been proved under weaker conditions by using an almost increasing sequence.

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1. Introduction

A positive sequence \((b_n)\) is said to be almost increasing if there exists a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that \(Ac_n \leq b_n \leq Bc_n\) (see [1]). Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example \(b_n = ne^{(-1)^n}\). Let \(\sum a_n\) be a given infinite series with the partial sums \((s_n)\). Let \((p_n)\) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).
\]

The sequence-to-sequence transformation

\[
\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
\]

defines the sequence \((\sigma_n)\) of the Riesz mean or simply the \((\vec{N}, p_n)\) mean of the sequence \((s_n)\), generated by the sequence of coefficients \((p_n)\) (see [4]). The series \(\sum a_n\) is said to be summable \( |\vec{N}, p_n|_k, \ k \geq 1, \) if (see [2])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty,
\]

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where

\[
\Delta \sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.
\]

Let \( A = (a_{nv}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( As = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \ldots
\]

The series \( \sum a_n \) is said to be summable \( |A, p_n| k \), \( k \geq 1 \), if (see [7])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{k-1} |\bar{A}A_n(s)|^k < \infty,
\]

and it is said to be summable \( |A, p_n; \delta| k \), \( k \geq 1 \) and \( \delta \geq 0 \), if (see [5])

\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k+k-1} |\bar{A}A_n(s)|^k < \infty,
\]

where

\[
\bar{A}A_n(s) = A_n(s) - A_{n-1}(s).
\]

If we take \( a_{nv} = \frac{p_v}{P_n} \) and \( \delta = 0 \), then \( |A, p_n; \delta|_k \) summability reduces to \( |\bar{N}, p_n|_k \) summability. In the special case \( \delta = 0 \) and \( p_n = 1 \) for all \( n \), \( |A, p_n; \delta|_k \) summability is the same as \( |A|_k \) summability. Also if we take \( a_{nv} = \frac{p_v}{P_n} \), then \( |A, p_n; \delta|_k \) summability is the same as \( |\bar{N}, p_n; \delta|_k \) summability.

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \bar{A} = (\bar{a}_{nv}) \) and \( \hat{A} = (\hat{a}_{nv}) \) as follows:

\[
\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots
\]

and

\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots
\]

It may be noted that \( \bar{A} \) and \( \hat{A} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v
\]
and

\[ \Delta A_n(s) = \sum_{v=0}^{n} \tilde{a}_{nv}a_v. \]

**2. Known result**

In [3], we have proved the following theorem dealing with \( |\tilde{N}, p_n|_k \) summability factors of an infinite series.

**Theorem A.** Let \( (p_n) \) be a sequence of positive numbers such that

\[ P_n = O(np_n) \quad as \quad n \to \infty. \]

Let \( (X_n) \) is a positive non-decreasing sequence and suppose that there exist sequences \( (\beta_n) \) and \( (\lambda_n) \) such that

\[ |\Delta \lambda_n| \leq \beta_n, \]

\[ \beta_n \to 0 \quad as \quad n \to \infty, \]

\[ \sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty \]

and

\[ |\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \]

If

\[ \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \]

where

\[ t_n = \frac{1}{n+1} \sum_{v=1}^{n} va_v, \]

then the series \( \sum a_n \lambda_n \) is summable \( |\tilde{N}, p_n|_k, k \geq 1. \)

**3. The main result**

The aim of this paper is to generalize Theorem A to \( |A, p_n; \delta|_k \) summability under weaker conditions. Now, we shall prove the following theorem.

**Theorem.** Let \( A = (a_{nv}) \) be a positive normal matrix such that

\[ \tilde{a}_{n0} = 1, \quad n = 0, 1, ..., \]

\[ a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v + 1, \]
(20) \[ a_{nn} = O \left( \frac{P_n}{P_n^2} \right), \]

(21) \[ |\hat{a}_{n,v+1}| = O \left( v |\Delta_v \hat{a}_{nv}| \right). \]

If \((X_n)\) is an almost increasing sequence. If the sequences \((\beta_n)\) and \((\lambda_n)\) are satisfied the conditions (12)-(16) of Theorem A and if the conditions

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k-1} |t_n|^k = O(X_m) \quad \text{as} \quad m \to \infty,
\]

\[
\sum_{n=v+1}^{\infty} \left( \frac{P_n}{P_n} \right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O \left\{ \left( \frac{P_v}{P_v} \right)^{\delta k-1} \right\}
\]

are satisfied, then the series \(\sum a_n \lambda_n\) is summable \(|A, p_n; \delta|, k \geq 1\) and \(0 \leq \delta < 1/k\).

It may be noted that, if we take \((X_n)\) as a positive non-decreasing sequence, \(a_{nv} = \frac{P_v}{P_n}\) and \(\delta = 0\) in this theorem, then we get Theorem A.

We need the following lemma for the proof of our theorem.

**Lemma ([6])**. Under the conditions on \((X_n), (\beta_n)\) and \((\lambda_n)\) as taken in the statement of the theorem, then we have the following:

(24) \[ n \beta_n X_n = O(1) \quad \text{as} \quad n \to \infty, \]

(25) \[ \sum_{n=1}^{\infty} \beta_n X_n < \infty. \]

4. **Proof of the theorem**

Let \((T_n)\) denotes A-transform of the series \(\sum a_n \lambda_n\). Then, by (10) and (11), we have

\[ \bar{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv} \lambda_v}{v} v a_v. \]

By Abel’s transformation, we have

\[
\bar{\Delta} T_n = \sum_{v=1}^{n-1} \Delta_v \left( \frac{\hat{a}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r a_r \\
= \frac{n+1}{n} a_{nn} \lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v (\hat{a}_{nv}) \lambda_v t_v \\
+ \sum_{v=1}^{n-1} \frac{v}{v} \hat{a}_{n,v+1} \Delta v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \\
= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say.} \]
To complete the proof of the theorem, by Minkowski’s inequality, it is enough to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + 1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4.$$ 

Firstly, we have that

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k + 1} |T_{n,1}|^k = O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n|^k |t_n|^k$$

$$= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |\lambda_n||\lambda_n|^{k-1} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m} \Delta |\lambda_n| \sum_{v=1}^{n} \left(\frac{P_v}{p_v}\right)^{\delta k - 1} |t_v|^k + O(1) |\lambda_m| \sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} |t_n|^k$$

$$= O(1) \sum_{n=1}^{m} |\Delta \lambda_n| X_n + O(1) |\lambda_m| X_m$$

$$= O(1) \beta_n X_n + O(1) |\lambda_m| X_m$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$

by virtue of the hypotheses of the theorem and lemma. Now, when $k > 1$, applying Hölder’s inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + 1} |T_{n,2}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k + 1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |t_v|^k \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |t_v|^k \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})||\lambda_v|^k |t_v|^k$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})|$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| \left(\frac{P_v}{p_v}\right)^{\delta k - 1} |t_v|^k$$

$$= O(1) \quad \text{as} \quad m \to \infty,$$
by virtue of the hypotheses of the theorem and lemma.

Now, since \( v\beta_v = O \left( \frac{1}{X_v} \right) = O(1) \), by (24), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\Delta v| |t_v| \right)^k
\]

\[
\begin{align*}
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\beta_v| t_v \right)^k \times \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \beta_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \sum_{v=1}^{n-1} v |\Delta v| \hat{a}_{nv} \beta_v |t_v|^k \\
&= O(1) \sum_{v=1}^{m} v \beta_v |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta v| \hat{a}_{nv} \\
&= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k-1} v \beta_v |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta (v\beta_v)| \sum_{i=1}^{v} \left( \frac{P_i}{p_i} \right)^{\delta k-1} |t_i|^k + O(1)m\beta_m \sum_{v=1}^{m} \left( \frac{P_v}{p_v} \right)^{\delta k-1} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta \beta_v| X_v + O(1) \sum_{v=1}^{m} \beta_{v+1} X_v + O(1)m\beta_m X_m \\
&= O(1) \text{ as } m \to \infty,
\]

by virtue of the hypotheses of the theorem and lemma.

Finally, as in \( T_{n,1} \), we have that

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} |T_{n,4}|^k = O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left( \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| |\lambda_{v+1}| |t_v| \right)^k
\]

\[
\begin{align*}
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left( \sum_{v=1}^{n-1} |\Delta v| \hat{a}_{nv} |\lambda_{v+1}| |t_v| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k + k-1} \left( \sum_{v=1}^{n-1} |\Delta v| \hat{a}_{nv} |\lambda_{v+1}| |t_v|^k \right)^k \times \left( \sum_{v=1}^{n-1} |\Delta v| \hat{a}_{nv} \right)^{k-1} \\
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| |t_v|^k \sum_{n=v+1}^{m+1} \left( \frac{P_n}{p_n} \right)^{\delta k} |\Delta v| \hat{a}_{nv} \\
\end{align*}
\]
\[ \begin{align*}
&= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| \left( \frac{p_v}{p_v} \right)^{\delta k-1} |t_v|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{align*} \]

by virtue of the hypotheses of the theorem and lemma. Therefore, we get that

\[ \sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right)^{\delta k+k-1} |T_{n,r}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \]

This completes the proof of the theorem.

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References


