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ON GENERALIZED ABSOLUTE MATRIX SUMMABILITY

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ABSTRACT. In this paper a general theorem on $|A, p_n; \delta|_k$ summability factors, which generalizes a theorem of Bor [3] on $\left|\bar{N}, p_n\right|_k$ summability factors, has been proved under weaker conditions by using an almost increasing sequence.

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1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

(1) $$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

(2) $$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (σ_n) of the Riesz mean or simply the $\left(\bar{N}, p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

(3) $$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta\sigma_{n-1}|^k < \infty,$$

where

$$(4) \qquad \Delta\sigma_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$(5) \qquad A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [7])

$$(6) \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\bar{\Delta} A_n(s)|^k < \infty,$$

and it is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [5])

$$(7) \qquad \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty,$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. In the special case $\delta = 0$ and $p_n = 1$ for all n, $|A, p_n; \delta|_k$ summability is the same as $|A|_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \delta|_k$ summability.

Before stating the main theorem we must first introduce some further notations.
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$(8) \qquad \bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ...$$

and

$$(9) \qquad \hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ...$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$(10) \qquad A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{11}$$

2. Known result

In [3], we have proved the following theorem dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of an infinite series.

Theorem A. Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad as \quad n \to \infty. \tag{12}$$

Let (X_n) is a positive non-decreasing sequence and suppose that there exist sequences (β_n) and (λ_n) such that

$$|\Delta \lambda_n| \leq \beta_n, \tag{13}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{14}$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty \tag{15}$$

and

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{16}$$

If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{17}$$

where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v,$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. The main result

The aim of this paper is to generalize Theorem A to $|A, p_n; \delta|_k$ summability under weaker conditions. Now, we shall prove the following theorem.

Theorem. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{18}$$

$$a_{n-1,v} \geq a_{nv}, \quad for \quad n \geq v + 1, \tag{19}$$

(20) $$a_{nn} = O\left(\frac{p_n}{P_n}\right),$$

(21) $$|\hat{a}_{n,v+1}| = O\left(v\left|\Delta_v \hat{a}_{nv}\right|\right).$$

If (X_n) is an almost increasing sequence. If the sequences (β_n) and (λ_n) are satisfied the conditions (12)-(16) of Theorem A and if the conditions

(22) $$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty,$$

(23) $$\sum_{n=v+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.
It may be noted that, if we take (X_n) as a positive non-decreasing sequence, $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$ in this theorem, then we get Theorem A.
We need the following lemma for the proof of our theorem.
Lemma ([6]). Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, then we have the following :

(24) $$n\beta_n X_n = O(1) \quad as \quad n \to \infty,$$

(25) $$\sum_{n=1}^{\infty} \beta_n X_n < \infty.$$

4. Proof of the theorem

Let (T_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (10) and (11), we have

$$\bar{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{a}_{nv}\lambda_v}{v} v a_v.$$

By Abel's transformation, we have

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \frac{n+1}{n} a_{nn}\lambda_n t_n + \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{a}_{nv}\right) \lambda_v t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} \lambda_{v+1} t_v \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad say.
\end{aligned}$$

To complete the proof of the theorem, by Minkowski's inequality, it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k<\infty, \quad for \quad r=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n||\lambda_n|^{k-1}|t_n|^k\\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}(\frac{P_v}{p_v})^{\delta k-1}|t_v|^k+O(1)|\lambda_m|\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k-1}|t_n|^k\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)|\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma. Now, when $k>1$, applying Hölder's inequality with indices k and $k^{'}$, where $\frac{1}{k}+\frac{1}{k^{'}}=1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of the theorem and lemma.
Now, since $v\beta_v = O\left(\frac{1}{X_v}\right) = O(1)$, by (24), we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,3}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}|\beta_v|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}|\beta_v\right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} v|\Delta_v\hat{a}_{nv}|\beta_v|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} v|\Delta_v\hat{a}_{nv}|\beta_v\right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} \sum_{v=1}^{n-1} v|\Delta_v\hat{a}_{nv}|\beta_v|t_v|^k$$

$$= O(1) \sum_{v=1}^{m} v\beta_v|t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v\hat{a}_{nv}|$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} v\beta_v|t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta\left(v\beta_v\right)| \sum_{i=1}^{v} \left(\frac{P_i}{p_i}\right)^{\delta k-1} |t_i|^k + O(1)m\beta_m \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1}X_v + O(1)m\beta_m X_m$$

$$= O(1) \quad as \quad m \to \infty,$$

by virtue of the hypotheses of the theorem and lemma.
Finally, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |T_{n,4}|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} \frac{1}{v}|\hat{a}_{n,v+1}||\lambda_{v+1}||t_v|\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v\hat{a}_{nv}||\lambda_{v+1}||t_v|\right)^k$$

$$= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left(\sum_{v=1}^{n-1} |\Delta_v\hat{a}_{nv}||\lambda_{v+1}|^k|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} |\Delta_v\hat{a}_{nv}|\right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_{v+1}||t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v\hat{a}_{nv}|$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k \\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and lemma. Therefore, we get that

$$
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|T_{n,r}|^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.
$$

This completes the proof of the theorem.

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