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g-COMPACTNESS LIKE PROPERTIES IN GENERALIZED TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we introduce the notions of g-compact spaces, g-semi compact spaces, g-precompact spaces, g-α compact spaces and g-β compact spaces and discuss the relation among these spaces. Also we characterize these five spaces and investigate their properties.

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Key words and phrases. g-compact spaces, g-semi compact spaces, g-precompact spaces, g-α-compact spaces, g-β-compact spaces and g-regular open set.

1. INTRODUCTION

Császár introduced the notions of generalized neighborhood systems and generalized topological spaces in [1]. Also he introduced the notion of continuous functions on generalized neighborhood systems and generalized topological spaces. Continuiy of functions on generalized topological spaces were also studied in [5]. Császár introduced and studied in [2], the notions of g-pre open sets, g-semi open sets, g-α-open sets, g-β-open sets in generalized topological spaces. Seperation axioms on generalized topological spaces were studied in [4] and [6]. In this paper, we introduce the definitions of g-compact spaces, g-precompact spaces, g-semi compact spaces, g-α-compact spaces, g-β-compact spaces and we investigate their characterizations and relationships among these g-compactness like properties.

2. PRELIMINARIES

Let X be a non empty set and g be a collection of subsets of X. Then g is called a generalized topology on X if and only if $\emptyset \in g$ and $U_i \in g$ for $i \in I$ implies $\bigcup_{i\in I} U_i \in g$. The pair (X, g) is called a generalized topological space. The members of g are called the g-open sets and their complements are called the g-closed sets. A generalized topological space (X, g) is said to be a strongly generalized topological space, if $X \in g$. The generalized closure of a subset A of X is defined as the intersection of all g-closed sets containing A and is denoted by $cl_g(A)$. The generalized interior of a subset A is the union of all g-open sets contained in A which is denoted by $int_g(A)$.

Theorem 2.1. [1] *Let (X, g) be a generalized topological space and $A \subseteq X$. Then*

(1) $cl_g(A) = X - int_g(X - A)$
(2) $int_g(A) = X - cl_g(X - A)$

Definition 2.2. [2] *Let (X, g) be a generalized topological space and $A \subseteq X$. Then A is said to be*

(1) *g-semi open if $A \subseteq cl_g(int_g(A))$*
(2) *g-pre open if $A \subseteq int_g(cl_g(A))$*
(3) *g-α-open if $A \subseteq int_g(cl_g(int_g(A)))$*
(4) *g-β-open if $A \subseteq cl_g(int_g(cl_g(A)))$.*

The complements of g-semi open subsets of (X, g) (respectively g-pre open, g-α-open and g-β-open) is called the g-semi closed subsets (respectively g-pre closed, g-α-closed and g-β-closed) in X.

Let $exp(X)$ denote the power set of X and let $\mathscr{B} \subseteq exp(X)$ satisfy $\emptyset \in \mathscr{B}$. Then all unions of some elements of $\mathscr{B}$ constitute a generalized topology g on X and $\mathscr{B}$ is said to be a base[3] for g. Let $\mathscr{M}_g$ denote the union of all elements of g.

3. Examples

Definition 3.1. *A generalized topological space X is said to be a*

(i) *g-compact if every g-open cover of X has a finite subcover.*
(ii) *g-semi compact if every g-semi open cover of X has a finite subcover.*
(iii) *g-precompact if every g-preopen cover of X has a finite subcover.*
(iv) *g-α-compact if every g-α-open cover of X has a finite subcover.*
(v) *g-β-compact if every g-βopen cover of X has a finite subcover.*

Example 3.2. *Every finite generalized topological space is a g-compact, g-precompact, g-semi compact, g-α-compact and g-β-compact space.*

From the defintion: 3.1, we easily see that every g-precompact, g-semi compact, g-α-compact, g-β-compact spaces are a g-compact space. This is shown in the following diagram.

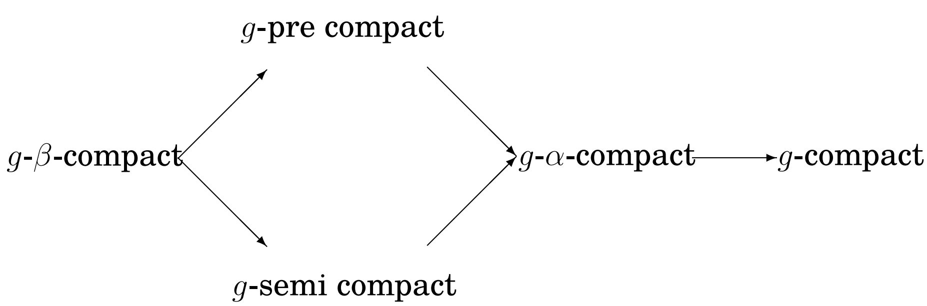

Fig. 3

The reverse implications of the above diagram Fig. 3 may not be true in general and we can verify this from the following examples.

Example 3.3. *Example for a g-semi compact space which is not a g-β-compact space. Consider $\mathbb{N}$, the set of all positive integers. Define a generalized topology on $\mathbb{N}$ as $g = \{\emptyset, A, B, \mathbb{N}\}$, where A and B are the set of all even and odd positive integers respectively.*

For any non empty subset M of $\mathbb{N}$ not containing A or B, $int_g(M) = \emptyset$ implies that $cl_g(int_g(M)) = \emptyset$. Also for any non empty subset $M(\neq \mathbb{N})$ of $\mathbb{N}$ and $M \supsetneq A$ or $M \supsetneq B$, we have $M \nsubseteq cl_g(int_g(M))$. Therefore the only g-semi open sets are the members of g. Hence $(\mathbb{N}, g)$ is a g-semi compact space.

For any non empty subset M of $\mathbb{N}$,

$$cl_g(M) = \begin{cases} \mathbb{N} & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B \neq \emptyset \\ A & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B = \emptyset \\ B & \text{if } M \cap A = \emptyset \text{ and } M \cap B \neq \emptyset \end{cases}$$

and $$cl_g(int_g(cl_g(M))) = \begin{cases} \mathbb{N} & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B \neq \emptyset \\ A & \text{if } M \cap A \neq \emptyset \text{ and } M \cap B = \emptyset. \\ B & \text{if } M \cap A = \emptyset \text{ and } M \cap B \neq \emptyset \end{cases}$$

This implies that each nonempty subset of $\mathbb{N}$ is a g-β-open subset of $\mathbb{N}$. Clearly $(\mathbb{N}, g)$ is not a g-β-compact subset of $\mathbb{N}$, because a g-β-open cover $\{\{x\} : x \in \mathbb{N}\}$ of $\mathbb{N}$ has no finite subcover.

Example 3.4. *Example for a g-α-compact space which is not a g - precompact space Consider $\mathbb{N}$, with a generalized topology $g = \{\emptyset, \mathbb{N}\}$.*

For any nonempty subset M of $\mathbb{N}$, $cl_g(M) = \mathbb{N}$ and $int_g(cl_g(M)) = \mathbb{N}$. This implies that every non empty subset of $\mathbb{N}$ is a g-preopen subset of $\mathbb{N}$. Since a g-preopen cover $\{\{x\} : x \in \mathbb{N}\}$ of $\mathbb{N}$ does not have a finite subcover, $(\mathbb{N}, g)$ is not a g-precompact space.

For any nonempty subset $M(\neq \mathbb{N})$ of $\mathbb{N}$, $int_g(M) = \emptyset$ and hence $int_g(cl_g(int_g(M))) = \emptyset$. Then $M \nsubseteq int_g(cl_g(int_g(M)))$. That is, every nonempty subset of $\mathbb{N}$ (except $\mathbb{N}$ itself) is *not a g-α-open set and the only non empty g-α-open subset of $\mathbb{N}$ is $\mathbb{N}$ itself. This implies that $(\mathbb{N}, g)$ is a g-α-compact spacce.*

Example 3.5. *Example for a g-compact space which is not a g-α-compact space. Define a generalized topology $g = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \mathbb{N} - \{2, 3\}, \mathbb{N} - \{3\}, \mathbb{N} - \{2\}, \mathbb{N}\}$.*

Since the number of g-open sets in $(\mathbb{N}, g)$ is finite, any g-open cover of $\mathbb{N}$ has itself as finite subcover. Therefore, $(\mathbb{N}, g)$ is a g-compact space.

If A is a proper finite subset of $\mathbb{N}$ containing either $\{1, 2\}$ or $\{1, 3\}$ or $\{1, 2, 3\}$, then $int_g(A)$ is either $\{1, 2\}$ or $\{1, 3\}$ or $\{1, 2, 3\}$. Then $cl_g(int_g(A)) = \mathbb{N}$ and $A \subseteq int_g(cl_g(int_g(A))) = \mathbb{N}$. Thus, if A is a proper finite subset of $\mathbb{N}$ containing either $\{1, 2\}$ or $\{1, 3\}$ or $\{1, 2, 3\}$, then A is a g-α-open subset of $(\mathbb{N}, g)$. Since a g-α-open cover $\mathscr{U} = \{\{a, b\} \cup \{1, 2\} : a, b \in \mathbb{N} - \{2, 3\}\} \cup \{\{1, 2, 3\}\}$ does not have a finite subcover, $(\mathbb{N}, g)$ is not a g-α-compact space.

Example 3.6. *Example for a g-precompact space which is not a g-β-compact space. Let us consider $\mathbb{N}$ with generalized topology*
$g = \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

If M is a nonempty subset of $\mathbb{N} - \{1, 2, 3\}$, then $cl_g(M) = N - \{1, 2, 3\}$ and $int_g(cl_g(M)) = \emptyset$. If $M = M_1 \cup \{2\}$ or $M = M_1 \cup \{3\}$, where $M_1 \subseteq \mathbb{N} - \{1, 2, 3\}$, then $cl_g(M)$ is either $\mathbb{N} - \{1, 2\}$ or $\mathbb{N} - \{1, 3\}$ and $int_g(cl_g(M)) = \emptyset$. If M is not a singleton subset of $\mathbb{N}$ containing $\{1\}$ and $M \notin g$, then $cl_g(M) = \mathbb{N}$ and $M \nsubseteq int_g(cl_g(M)) = \{1, 2, 3\}$. If $M = \{1\}$ or $\{2, 3\}$, then $cl_g(M) = \mathbb{N}$ and $int_g(cl_g(M)) = \{1, 2, 3\} \supseteq \{1\}$ and $int_g(cl_g(M)) = \{1, 2, 3\} \supseteq \{2, 3\}$. Therefore the subsets $\{1\}, \{2, 3\}$ and g-open subsets of $\mathbb{N}$ are the only g-preopen subsets of $\mathbb{N}$. Since the number of g-preopen subsets of $\mathbb{N}$ is finite, any g-preopen cover of $\mathbb{N}$ has itself as finite subcover. Therefore, $(\mathbb{N}, g)$ is a g-precompact space.

If $M \subseteq N$ and $M \supsetneq \{1\}$, then $cl_g(int_g(cl_g(M))) = cl_g(int_g(\mathbb{N})) = cl_g(\{1, 2, 3\}) = \mathbb{N}$. Here $M \subseteq cl_g(int_g(cl_g(M)))$. Thus any subset of $\mathbb{N}$ which contains $\{1\}$ is a g-β-open subset of $\mathbb{N}$. Then $(\mathbb{N}, g)$ is not a g-β-compact space, because the g-β-open cover $\{\{1, a, b\} : a, b \in \mathbb{N} - \{1, 2\}\} \cup \{1, 2\}$ has no finite subcover. Therefore $(\mathbb{N}, g)$ is not a g-β-compact space.

Example 3.7. *Example for a g-α-compact space which is not a g-semi compact space. In the above example 3.6, If $M \subseteq N$ and $M \supseteq \{1, 2\}$ or $M \supseteq \{1, 3\}$, then $int_g(M)$ is either $\{1, 2\}$ or $\{1, 3\}$ or $\{1, 2, 3\}$ and $cl_g(int_g(M)) = \mathbb{N}$. If $M = \{2\}$ or $M = \{3\}$ or $M = \{1\}$ or $M = \{2, 3\}$ then $cl_g(int_g(M)) = cl_g(\emptyset) = \mathbb{N} - \{1, 2, 3\}$. Thus any subset of $\mathbb{N}$ other than $\{2\}$, $\{3\}$, $\{1\}$, $\{2, 3\}$ is a g-semi open subset of $\mathbb{N}$. Thus a g-semi open cover $\{\{1, 2, 3, a, b\} : a, b \in \mathbb{N} - \{2, 3\}$ has no finite subcover. Therefore $(\mathbb{N}, g)$ is not a g-semi compact space.*

Since $(\mathbb{N}, g)$ is a g-precompact space, $(\mathbb{N}, g)$ is a g-α-compact space.

4. Properties

Theorem 4.1. *A generalized topological space X is g-semi compact (respectively, g-precompact space, g-α-compact, g-β-compact, g-compact) if and only if any collection of g-semi closed (respectively, g-preclosed, g-α-closed, g-β-closed, g-closed) sets with finite intersection property has non empty intersection.*

Proof. Assume that X is a g-semi compact space. Let $\mathscr{F}$ be any collection of g-semi closed sets with finite intersection property. We have to prove that $\underset{F\in\mathscr{F}}{\cap} F \neq \emptyset$. If $\underset{F\in\mathscr{F}}{\cap} F = \emptyset$, then $X - \underset{F\in\mathscr{F}}{\cap} F = X$. This implies $\underset{F\in\mathscr{F}}{\cup}(X-F) = X$. That is, $\{X-F : F \in \mathscr{F}\}$ is a collection of g-semi open sets which covers X. By the assumption, there exists $\{X - F_i : F_i \in \mathscr{F}, i = 1, 2, 3, \cdots, n\}$ such that $\cup\{X - F_i : F_i \in \mathscr{F}, i = 1, 2, 3, \cdots, n\} = X$. That is, $X - \cap\{F_i : F_i \in \mathscr{F}, i = 1, 2, 3, \cdots n\} = X$ and $\cap\{F_i : F_i \in \mathscr{F}, i = 1, 2, 3, \cdots, n\} = \emptyset$, which is a contradiction to our assumption.

Conversely, assume that any collection of g-semi closed sets with finite intersection property has non empty intersection. Suppose for some g-semi open cover $\mathscr{U}$, there is no finite subcover. That is, for any finite subcollection of $\mathscr{U}$, we have $\overset{n}{\underset{i=1}{\cup}} U_i \neq X$. Then $X - \overset{n}{\underset{i=1}{\cup}} U_i \neq \emptyset$ and $\overset{n}{\underset{i=1}{\cap}} (X - U_i) \neq \emptyset$. Now, $\{X - U_i : U_i \in \mathscr{U}\}$ is a collection of g-semi closed sets having finite intersection property. By the assumption, $\underset{U_i\in\mathscr{U}}{\cap} (X - U_i) \neq \emptyset$. Then $X - \underset{U_i\in\mathscr{U}}{\cup} U_i \neq \emptyset$ and hence $\underset{U_i\in\mathscr{U}}{\cup} U_i \neq X$, which is a contradiction to our assumption. This completes the proof of this theorem. □

Theorem 4.2. *Every g-semi closed (respectively g-preclosed, g-α-closed, g-β-closed, g-closed) subset of a g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact) space is g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact).*

Proof. Let A be any g-semi closed subset of a g-semi compact space X and $\mathscr{U}$ be any g-semi open cover for A. Now $\mathscr{U} \cup \{X - A\}$ is a g-semi open cover for X. Since X is g-semi compact, this g-semi open cover $\mathscr{U} \cup \{X - A\}$ has a finite subcover $\mathscr{V}$. Then $\mathscr{V} - \{X - A\}$ is a finite subcover for A and hence A is g-semi compact. □

Corollary 4.3. *let A be a g-α-closed set in X, then*

(1) *If X is g-semi compact, then A is g-semi compact.*
(2) *If X is g-precompact, then A is g-precompact.*
(3) *If X is g-β-compact, then A is g-β-compact.*

Proof. Every g-α-closed set in X is a g-semi closed set, g-preclosed set and g-β-closed set and the proof follows from the previous theorem: 4.2 □

Corollary 4.4. (1) *If A is a g-semi closed subset of a g-β-compact space is g-β compact.*
(2) *If A is a g-preclosed subset of a g-β-compact space is g-β compact.*

Theorem 4.5. *Union of finite number of g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact) subsets of a generalized topological space is g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact).*

Theorem 4.6. *Let A be a g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact) subset of a generalized topological space X.If B is a g-semi closed (respectively, g-preclosed, g-α-closed, g-β-closed, g-closed) subset of X, then $A \cap B$ is a g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact, g-compact) subset of X.*

Proof. We prove this theorem for A is g-semi compact and B is a g-semi closed. The other cases are similar.

Let $\mathscr{U}$ be any g-semi open cover of $A \cap B$. Then $\mathscr{U} \cup \{X - B\}$ is a g-semi open cover of A in X. Since A is a g-semi compact subset of X,$\mathscr{U} \cup \{X - B\}$ has a finite subcover $\mathscr{U}_1$. Then $\mathscr{U}_1 - \{X - B\}$ is a finite subcover for $A \cap B$. □

5. g-Compactness Related Properties in Generalized Topological Spaces

Definition 5.1. [6] *Let (X, g) be a generalized topological space. Then X is called a relative GT_1-space (simply, GT_1-space) if for $x_1, x_2 \in \mathscr{M}_g$ with $x_1 \neq x_2$, there exist $U, V \in g$ such that $x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$.*

Definition 5.2. [6] *Let (X, g) be a generalized topological space. Then X is called a relative GT_2-space (simply, GT_2-space) if for $x_1, x_2 \in \mathscr{M}_g$ with $x_1 \neq x_2$, there exist $U, V \in g$ such that $x_1 \in U, x_2 \in V$ and $U \cap V = \emptyset$.*

It is not necessary that if A is a g-compact subset of a generalized topological space X and if X is a GT_2-space, then A is a g-closed set in X. The following example shows this.

Example 5.3. *Consider a generalized topological space $(\mathbb{N}, g)$, where $g = \{\emptyset, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$. Consider the set $A = \{1,2,4\}$. Then A is a g-compact subset of $\mathbb{N}$. But A is not a closed set, because its complement $\mathbb{N} - \{1,2,4\}$ is not g-open.*

Definition 5.4. *Let X be a generalized topological space and $A \subseteq X$. Then an element $x \in X$ is said to be a g-semicluster point (g-precluster point, g-α-cluster point, g-β-cluster point) if for each g-semi open (g-preopen, g-α-open, g-β-open) neighborhood U of x, $U \cap (A - \{x\}) \neq \emptyset$.*

Theorem 5.5. *If $A \subseteq X$ has no g-cluster (respectively g-semi cluster points, g-α-cluster points) points, then X is a strongly generalized topological space.*

Proof. If A has no g-cluster points, then for each $x \in X$ has a g-open neighborhood U such that $U \cap (A - \{x\}) = \emptyset$. This implies that $\mathscr{M}_g = X$ and hence X is strongly generalized topological space. $\square$

Theorem 5.6. *If A is a subset of X which contains more than one point snd has no g-pre cluster (g-α-cluster points) points, then X is a strongly generalized topological spacce.*

Proof. If V is a g-preopen (g-α-open) set of x, then $x \in V \subseteq int_g(cl_g(V))$ ($x \in V \subseteq int_g(cl(int_g(V)))$). In this case $int_g(cl_g(V))$ ($int_g(cl_g(int_g(V)))$) is a g-open set containing x. This proves that $\mathscr{M}_g = X$ and hence $X \in \tau_g$ $\square$

But the above theorem:5.6 is not true if $A \subseteq X$ has no g-semi cluster points or g-β-cluster points. The following example shows this.

Example 5.7. *Let $X = \mathbb{N}$, the set of all positive integers. Define a generalized topology τ_g on X by $\tau_g = \{\emptyset, \{5\}\}$. Note that $\mathscr{M}_g = \{5\} \neq X$. But for each $x \in X$, the set $\{x, 5\}$ is a g-semi open neighborhood and g-β-open neighborhood of x. Since $\{5\}$ is a g-open set, $\{5\}$ is a g-semi open neighborhood and g-β-open neighborhood of 5. Thus $A = 2\mathbb{N}$(Set of all positive even integers) has no g-semi cluster points or g-β-cluster points.*

Theorem 5.8. *Let X be a g-semi compact (respectively, g-precompact, g-α-compact, g-β-compact) space and $A \subseteq X$. If A is an infinite set, then A has atleast one g-semi cluster point (respectively, g-precluster point, g-α-cluster point, g-β-cluster point) in X.*

Proof. If A has no g-semi cluster points in X, then for each $x \in X$, there is a g-semi open neighborhood U_x of x such that $U \cap (A - \{x\}) = \emptyset$. Now the collection $\{U_x : x \in X\}$ is a g-semi open cover for X. Since X is g-semi compact, there exist $x_1, x_2, \cdots, x_n$ such that $\bigcup\limits_{i=1}^{n} U_{x_i} = X$. This implies $(\bigcup\limits_{i=1}^{n} U_{x_i}) \cap A = X \cap A$. That is $(U_{x_1} \cap A) \cap (U_{x_2} \cap A) \cap \cdots \cap (U_{x_n} \cap A) = A$. But each one of $(U_{x_1} \cap A), (U_{x_2} \cap A), \cdots, (U_{x_n} \cap A)$ is either a finite set or an empty set. This implies A is either a finite set or an empty set, which is a contradiction to our assumption that A is an infinite set. Thus A has atleast one g-semi cluster points in X. $\square$

Definition 5.9. [6] *Let (X, g) be a generalized topological space. Then X is said to be relative G-regular (simply, G-regular) if for $x \in \mathscr{M}_g$ and g-closed set F with $x \notin F$, there exist $U, V \in g$ such that $x \in U, F \cap \mathscr{M}_g \subseteq V$ and $U \cap V = \emptyset$. If X is GT_1 and G-regular, then it is said to be GT_3.*

Definition 5.10. *A generalized topological space X is said to be G-semi regular(G-pre regular, G-α-Regular, G-β-Regular) if for $x \in \mathscr{M}_g$ and g-semi closed(pre closed, G-α-closed, G-β-closed) set F with $x \notin F$, there exist g-semi open(pre open, G-α-open, G-β-open) sets U and V such that $x \in U, F \cap \mathscr{M}_g \subseteq V$ and $U \cap V = \emptyset$. If X is GT_1 and G-semi regular(G-pre regular, G-α-Regular, G-β-Regular), then it is said to be G-semi T_3 (G-pre T_3, G-α-T_3, G-β-T_3) space.*

Here after we use $Sint_g(A)$ and $Scl_g(A)$ to notate semi interior and semi closure of set A. Similarly, we use $Pint_g(A)$, $Pcl_g(A)$, $\alpha int_g(A)$, $\alpha cl_g(A)$, $\beta int_g(A)$ and $\beta cl_g(A)$ to notate pre interior, pre closure, α-interior, α-closure, β-interior and β- closure of set A respectively.

Theorem 5.11. *Let X be a g-semi compact, G-semi T_3-space and $\{A_n\}$ be a countable collection of g-semi closed subsets of X such that $Sint_g(A_n) = \emptyset$, for each n. If $cl_g(A_i \cap A_j) = cl_g(A_i) \cap cl_g(A_j)$ for $i \neq j$ and X is a strongly generalized topological space, then $Sint_g(\bigcup_{i=1}^{\infty} A_n) = \emptyset$.*

Proof. Let $\{A_n\}$ be a collection of g-semi closed subsets of X such that $Sint_g(A_n) = \emptyset$, for each n. Let U_0 be any g-semi open set. Since $Sint_g(A_1) = \emptyset$, $U_0 \nsubseteq A_1$. Choose a point $z \in U_0$ and $z \notin A_1$. Since X is a strongly generalized topological space, $\mathscr{M}_g = X$. Since X is g-semi T_3-space, there are g-semi open set V_1 and V_2 such that $z \in V_1 \subseteq Scl_g(V_1) \subseteq X - A_1$ and $z \in V_2 \subseteq Scl_g(V_2) \subseteq U_0$. By the assumption $cl_g(A_i \cap A_j) = cl_g(A_i) \cap cl_g(A_j)$, intersection of two g-semi closed sets is a g-semi closed set. Since X is g-semi T_3 space, there is a g-semi open set U_1 such that $z \in U_1 \subseteq Scl_g(U_1) \subseteq U_0$ and $Scl_g(U_1) \cap A_1 = \emptyset$. Proceeding like this, we get sequence of g-semi open sets U_n such that $Scl_g(U_{n+1}) \subseteq U_n$ and $Scl_g(U_n) \cap A_n = \emptyset$. Now the collection $\{Scl_g(U_n) : n \in \mathbb{N}\}$ is a sequence of g-semi closed sets with finite intersection property. By theorem 4.1, $\bigcap_{i=1}^{\infty} Scl_g(U_n) \neq \emptyset$. Let $x \in \bigcap_{i=1}^{\infty} Scl_g(U_n)$. This implies $x \in Scl_g(U_n) \subseteq U_0$ and hence $x \notin A_n$, for every n. Thus we have found a point x in a given g-semi open set U_0 such that $x \notin A_n$. Therefore $Sint_g(\bigcup_{i=1}^{\infty} A_n) = \emptyset$. □

We can replace g-α-compact spaces (respectively, g-pre compact spaces and g-β-compact spaces), g-α-closed sets (respectively, g-pre closed sets and g-β-closed sets) and g-α-T_3 space (respectively, g-pre -T_3 space and g-β-T_3 space) instead of g-semi compact spaces, g-semi closed sets and g-semi T_3 space in the above theorem 5.11.

Theorem 5.12. *Let A be a g-α-compact subset of a generalized topological space X. Then A is g-semi compact subset of X if the g-closure of each g-open set is g-open.*

Proof. Let $\mathscr{U}$ be any g-semi open cover of A. For each $U \in \mathscr{U}$, $U \subseteq cl_g(int_g(U))$. $cl_g(int_g(U))$ is a g-closure of an g-open set $int_g(U)$, by assumption $cl_g(int_g(U))$ is a g-open set and hence $cl_g(int_g(U)) = int_g(cl_g(int_g(U)))$. Therefore U is a g-α-open set and $\mathscr{U}$ is a g-α-open cover for A. Since A is g-α compact, $\mathscr{U}$ has a finite subcover. This completes the proof of the theorem. □

Theorem 5.13. *g-pre compact subset of a generalized topological space is g-β-compact if either one of the following holds*

(i) *g-interior of each g-closed set is g-closed*
(ii) *g-closure of each g-open set is g-open.*

Proof. Let A be a g-pre compact subset of a generalized topological space X and let $\mathscr{V}$ be any g-β-open cover of A. If $V \in \mathscr{V}$, then $V \subseteq cl_g(int_g(cl_g(V)))$.

Case: 1 (Suppose if the g-interior of each g-closed set is g-closed)

Since $int_g(cl_g(V))$ is the g-interior of a g-closed set $cl_g(V)$, by assumption, $int_g(cl_g(V))$ is g-closed and hence $cl_g(int_g(cl_g(V))) = int_g(cl_g(V))$. Therefore V is the g-pre open set and $\mathscr{V}$ is a g-pre open cover of A. Since A is g-pre compact, $\mathscr{V}$ has a finite subcover. This completes the proof of case:1.

Case: 2 (Suppose the g-closure of each g-open set is g-open)

Since $cl_g(int_g(cl_g(V)))$ is the g-closure of the g-open set $int_g(cl_g(V))$, by our assumption $cl_g(int_g(cl_g(V)))$ is g-open. This implies $cl_g(int_g(cl_g(V))) = int_g(cl_g(int_g(cl_g(V))))$. Now, $int_g(cl_g(V)) \subseteq cl_g(V)$ and
$int_g(cl_g(int_g(cl_g(V)))) \subseteq int_g(cl_g(V))$. Thus $cl_g(int_g(cl_g(V))) =$
$int_g(cl_g(int_g(cl_g(V)))) \subseteq int_g(cl_g(V))$ and hence $\mathscr{V}$ is a g-pre open cover of A. Since A is g-pre compact, $\mathscr{V}$ has a finite subcover. This completes the proof of case:2. □

Definition 5.14. *Let A be a subset of a generalized topological spae X. Then A is said to be g-regular open if $A = int_g(cl_g(A))$. The complement of a g-regular open set is a g-regular closed set.*

Theorem 5.15. *g-α-compact subset A of a generalized topological space X is g-semi compact if the g-regular closed subsets of X is g-open.*

Proof. Let $\mathscr{U}$ be any g-semi open cover of A. For each $U \in \mathscr{U}$, $U \subseteq cl_g(int_g(U))$. $cl_g(int_g(U))$ is a g-regular closed set, by assumption $cl_g(int_g(U))$ is a g-open set and hence $U \subseteq cl_g(int_g(U)) = int_g(cl_g(int_g(U)))$. This means that U is a g-$\alpha-$open set and $\mathscr{U}$ is a g-$\alpha-$open cover of A. Since A is g-$\alpha-$compact, $\mathscr{U}$ has a finite subcover. Therefore A is g-semi compact. □

Theorem 5.16. *If all the g-regular open subsets of a generalized topological space X are g-closed, then every g-pre compact subset is g-β-compact.*

Proof. Let A be a g-pre compact subset of a generalized topological space X and let $\mathscr{V}$ be any g-β-open cover of A. If $v \in \mathscr{V}$, then $V \subseteq cl_g(int_g(cl_g(V)))$. Since $int_g(cl_g(V))$ is a g-regular open set, by assumption $int_g(cl_g(V))$ is g-closed and hence $V \subseteq cl_g(int_g(cl_g(V))) = int_g(cl_g(V))$. Thus $\mathscr{V}$ is a g-pre open cover of A. Since A is g-pre compact, $\mathscr{V}$ has a finite subcover. Thus A is g-β-compact. □

Theorem 5.17. *A g-compact subset of a generalized topological space is g-α-compact if U^c is g-open whenever U is g-open.*

Proof. Assume that U^c is g-open whenever U is g-open. Then all the g-open sets are g-closed and all the g-closed sets are g-open. Assume that A is g-compact in X. Let $\mathscr{U}$ be an g-α-open cover for A. For each $U \in \mathscr{U}$, $U \subseteq cl_g(int_g(U))$. By assumption, $U \subseteq cl_g(int_g(U)) = int_g(U)$. U is g-open in X and hence $\mathscr{U}$ is a g-open cover of A. Since A is g-compact, this g-open cover has a finite sub cover. Hence A is g-α-compact. □

Theorem 5.18. *A g-α-compact subset of a generalized topological space X is g-precompact if each g-preopen subset of X is a g-open set in X.*

Proof. Let $\mathscr{U}$ be any g-preopen cover of A. If $u \in \mathscr{U}$, then $u \subseteq int_g(cl_g(u))$. By assumption, u is a g-open set. That is $u = int_g(u)$. Now $u \subseteq int_g(cl_g(u)) = int_g(cl_g(int_g(u)))$. Thus $\mathscr{U}$ is a g-α-open cover for A and this $\mathscr{U}$ has a finite sub cover because A is g-α-compact subset of X. □

Theorem 5.19. *A g-semi compact subset of a generalized topological space X is g-β-compact if each g-β-open subset of X is a g-open set in X.*

Proof. Let $\mathscr{U}$ be any g-preopen cover of A. If $u \in \mathscr{U}$, then $u \subseteq cl_g(int_g(cl_g(u)))$. By hypothesis, u is a g-open set. That is $u = int_g(u)$. Now $u \subseteq cl_g(int_g(cl_g(u))) = cl_g(int_g(cl_g(int_g(u)))) \subseteq cl_g(int_g(u))$. Thus $\mathscr{U}$ is a g-semi open cover for A and this $\mathscr{U}$ has a finite sub cover because A is g-semi compact subset of X. □

Theorem 5.20. (i) *A g-α-compact subset of a generalized topological space X is g-precompact if each g-preopen subset of X is a g-regular open set in X.*
(ii) *A g-semi compact subset of a generalized topological space X is g-β-compact if each g-β-open subset of X is a g-regular open set in X.*

Proof. (i) Let $\mathscr{V}$ be any g-preopen cover of A. If $v \in \mathscr{V}$, then v is a g-preopen set and $v \subseteq int_g(cl_g(v))$. By assumption, v is a g-regular open set so that

$int_g(cl_g(v)) = int_g(v)$. This implies $int_g(cl_g(int_g(cl_g(v)))) = int_g(cl_g(int_g(v)))$. That is $v \subseteq int_g(cl_g(v)) \subseteq int_g(cl_g(int_g(cl_g(v)))) = int_g(cl_g(int_g(v)))$. Thus $\mathscr{V}$ is a g-α-open cover for A and this has a finite sub cover because A is g-α-compact subset of X.

(ii) If $\mathscr{V}$ is a g-β-open cover of A and $v \in \mathscr{V}$, then v is a g-β-open set and $v \subseteq cl_g(int_g(cl_g(v)))$. By assumption, v is a g-regular open set and hence $int_g(cl_g(v)) = int_g(v)$. This implies $cl_g(int_g(cl_g(v))) = cl_g(int_g(v))$. That is $v \subseteq cl_g(int_g(cl_g(v))) = cl_g(int_g(v))$. Then v is a g-semi open set and $\mathscr{V}$ is a g-semi open cover of A. Since A is g-semi compact, this $\mathscr{V}$ has a finite sub cover. Hence A is g-β-compact.

□

Theorem 5.21. *Let A be a subset of a generalized topological space X such that each g-β-open subset of X is a g-open set in X. Then the following are equivalent.*

(i) *A is g-compact*
(ii) *A is g-α-compact*
(iii) *A is g-semi compact*
(iv) *A is g-β-compact*
(v) *A is g-pre compact*

Proof. $(i) \implies (ii)$. If $\mathscr{V}$ is a g-α-open cover of A and if $v \in \mathscr{V}$, then v is g-α-open set in X. We know that every g-α-open set is a g-β-open set in X. By our assumption, v is a g-open set in X so that, $\mathscr{V}$ is g-open cover of A. Since A is g-compact, this $\mathscr{V}$ has a finite subcover.
$(ii) \implies (i)$. Trivial.
$(ii) \implies (iii)$. We know that every g-semi open set is a g-β-open set in X. By our assumption, each g-β-open set is a g-open set in X. Since each g-open set is a g-α-open set, if $\mathscr{V}$ is a g-semi open cover of A, it is a g-α-open cover of A. Since A is g-α-compact, this $\mathscr{V}$ has a finite subcover.
$(iii) \implies (ii)$. Trivial.
$(iii) \implies (iv)$. By theorem 5.19.
$(iv) \implies (iii)$. Trivial.
$(iv) \implies (v)$. Trivial.
$(v) \implies (iv)$. If $\mathscr{V}$ is a g-β-open cover of A and if $v \in \mathscr{V}$, then v is a g-β open set in X. By our assumption, v is a g-open set in X. We know that each g-open set is a g-pre open set in X. Since A is g-pre compact, this g-pre open cover $\mathscr{V}$ has a finite subcover. This completes the proof of this theorem. □

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