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A SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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Abstract. A new class of harmonic univalent functions defined by an integral operator is introduced. Coefficient inequalities, extreme points, distortion bounds, inclusion results and closure under an integral operator for this class are obtained.
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1. Introduction

The pioneering work of Clunie and Sheil-Small [4] on harmonic univalent mappings gave rise to the birth of theory of harmonic univalent mappings. This theory has attracted the function theorists to look at the harmonic analogue of the theory of analytic univalent functions in the open unit disc.

In any simply connected domain D, we can write $f = h+\overline{g}$ where h and g are analytic in D.

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions $f = h + \overline{g}$ which are univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$, so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k; \;\; |b_1| < 1. \tag{1.1}$$

Subclasses of harmonic functions have been studied by many authors including [2, 5, 6, 7, 8, 9].

Denote by $\mathcal{H}(\Delta)$ the space of holomorphic functions in Δ and let

$$A_n = \{f \in \mathcal{H}(\Delta), f(z) = z + a_{n+1}z^{n+1} + \cdots, \;\; z \in \Delta\}.$$

When $n=1$, we have $A_1 = A = \left\{ f(z) \in \mathcal{H}(\Delta) : f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta \right\}$.

The integral operator I^n was introduced by Salagean [11] which is defined as

(i) $I^0 f(z) = f(z)$

(ii) $I^1 f(z) = If(z) = \int_0^z f(t)t^{-1}dt$

(iii) $I^n f(z) = I(I^{n-1}f(z)), \quad n = 1, 2, \dots, f \in A$.

Ahuja and Jahangiri [1] defined the class $H(p)$, ($p \in N = \{1, 2, 3, \dots\}$), consisting of all multivalent harmonic functions $f = h + \overline{g}$ that are sense preserving in Δ and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}; \quad |b_p| < 1.$$

The class $H(1)$ of harmonic univalent functions was studied in detail by Clunie and Sheil-Small [4].

The modified Salagean integral operator I^n of $f = h + \overline{g}$ given by (1.1) is defined as:

$$I^n f(z) = I^n h(z) + (-1)^n \overline{I^n g(z)}; \quad z \in \Delta, \tag{1.2}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k$$

$$I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

Let $H(n, \alpha, \beta, t)$ denote the class of univalent harmonic functions of the form (1.1) that satisfy the condition:

$$Re \left\{ \frac{I^n f(z)}{(1-t)z + tI^{n+1} f(z)} \right\} > \beta \left| \frac{I^n f(z)}{(1-t)z + tI^{n+1} f(z)} - 1 \right| + \alpha \tag{1.3}$$

or equivalently

$$Re \left[\frac{(\beta e^{i\theta} + 1) I^n f(z)}{(1-t)z + tI^{n+1} f(z)} - \beta e^{i\theta} \right] > \alpha \tag{1.4}$$

where n is a fixed positive integer, $0 \le \alpha < 1$, $0 \le t \le 1$, θ real and $\beta \ge 0$.

We further let $\overline{H}(n, \alpha, \beta, t)$ denote the subclass of $H(n, \alpha, \beta, t)$ consisting of functions $f_n = h + \overline{g_n}$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} b_k z^k, \tag{1.5}$$

where $a_k, b_k \ge 0$ and $|b_1| < 1$.

Remark 1.1.

(1) *The class* $\overline{H}(n, \alpha, \beta, t)$ *reduces to the class* $\overline{H}(n, \alpha, \beta)$ [7]*, when* $t = 1$.

(2) *It is noted that the class* $\overline{H}(n, \alpha, \beta, t)$*, when* $t = 1$ *yields* $\overline{H}(n, \alpha, \beta, 1)$. *With reference to* [6]*, when* $p = 1$*,* $\overline{H_p}(n + 1, n, \alpha, \beta)$ *yields* $\overline{H_1}(n + 1, n, \alpha, \beta) = \overline{H}(n, \alpha, \beta, 1)$.

In this paper motivated by the study in [7], coefficient inequalities, extreme points, distortion bounds, inclusion results and closure under an integral operator are obtained for functions in the class $\overline{H}(n, \alpha, \beta, t)$.

2. Main Results

We begin with a sufficient coefficient condition for functions in $H(n, \alpha, \beta, t)$.

Theorem 2.1. *Let the function* $f = h + \overline{g}$ *be so that* h *and* g *are given by (1.1). Furthermore let*

$$\sum_{k=2}^{\infty} \psi(n, \alpha, \beta, t)|a_k| + \sum_{k=1}^{\infty} \mu(n, \alpha, \beta, t)|b_k| \leq 1, \tag{2.1}$$

where

$$\psi(n, \alpha, \beta, t) = \frac{\frac{1}{k^n}\left[1 + \beta - \left(\frac{1}{k}\right) t(\alpha + \beta)\right]}{1 - \alpha},$$

$$\text{and} \qquad \mu(n, \alpha, \beta, t) = \frac{\frac{1}{k^n}\left[1 + \beta + \left(\frac{1}{k}\right) t(\alpha + \beta)\right]}{1 - \alpha},$$

$0 \leq \alpha < 1$*,* $\beta \geq 0$*,* $0 \leq t \leq 1$*,* θ *real,* $n \in N$*, then* $f \in H(n, \alpha, \beta, t)$.

Proof. To show that $f \in H(n, \alpha, \beta, t)$, according to the condition (1.3), we only need to show that if (2.1) holds, then

$$Re\left[\frac{(1 + \beta e^{i\theta})I^n f(z) - \beta e^{i\theta}[(1 - t)z + tI^{n+1} f(z)]}{(1 - t)z + tI^{n+1} f(z)}\right] = Re\left(\frac{A(z)}{B(z)}\right) \geq \alpha,$$

where $z = re^{i\xi}$, $0 \leq r < 1$, ξ, θ real and $0 \leq \alpha < 1$.

Note that

$$A(z) = (1 + \beta e^{i\theta})I^n f(z) - \beta e^{i\theta}[(1 - t)z + tI^{n+1} f(z)]$$

$$\text{and} \qquad B(z) = (1 - t)z + tI^{n+1} f(z).$$

Using the fact that $Re\ w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0 \tag{2.2}$$

Substituting $A(z)$ and $B(z)$ in (2.2), we obtain,

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&= \left|(1+\beta e^{i\theta})I^n f(z)-\beta e^{i\theta}[(1-t)z+tI^{n+1}f(z)]+(1-\alpha)[(1-t)z+tI^{n+1}f(z)]\right| \\
&- \left|(1+\beta e^{i\theta})I^n f(z)-\beta e^{i\theta}[(1-t)z+tI^{n+1}f(z)]-(1+\alpha)[(1-t)z+tI^{n+1}f(z)]\right| \\
&= \Bigg|(1+\beta e^{i\theta})\left[z+\sum_{k=2}^{\infty}k^{-n}a_k z^k+(-1)^n\sum_{k=1}^{\infty}k^{-n}\overline{b_k z^k}\right]+z(1-t)[1-\alpha-\beta e^{i\theta}] \\
&+t[1-\alpha-\beta e^{i\theta}]\left[z+\sum_{k=2}^{\infty}k^{-(n+1)}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}k^{-(n+1)}\overline{b_k z^k}\right]\Bigg| \\
&-\Bigg|(1+\beta e^{i\theta})\left[z+\sum_{k=2}^{\infty}k^{-n}a_k z^k+(-1)^n\sum_{k=1}^{\infty}k^{-n}\overline{b_k z^k}\right]+z(1-t)[-1-\alpha-\beta e^{i\theta}] \\
&+t[-1-\alpha-\beta e^{i\theta}]\left[z+\sum_{k=2}^{\infty}k^{-(n+1)}a_k z^k+(-1)^{n+1}\sum_{k=1}^{\infty}k^{-(n+1)}\overline{b_k z^k}\right]\Bigg| \\
&= \Bigg|(2-\alpha)z+\sum_{k=2}^{\infty}\frac{1}{k^n}\left[1+\left(\frac{1}{k}\right)t(1-\alpha)+\beta e^{i\theta}\left(1-\left(\frac{1}{k}\right)t\right)\right]a_k z^k \\
&\quad -(-1)^{n+1}\sum_{k=1}^{\infty}\frac{1}{k^n}\left[1-\left(\frac{1}{k}\right)t(1-\alpha)+\beta e^{i\theta}\left(1+\left(\frac{1}{k}\right)t\right)\right]\overline{b_k z^k}\Bigg| \\
&\quad -\Bigg|-\alpha z+\sum_{k=2}^{\infty}\frac{1}{k^n}\left[1+\left(\frac{1}{k}\right)t(-1-\alpha)+\beta e^{i\theta}\left(1-\left(\frac{1}{k}\right)t\right)\right]a_k z^k \\
&\quad -(-1)^{n+1}\sum_{k=1}^{\infty}\frac{1}{k^n}\left[1-\left(\frac{1}{k}\right)t(-1-\alpha)+\beta e^{i\theta}\left(1+\left(\frac{1}{k}\right)t\right)\right]\overline{b_k z^k}\Bigg| \\
&\geq (2-\alpha)|z|-\sum_{k=2}^{\infty}\frac{1}{k^n}\left[1+\left(\frac{1}{k}\right)t(1-\alpha)+\beta\left(1-\left(\frac{1}{k}\right)t\right)\right]|a_k||z|^k \\
&\quad -\sum_{k=1}^{\infty}\frac{1}{k^n}\left[1-\left(\frac{1}{k}\right)t(1-\alpha)+\beta\left(1+\left(\frac{1}{k}\right)t\right)\right]|b_k||z|^k \\
&\quad -\alpha|z|-\sum_{k=2}^{\infty}\frac{1}{k^n}\left[1+\left(\frac{1}{k}\right)t(-1-\alpha)+\beta\left(1-\left(\frac{1}{k}\right)t\right)\right]|a_k||z|^k \\
&\quad -\sum_{k=1}^{\infty}\frac{1}{k^n}\left[1-\left(\frac{1}{k}\right)t(-1-\alpha)+\beta\left(1+\left(\frac{1}{k}\right)t\right)\right]|b_k||z|^k \\
&\geq 2(1-\alpha)|z|-\sum_{k=2}^{\infty}2\left(\frac{1}{k^n}\right)\left[1+\beta-\left(\frac{1}{k}\right)t(\alpha+\beta)\right]|a_k||z|^k \\
&\quad -\sum_{k=1}^{\infty}2\left(\frac{1}{k^n}\right)\left[1+\beta+\left(\frac{1}{k}\right)t(\alpha+\beta)\right]|b_k||z|^k
\end{aligned}$$

$$\geq 2(1-\alpha)\left[1-\left(\sum_{k=2}^{\infty}\frac{\frac{1}{k^n}\left[1+\beta-\left(\frac{1}{k}\right)t(\alpha+\beta)\right]}{1-\alpha}|a_k|\right.\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\frac{\frac{1}{k^n}\left[1+\beta+\left(\frac{1}{k}\right)t(\alpha+\beta)\right]}{1-\alpha}|b_k|\right)\right]$$
$$\geq 0.$$

This completes the proof. □

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1}{\psi(n,\alpha,\beta,t)}x_k z^k+\sum_{k=1}^{\infty}\frac{1}{\mu(n,\alpha,\beta,t)}\overline{y_k z^k}, \tag{2.3}$$

where $n\in N$, $0\leq\alpha<1$, $\beta\geq 0$, $0\leq t\leq 1$, θ real and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp.

This is because,

$$\sum_{k=2}^{\infty}\psi(n,\alpha,\beta,t)|a_k|+\sum_{k=1}^{\infty}\mu(n,\alpha,\beta,t)|b_k|$$
$$=\sum_{k=2}^{\infty}\psi(n,\alpha,\beta,t)\frac{1}{\psi(n,\alpha,\beta,t)}|x_k|+\sum_{k=1}^{\infty}\mu(n,\alpha,\beta,t)\frac{1}{\mu(n,\alpha,\beta,t)}|y_k|$$
$$=\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1.$$

In the next theorem, we show that the condition (2.1) is both necessary and sufficient for $f_n=h+\overline{g_n}\in\overline{H}(n,\alpha,\beta,t)$ where h and g_n are of the form (1.5).

Theorem 2.2. *Let $f_n=h+\overline{g_n}$ be given by (1.5). Then $f_n\in\overline{H}(n,\alpha,\beta,t)$ if and only if*

$$\sum_{k=2}^{\infty}\psi(n,\alpha,\beta,t)a_k+\sum_{k=1}^{\infty}\mu(n,\alpha,\beta,t)b_k\leq 1 \tag{2.4}$$

where $0\leq\alpha<1$, $\beta\geq 0$, $0\leq t\leq 1$, θ real and $n\in N$, with $b_k>a_k$, for every $k\geq 2$ and $\psi(n,\alpha,\beta,t)$, $\mu(n,\alpha,\beta,t)$ are as defined in the statement of Theorem 2.1.

Proof. Since $\overline{H}(n,\alpha,\beta,t)\subset H(n,\alpha,\beta,t)$, we only need to prove the "only if" part of the theorem. For functions f_n of the form (1.5), we note that the condition

$$Re\left\{\frac{I^n f(z)}{(1-t)z+tI^{n+1}f(z)}\right\}>\beta\left|\frac{I^n f(z)}{(1-t)z+tI^{n+1}f(z)}-1\right|+\alpha$$

is equivalent to

$$Re\left\{\frac{\begin{array}{c}(1-\alpha)z-\displaystyle\sum_{k=2}^{\infty}\left[\frac{1}{k^n}-\alpha t\frac{1}{k^{n+1}}\right]a_kz^k+(-1)^{2n-1}\sum_{k=1}^{\infty}\left[\frac{1}{k^n}+\alpha t\frac{1}{k^{n+1}}\right]b_k\overline{z}^k\\ -\beta e^{i\theta}\displaystyle\sum_{k=2}^{\infty}\left[\frac{1}{k^n}-t\frac{1}{k^{n+1}}\right]a_kz^k+(-1)^{2n-1}\beta e^{i\theta}\sum_{k=1}^{\infty}\left[\frac{1}{k^n}+t\frac{1}{k^{n+1}}\right]b_k\overline{z}^k\end{array}}{z-t\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^{n+1}}a_kz^k+(-1)^{2n}t\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}b_k\overline{z}^k}\right\}$$

$$\geq 0. \tag{2.5}$$

The above required condition (2.5) must hold for all values of $z\in\Delta$ and for real θ, so that on taking the values of z on the positive real axis where $0\leq z=r<1$, $\theta=0$ and $b_k>a_k$, for every $k\geq 2$, the above inequality reduces to

$$\frac{\begin{array}{c}\left[(1-\alpha)-\displaystyle\sum_{k=2}^{\infty}\left[\frac{1}{k^n}(1+\beta)-(\alpha+\beta)t\frac{1}{k^{n+1}}\right]a_kr^{k-1}\right.\\ \left.-\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k^n}(1+\beta)+(\alpha+\beta)t\frac{1}{k^{n+1}}\right]b_kr^{k-1}\right]\end{array}}{1-t\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^{n+1}}a_kr^{k-1}+t\sum_{k=1}^{\infty}\frac{1}{k^{n+1}}b_kr^{k-1}}\geq 0 \tag{2.6}$$

If the condition (2.5) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Hence there exist $z_0=r_0$ in (0, 1) for which this quotient in (2.6) is negative. This contradicts the required condition for $f_n\in\overline{H}(n,\alpha,\beta,t)$ and so the proof is complete. □

We now determine the distortion bounds for functions in $\overline{H}(n,\alpha,\beta,t)$.

Theorem 2.3. *Let $f_n\in\overline{H}(n,\alpha,\beta,t)$. Then for $|z|=r<1$, we have*

$$|f_n(z)|\geq(1-b_1)r-[\phi(n,\alpha,\beta,t)-\Omega(n,\alpha,\beta,t)b_1]r^2$$

where

$$\phi(n,\alpha,\beta,t)=\frac{1-\alpha}{\left(\frac{1}{2^n}\right)\left[1+\beta-\left(\frac{1}{2}\right)t(\alpha+\beta)\right]}$$

and $$\Omega(n,\alpha,\beta,t)=\frac{1+\beta+t(\alpha+\beta)}{\left(\frac{1}{2^n}\right)\left[1+\beta-\left(\frac{1}{2}\right)t(\alpha+\beta)\right]}.$$

Proof. Let $f_n(z) \in \overline{H}(n,\alpha,\beta,t)$, then by Theorem 2.2, we have

$$\begin{aligned}
|f_n(z)| &\geq (1-b_1)r - \sum_{k=2}^{\infty}(a_k+b_k)r^k \\
&\geq (1-b_1)r - \sum_{k=2}^{\infty}(a_k+b_k)r^2 \\
&\geq (1-b_1)r - \phi(n,\alpha,\beta,t)\sum_{k=2}^{\infty}\frac{1}{\phi(n,\alpha,\beta,t)}(a_k+b_k)r^2 \\
&\geq (1-b_1)r - \phi(n,\alpha,\beta,t)\sum_{k=2}^{\infty}[\psi(n,\alpha,\beta,t)a_k+\mu(n,\alpha,\beta,t)b_k]r^2 \\
&\geq (1-b_1)r - \phi(n,\alpha,\beta,t)\left[1-\frac{[(1+t\alpha)+\beta(1+t)]}{1-\alpha}b_1\right]r^2 \\
&= (1-b_1)r - [\phi(n,\alpha,\beta,t)-\Omega(n,\alpha,\beta,t)b_1]r^2,
\end{aligned}$$

where $\psi(n,\alpha,\beta,t), \mu(n,\alpha,\beta,t)$ are as defined in the statement of Theorem 2.1. □

Here, we determine the extreme points of the closed convex hulls of $\overline{H}(n,\alpha,\beta,t)$ denoted by $clco\,\overline{H}(n,\alpha,\beta,t)$.

Theorem 2.4. *Let f_n be given by (1.5). Then $f_n \in \overline{H}(n,\alpha,\beta,t)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{n_k}(z)],$$

where $h_1(z) = z$,

$$h_k(z) = z - \frac{(1-\alpha)}{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}z^k, \quad (n = 2,3,\dots),$$

and

$$g_{n_k}(z) = z + (-1)^{n-1}\frac{(1-\alpha)}{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}\overline{z}^k, \quad (n = 1,2,\dots),$$

$x_k \geq 0$, $y_k \geq 0$, $\sum_{k=1}^{\infty}(x_k+y_k) = 1$.
In particular, the extreme points of $\overline{H}(n,\alpha,\beta,t)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. Let

$$\begin{aligned}
f_n(z) &= \sum_{k=1}^{\infty}[x_k h_k + y_k g_{n_k}] \\
&= \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}x_k z^k \\
&\quad + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}y_k \overline{z}^k \\
&= z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}x_k z^k \\
&\quad + (-1)^{n-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}y_k \overline{z}^k.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}|a_k| + \sum_{k=1}^{\infty}\frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}|b_k| \\
&= \sum_{k=2}^{\infty}\frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}.\frac{1-\alpha}{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}x_k \\
&+ \sum_{k=1}^{\infty}\frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}.\frac{1-\alpha}{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}y_k \\
&= \sum_{k=2}^{\infty}x_k + \sum_{k=1}^{\infty}y_k = \sum_{k=1}^{\infty}(x_k + y_k) - x_1 \le 1,
\end{aligned}$$

and so $f_n(z) \in \overline{H}(n, \alpha, \beta, t)$.

Conversely, suppose $f_n(z) \in \overline{H}(n, \alpha, \beta, t)$.

Let $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k$.

Let

$$x_k = \frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}a_k, \quad k = 2, 3, \ldots,$$

and

$$y_k = \frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}b_k, \quad k = 1, 2, 3, \ldots.$$

The required representation is obtained as

$$\begin{aligned}
f_n(z) &= z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n+1} \sum_{k=1}^{\infty} b_k \overline{z}^k \\
&= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}} x_k z^k \\
&\quad + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}} y_k \overline{z}^k \\
&= z - \sum_{k=2}^{\infty} [z - h_k(z)] x_k - \sum_{k=1}^{\infty} [z - g_{n_k}(z)] y_k \\
&= \left[1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k\right] z + \sum_{k=2}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_{n_k}(z) \\
&= \sum_{k=1}^{\infty} [x_k h_k(z) + y_k g_{n_k}(z)].
\end{aligned}$$

□

3. Closure Property of the Class $\overline{H}(n, \alpha, \beta, t)$

In the next two theorems, we prove that the class $\overline{H}(n, \alpha, \beta, t)$ is invariant under convolution and convex combinations of its members. The convolution of two harmonic functions,

$$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_k \overline{z}^k \tag{3.1}$$

and

$$F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} B_k \overline{z}^k \tag{3.2}$$

is defined as

$$\begin{aligned}
(f_n * F_n)(z) &= f_n(z) * F_n(z) \\
&= z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_k B_k \overline{z}^k
\end{aligned} \tag{3.3}$$

Using this definition, we first show that the class $\overline{H}(n, \alpha, \beta, t)$ is closed under convolution.

Theorem 3.1. *For* $0 \le \gamma \le \alpha < 1$, $0 \le t \le 1$, *let* $f_n \in \overline{H}(n, \alpha, \beta, t)$ *and* $F_n \in \overline{H}(n, \gamma, \beta, t)$, *then* $f * F \in \overline{H}(n, \alpha, \beta, t) \subset \overline{H}(n, \gamma, \beta, t)$.

Proof. Let $f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_k \overline{z}^k$ be in $\overline{H}(n, \alpha, \beta, t)$ and $F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} B_k \overline{z}^k$ be in $\overline{H}(n, \gamma, \beta, t)$.

Then the convolution $f_n * F_n$ is given by (3.3). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{H}(n, \gamma, \beta, t)$, we note that $A_k < 1$ and $B_k < 1$. Now for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\gamma)k^{-(n+1)}}{1-\gamma} a_k A_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\gamma)k^{-(n+1)}}{1-\gamma} b_k B_k \\
&\leq \sum_{k=2}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\gamma)k^{-(n+1)}}{1-\gamma} a_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\gamma)k^{-(n+1)}}{1-\gamma} b_k \\
&\leq \sum_{k=2}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} b_k \\
&\leq 1,
\end{aligned}$$

since $0 \leq \gamma \leq \alpha < 1$ and $f_n \in \overline{H}(n, \alpha, \beta, t)$.

Now, we show that $\overline{H}(n, \alpha, \beta, t)$ is closed under convex combination of its members. □

Theorem 3.2. *The class $\overline{H}(n, \alpha, \beta, t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, suppose $f_{n_i}(z) \in \overline{H}(n, \alpha, \beta, t)$, where f_{n_i} is given by

$$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k + (-1)^{n-1} \sum_{k=1}^{\infty} b_{i,k} \overline{z^k}.$$

Then by (2.4),

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} a_{i,k} + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} b_{i,k} \leq 1 \tag{3.4}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of $f_{n_i}(z)$ may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i,k} \right) z^k + (-1)^{n-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i,k} \right) \overline{z}^k.$$

Then by (3.4), we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i,k} \right) \\
&\quad + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i,k} \right) \\
&= \sum_{i=1}^{\infty} t_i \left(\sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} - t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} a_{i,k} \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \frac{(1+\beta)k^{-n} + t(\beta+\alpha)k^{-(n+1)}}{1-\alpha} b_{i,k} \right) \\
&\leq \sum_{i=1}^{\infty} t_i = 1,
\end{aligned}$$

which is the required coefficient condition. □

Finally, we examine the closure property of the class $\overline{H}(n, \alpha, \beta, t)$ under the generalized Bernardi-Libera-Livingston integral operator [3, 10], $\mathcal{L}_c(f)$ which is defined by

$$\mathcal{L}_c(f) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -1.$$

Theorem 3.3. *Let* $f_n(z) \in \overline{H}(n, \alpha, \beta, t)$. *Then*

$$\mathcal{L}_c(f_n(z)) \in \overline{H}(n, \alpha, \beta, t)$$

Proof. From the representation of $\mathcal{L}_c(f_n(z))$, it follows that

$$\begin{aligned}
\mathcal{L}_c(f_n(z)) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left[t - \sum_{k=2}^{\infty} a_k t^k + \overline{(-1)^{n-1} \sum_{k=1}^{\infty} b_k t^k} \right] dt \\
&= z - \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} \frac{c+1}{c+k} b_k z^k \\
&= z - \sum_{k=2}^{\infty} x_k z^k + (-1)^{n-1} \sum_{k=1}^{\infty} y_k z^k
\end{aligned}$$

where

$$x_k = \frac{c+1}{c+k} a_k \text{ and}$$
$$y_k = \frac{c+1}{c+k} b_k.$$

Hence

$$\sum_{k=2}^{\infty}\frac{(1+\beta)k^{-n}-t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}\left(\frac{c+1}{c+k}a_k\right)$$
$$+\sum_{k=1}^{\infty}\frac{(1+\beta)k^{-n}+t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}\left(\frac{c+1}{c+k}b_k\right)$$
$$\leq\sum_{k=2}^{\infty}\frac{(1+\beta)k^{-n}-t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}a_k+\sum_{k=1}^{\infty}\frac{(1+\beta)k^{-n}+t(\beta+\alpha)k^{-(n+1)}}{1-\alpha}b_k$$
$$\leq 1, \quad \text{by } (2.4).$$

Hence by Theorem 2.2, $\mathcal{L}_c(f_n(z)) \in \overline{H}(n,\alpha,\beta,t)$. □

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