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SOBOLEV SPACES FOR THE DUNKL OPERATOR ON THE REAL LINE

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Abstract. In this paper we shall introduce functions spaces as subspaces of L^p that we call Sobolev spaces. We prove the generalization of Titchmarsh's theorem of these spaces.

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1. Introduction and preliminaries

We consider the differential-difference operator D_α on $\mathbb{R}$ introduced by Dunkl [4] and called in the literature Dunkl operator on $\mathbb{R}$ on index $(\alpha+\frac{1}{2})$ associated with the reflection group $\mathbb{Z}_2$ given by

$$\mathrm{D}f(x)=\mathrm{D}_\alpha f(x)=\frac{df(x)}{dx}+(\alpha+\frac{1}{2})\frac{f(x)-f(-x)}{x},\ \alpha>-\frac{1}{2},$$

where $f\in \mathrm{L}^p(\mathbb{R},|x|^{2\alpha+1}dx),\ (1<p\leq 2)$.

These operators are very important in mathematics and physics.

For $\alpha>-\frac{1}{2}$, Dunkl kernel function e_α is defined as the unique solution of a differential-difference equation related to D and satisfying $e_\alpha(0)=0$. This kernel is used to defined Dunkl transform from which was introduced by Dunkl in [3].

Let $j_\alpha(x)$ is a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(x)}{x^\alpha},$$

where $J_\alpha(x)$ is a Bessel function of the first kind ([2], Chap. 7). The function $j_\alpha(x)$ is infinitely differentiable and even, in addition, $j_\alpha(0) = 1$.

The kernel Dunkl function is defined by the formula

$$e_\alpha(x) = j_\alpha(x) + i c_\alpha x j_{\alpha+1}(x),$$

where $c_\alpha = (2\alpha+2)^{-1}$.

The function $y = e_\alpha(x)$ satisfies the equation $\mathrm{D}y = iy$ with the initial condition $y(0) = 1$. In the limit case with $\alpha = -\frac{1}{2}$ the kernel Dunkl function coincides with the usual exponential function e^{ix}

Lemma 1.1. *For $x \in \mathbb{R}$ the following inequalities are fulfilled*

(1) $|e_\alpha(x)| \leq 1$.
(2) $|1 - e_\alpha(x)| \leq 2|x|$
(3) $|1 - e_\alpha(x)| \geq 1$ *with* $|x| \geq 1$, *where* $c > 0$ *is a certain constant which depends only on* α

Proof. (See Lemma 2.9 in [1]).

Let $\mathrm{L}_{p,\alpha} = \mathrm{L}^p(\mathbb{R}, |x|^{2\alpha+1}dx)$ stand for the space consits of measurable functions $f(x)$ defined on $\mathbb{R}$ with the finite norm

$$\|f\|_{p,\alpha} = \left(\int_{-\infty}^{+\infty} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.$$

The Dunkl transform is defined by

$$\widehat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x) e_\alpha(\lambda x) |x|^{2\alpha+1} dx.$$

The inverse Dunkl transform is defined by the formula

$$f(x)=A\int_{-\infty}^{+\infty}\widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$A=(2^{\alpha+1}\Gamma(\alpha+1))^{-2}.$$

Plancherel's theorem and the Marcinkiewics interpolation theorem (See [7]) we get for $f\in \mathrm{L}_{p,\alpha}$ with $1<p\leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$,

(1) $$\|\widehat{f}\|_{q,\alpha}\leq C\|f\|_{p,\alpha},$$

where C is a positive constant.

Let $\mathrm{W}^r_{p,\alpha}$ be Sobolev space constructed by the operator D, i.e.,

$$\mathrm{W}^r_{p,\alpha}=\{f\in \mathrm{L}_{p,\alpha};\ \mathrm{D}^j f\in \mathrm{L}_{p,\alpha},\ j=1,2,...,r\},$$

where $r\in\{1,2,....\}$.

We have

(2) $$\widehat{(\mathrm{D}^r f)}(\lambda)=(-i\lambda)^r\widehat{f}(\lambda),$$

where $f\in \mathrm{W}^r_{p,\alpha}$.

K. Trimèche has introduced in [8] the Dunkl translation operator, T_h and in [6], we have

(3) $$\widehat{(\mathrm{T}_h f)}(\lambda)=e_\alpha(\lambda h)\widehat{f}(\lambda).$$

We define the finite differences of the first and higher orders are defined as follows:

$$\Delta_h f(x)=\mathrm{T}_h f(x)-f(x)=(\mathrm{T}_h-\mathrm{I})f(x),$$

(4) $$\Delta_h^k f(x)=\Delta_h(\Delta_h^{k-1}f(x))=(\mathrm{T}_h-\mathrm{I})^k f(x)=\sum_{i=0}^{k}(-1)^{k-i}\tbinom{k}{i}\mathrm{T}_h^i f(x),$$

where $\mathrm{T}_h^0 f(x)=f(x)$, $\mathrm{T}_h^i f(x)=\mathrm{T}_h(\mathrm{T}_h^{i-1}f(x))$, $(i=1,2,...,k\ ;\ k=1,2,....)$ and I is a unit operator in $\mathrm{L}_{p,\alpha}$.

In [5], we have

Theorem 1.2. *Let* $f(x)\in\mathrm{L}_{p,\alpha}$*, and let*

$$\int_{-\infty}^{+\infty}|\mathrm{T}_h f(x)-f(x)|^p|x|^{2\alpha+1}dx=O(h^{\gamma p});\ \ as,\ h\longrightarrow 0$$

Then $\widehat{f}\in\mathrm{L}_{\beta,\alpha}$ *for*

$$\frac{2p\alpha+2p}{2p+2\alpha(p-1)+\gamma p-2}<\beta\le\frac{p}{p-1}$$

The main aim of this paper is establish the generalization of Theorem 1.2 in the Dunkl operator setting by means of the differences of higher orders.

2. Main Results

Lemma 2.1. *Let* $f\in\mathrm{W}^r_{p,\alpha}$*. Then*

$$\int_{-\infty}^{+\infty}|\lambda|^{qr}|1-e_\alpha(\lambda h)|^{qk}|\widehat{f}(x)|^q|\lambda|^{2\alpha+1}d\lambda\le C\|\Delta_h^k\mathrm{D}^r f(x)\|^q_{p,\alpha},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Formulas (2) and (3) give

(5) $$(\widehat{\mathrm{T}_h^i\mathrm{D}^r f})(\lambda)=(-i\lambda)^r e^i_\alpha(\lambda h)\widehat{f}(\lambda).$$

From (4) and (5), we obtain

$$(\widehat{\Delta_h^k\mathrm{D}^r f})(\lambda)=(-i\lambda)^r(1-e_\alpha(\lambda h))^k\widehat{f}(\lambda).$$

By formula (1), we have the poof of Lemma 2.1.

Theorem 2.2. *Let* $f\in\mathrm{W}^r_{p,\alpha}$*, and let*

$$\|\Delta_h^k\mathrm{D}^r f(x)\|_{p,\alpha}=O(h^\gamma);\ as\ h\longrightarrow 0,$$

$(0<\gamma\le k)$*. Then* $\widehat{f}\in\mathrm{L}_{\beta,\alpha}$ *for*

$$\frac{2\alpha p+2p}{2p+2\alpha(p-1)-2+rp}<\beta\le\frac{p}{p-1}.$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}\int_{-\infty}^{+\infty}|\lambda|^{qr}|1-e_\alpha(\lambda h)|^{qk}|\widehat{f}(x)|^q|\lambda|^{2\alpha+1}d\lambda &\leq C\|\Delta_h^k\mathrm{D}^r f(x)\|_{p,\alpha}^q\\ &\leq C_1h^{\gamma q},\end{aligned}$$

where C_1 is a positive contant.

If $|x|\in[\frac{1}{h},\frac{2}{h}]$ then $|xh|\geq 1$ and formula (3) of Lemma 1.1, we obtain

$$1\leq\frac{1}{c}|1-e_\alpha(hx)|$$

then

$$|xh|^{qk}\leq\frac{1}{c^{qk}}|1-e_\alpha(xh)|^{qk}.$$

Therefore

$$\begin{aligned}\int_0^{2/h}|xh|^{qk}|x|^{qr}|\widehat{f}(x)|^q|x|^{2\alpha+1}dx &\leq \frac{1}{c^{qk}}\int_0^{2/h}|x|^{qr}|1-e_\alpha(xh)|^{qk}|\widehat{f}(x)|^q|x|^{2\alpha+1}dx\\ &\leq \frac{1}{c^{qk}}\int_{-\infty}^{+\infty}|x|^{qr}|1-e_\alpha(xh)|^{qk}|\widehat{f}(x)|^q|x|^{2\alpha+1}dx\\ &\leq C_2\|\Delta_h^k\mathrm{D}^r f(x)\|_{p,\alpha}^q\\ &= O(h^{\gamma q}).\end{aligned}$$

Then

$$\int_0^{2/h}|x|^{qk+qr}|\widehat{f}(x)|^q|x|^{2\alpha+1}dx=O(h^{(\gamma-k)q}).$$

Let

$$\psi(t)=\int_1^t|x^{k+r}\widehat{f}(x)|^\beta x^{(2\alpha+1)\frac{\beta}{q}}dx.$$

Then, if $\beta<q$, by Hölder inequality we have

$$\begin{aligned}\psi(t) &\leq \left(\int_1^t|x^{k+r}\widehat{f}(x)|^q|x|^{2\alpha+1}dx\right)^{\beta/q}\left(\int_1^t dx\right)^{1-\beta/q}\\ &= O(t^{(k-\gamma)\beta}t^{1-\frac{\beta}{q}})\\ &= O(t^{1-\gamma\beta+k\beta-\frac{\beta}{q}}).\end{aligned}$$

Then

$$\int_1^t |\widehat{f}(x)|^\beta x^{2\alpha+1}dx = \int_1^t x^{-k\beta-r\beta-(2\alpha+1)\frac{\beta}{q}}\psi'(x)x^{2\alpha+1}dx$$

$$\begin{aligned}
&= t^{-k\beta-r\beta-(2\alpha+1)\frac{\beta}{q}}t^{2\alpha+1}\psi(t) + (k\beta + r\beta + (2\alpha+1)\frac{\beta}{q} \\
&- (2\alpha+1))\int_1^t x^{-k\beta-r\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha}\psi(x)dx \\
&= O(t^{-r\beta-(2\alpha+1)\frac{\beta}{q}+2\alpha+2-\gamma\beta-\frac{\beta}{q}}).
\end{aligned}$$

and this bounded as $t \longrightarrow \infty$ if $-r\beta - (2\alpha+1)\frac{\beta}{q} + 2\alpha + 2 - \gamma\beta - \frac{\beta}{q} < 0$, i.e. if

$$\beta > \frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \gamma p - 2 + rp}.$$

Similarly for the integral over $(-t, -1)$. This proves the theorem.

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