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CERTAIN REGULARITY RESULTS OF $p(x)$-PARABOLIC PROBLEMS WITH MEASURE DATA

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Abstract. In this paper, we establish certain existence and regularity results for a class of nonlinear $p(x)$-parabolic equations

$$u_t - div\big[\phi(t,x,u)(1+|u|)^{s(x)}|\nabla u|^{p(x)-2}\nabla u\big] = \mu,$$

where $Q_T := (0,T)\times\Omega \; with \; \Omega\subset\mathbb{R}^N(N\geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $T>0$, the vector filed $\phi(t,x,u)$ verified some appropriate assumptions, μ is a bounded Radon measure on Q_T and the initial data $u_0\in L^1(\Omega)$. The main results of this paper are proved by using approximation problems, certain a priori estimates, compactness results and the truncation method in the framework of Sobolev spaces with variable exponents.

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1. Introduction

In this article, we demonstrate the existence and regularity results to nonlinear $p(x)-$parabolic equations that contain a low order term with natural growth. We are specifically interested in the following problem

$$(\mathcal{P})\begin{cases} u_t - div\big[\phi(t,x,u)(1+|u|)^{s(x)}|\nabla u|^{p(x)-2}\nabla u\big] = \mu & \text{in } Q_T,\\ u(t,x)=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0,x)=u_0(x) & \text{in } \Omega,\end{cases}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain, with a smooth boundary $\partial\Omega$ and $Q_T := (0,T)\times\Omega$, $T > 0$. The vector filed $\phi(t,x,u)$ verified certain appropriate hypotheses, μ is a bounded Radon measure on Q_T and the initial data $u_0 \in L^1(\Omega)$.

The analysis of general Leray-Lions problems with non-regular source terms (L^1 or measures) has received a lot of attention in the literature. For exemple, in [13], the authors has been showed the existence results in the sense of distributions. However, the distributional formulation is not strong enough to give uniqueness, because the solution lacks regularity as demonstrated by Serrin's counter-example in the parabolic case. To solve this obstacle, it is permissible to employ a different kind of solution, namely "renormalized-entropy" solutions, which require less regularity than weak solutions. On the other side, the notion of existence and regularity results was introduced by L. Boccardo and al [13] when the right hand side is function in $L^m(Q_T) \times (0,T)$, with m "small" and $p > 1$. In [15], the existence of finite energy solutions of the following quasi-linear elliptic problem

$$(1) \qquad \begin{cases} -div(\phi(x) + |u|)^s \nabla u) + b(x)|u|^{p-1}u|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has been studied when $f \in L^1(\Omega)$ is non-negative and $p \geq 2q$. Moreover, in [21] a similar results concerning the existence and regularity have also been proved by considering the parameters $p,\ q$ and the summability of the datum f. In addittion the author reinforced the results of [15] by proving the existence of solutions of the problem (1) without any condition on the values of $p,\ q$ and on the sign of f. We draw attention to the fact that the key factor of the existence result of [21] is to prove that $|\nabla u||u|^q \in L^1(\Omega)$ for any $q > 0$.
At present, we shell consider the general form of problem $(\mathcal{P})$ as follows

$$(2) \qquad \begin{cases} u_t - div(\phi(x,t,u,\nabla u)) = \mu & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(x,0) = u_0(x) & in\ \Omega, \end{cases}$$

which was examined by D. Blanchard and F. Murat [10], they established the existence and uniqueness of a solution for this problem with the data μ and u_0 are still taken in $L^1(Q_T)$ and $L^1(\Omega)$ respectively. However, for nonlinear operators with L^1 data, A. Prignet and D. Blanchard [11, 26] developed a new concept of solutions, which introduced the notions of renormalized solution and entropy solution, respectively. Subsequently, D. Boccardo, T. Gallouet, and L. Orsina, [13], proved the existence of a solution where μ belongs to $L^1(\Omega) + H^{-1}(\Omega)$ and μ does not load sets of zero capacity. In the case where $\phi(x,t,s,\xi)$ is independent of s the existence and uniqueness of renormalized solution has been established by D. Blanchard in [8]. In the case where $\phi(x,t,s,\xi)$ is replaced by $\phi(x,t,s,\xi)+U(s)$ and $\mu = f + div(F)$, has been solved in [9,24]. For the degenerated parabolic equations, the existence of weak solutions have been proved by L. Aharouch and al in [4] when $\phi(x,t,u,\nabla u)$ is strictly monotone and $\mu \in L^{p'}(0,T,W^{-1,p'}(\Omega,\omega^*))$ see also the existence and uniqueness of a renormalized solution proved by Y. Akdim and al [5] in the case where $\phi(x,t,s,\xi)$ is independent of s. The work of M. Bendahmane and A. Zimmermann [7] for $p(\cdot)$-Laplacian is presented as an example in the setting of Sobolev spaces with variable exponents, particularly when the operator growth is nonstandard and nonhomogeneous nonlinearities. Recently, in the case when $\phi(x,t,s,\xi) = |\nabla\xi|^{p(x)-2}\nabla\xi$, this study can be considered as a continuation of [27]. Many results have been obtained on this kind of problems, for example [1–3,19,22].

The aim of this paper is to study the existence and regularity of weak solutions to a nonlinear $p(x)$-parabolic problem. Our approach is based on an approximation problem combined with a priori estimates, and compactness results in variable exponent Sobolev spaces.

The structure of this paper will be broken down in the following sections. In section 2, we provide some preliminaries. In particular, we collect some important properties and results on Lebesgue-Sobolev spaces with variable exponents and the generalized $p(\cdot)$-parabolic capacity. Furthermore, we present the fundamental assumptions that must be made on $\phi,\ \mu,\ L,\ u_0$ and to provide our general definition, when the operator is modified in some way in section 3, finally a new result is proved in section 4.

2. Preliminaries

In the analysis of the problem $(\mathcal{P})$, we will use some definitions and basic properties of generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and the theory of parabolic capacities. Moreover, we only recall some basic results which will be used later, we refer to [17,18] for more details.

2.1. **Sobolev spaces with variable exponents.**

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we say that a real-valued continuous function p is $\log$-Höder continuous in Ω if

$$|p(x)-p(y)| \leq \frac{C}{|\log|x-y||} \text{ for all } x,\ y \in \overline{\Omega} \text{ such that } |x-y| < \frac{1}{2},$$

where C is a constant. We designate by

$$C_+(\overline{\Omega}) = \Big\{ \text{ log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p(x) \leq p^+ < N \Big\},$$

where

$$p^- = \min \Big\{ p(x) : x \in \overline{\Omega} \Big\} \text{ and } p^+ = \max \Big\{ p(x) : x \in \overline{\Omega} \Big\}.$$

The Lebesgue space with variable exponent is defined by

$$L^{p(x)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\},$$

which is endowed with the "Luxembourg" norm given by

$$\|u\|_{p(\cdot)} = \inf \Big\{ \Lambda > 0; \int_\Omega |\frac{u(x)}{\Lambda}|^{p(x)} \mathrm{d}x \leq 1 \Big\}.$$

Recall that the inequality below will be used later

$$\min \Big\{ \|u\|_{p(\cdot)}^{p^-} \, ; \, \|u\|_{p(\cdot)}^{p^+} \Big\} \leq \int_\Omega |u(x)|^{p(x)} \mathrm{d}x \leq \max \Big\{ \|u\|_{p(\cdot)}^{p^-} \, ; \, \|u\|_{p(\cdot)}^{p^+} \Big\}.$$

Remark that, if $1 < p^- < \infty$, then $L^{p(\cdot)}(\Omega)$ is reflexive and its dual denoted by $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{p(\cdot)} = 1$, and then for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the inequality of type Hölder given by

$$\int_\Omega |uv| \mathrm{d}x \leq \Big(\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} \Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

Next, if $p(\cdot), p'(\cdot) \in C_+(\overline{\Omega})$, then the Young's type inequality setting by the formula

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p(x)},$$

such that $\frac{1}{p(\cdot)} + \frac{1}{p(\cdot)} = 1$ and for each $a, b > 0$. Extending a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q}_T = \overline{\Omega} \times [0, T]$ by setting $p(x) := p(t, x)$ for every $(x, t) \in \overline{Q}_T$, we may also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q_T) = \Big\{ u : Q_T \to \mathbb{R}; \text{ measurable such that } \int_{Q_T} \Big| u(x,t) \Big|^{p(x)} dxdt < \infty \Big\},$$

under the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} = \inf \Big\{ \Lambda > 0; \int_{Q_T} \Big| \frac{u(x,t)}{\Lambda} \Big|^{p(x)} dxdt < 1 \Big\},$$

retains the same properties as $L^{p(\cdot)}(\Omega)$. Furthermore, the variable exponent Sobolev space given by

$$W^{1,p(\cdot)}(\Omega) = \Big\{ u \in L^{p(\cdot)}(\Omega) \, ; \, |\nabla u| \in L^{p(\cdot)}(\Omega) \Big\},$$

is a Banach space with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)},$$

such that

$$\|u\|_{1,p(\cdot)} = \inf\Big\{\Lambda > 0; \int_\Omega \Big(\Big|\frac{\nabla u(x)}{\Lambda}\Big|^{p(x)} + \Big|\frac{u(x)}{\Lambda}\Big|^{p(x)}\Big)dx \le 1\Big\}. \tag{3}$$

We define the functional space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ with respect to the norm (3). Note that $W_0^{1,p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces if $1 \le p^- < \infty$ and $1 < p^- < \infty$ respectively. At last, we shall employ the standard notation for Bochner spaces, i.e., $L^q(0,T;X)$ is the space of strongly measurable function $u : (0,T) \to X$ for which $t \mapsto \|u(t)\|_X \in L^q(0,T)$. In addition, $C([0,T];X)$ represents the space of continuous function u: $[0,T] \to X$ according to the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X$, where X is a Banach space and $q \ge 1$.

$$L^{p^-}(0,\ T;W_0^{1,p(x)}(\Omega)) = \Big\{u : (0,T) \to W_0^{1,p(x)}(\Omega) \text{ measurable with } \Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega)}\Big)^{\frac{1}{p^-}} dt < +\infty\Big\}.$$

2.2. **Measures and Parabolic capacity.**

Let $Q_T = \Omega \times (0,T)$ for each fixed $T > 0$ and let us recall that $V = W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$ equipped with its appropriate norm $\|.\|_{W_0^{1,p(\cdot)}} + \|.\|_{L^2(\Omega)}$, the space $W_{p(\cdot)}(0,T)$ setting by

$$W_{p(\cdot)}(0,T) = \Big\{u \in L^{p^-}(0,T,V);\ \nabla u \in (L^{p(\cdot)}(Q_T))^N \text{ and } u_t \in L^{(p^-)'}(0,T,V')\Big\}$$

equipped with the following norm

$$\|u\|_{W_{p(\cdot)}(0,T)} = \|u\|_{L(0,T,V)} + \|\nabla u\| + \|u_t\|_{L(0,T,V')}.$$

Note that $W_{p(\cdot)}(0,T) \hookrightarrow C([0,T],L^2(\Omega))$ continuously. Let $O \subseteq Q_T$ be an open set, we define the (generalized) parabolic capacity of O as

$$cap_{p(\cdot)}(O) = \inf\Big\{\|u\|_{W_{p(\cdot)}(0,T)} : O \in W_{p(\cdot)}(0,T),\ s \ge \chi_O \text{ a.e. in } Q_T\Big\},$$

where as usual we set $\inf\{\emptyset\} = +\infty$, then for any Borel set $B \subseteq Q_T$, the definition of (generalized) parabolic capacity can be extended by setting

$$cap_{p(\cdot)}(B) = \inf\Big\{cap_{p(\cdot)}(O) : O \text{ open subset of } Q_T, B \subseteq O\Big\}.$$

Since, we are interested by using some regular properties, we need to define the following space

$$\mathcal{V} = \Big\{u \in L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega)) \ : \ \nabla u \in (L^{p(\cdot)}(Q_T))^N and\ u_t \in L^{(p^-)'}(0,T,W^{1,p'(\cdot)}(\Omega)) + L^1(Q_T)\Big\},$$

endowed with its natural norm

$$\|u\|_{\mathcal{V}} = \|u\|_{L^{p^-}(0,T,W_0^{1,p(\cdot)}(\Omega))} + \|\nabla u\|_{(L^{p(\cdot)}(Q_T))^N} + \|u_t\|_{L(0,T,W^{1,p'(\cdot)}(\Omega))+L^1(Q_T)}.$$

In the following, $\mathcal{M}_b(Q_T)$ denotes the set of all Radon measures with bounded variation on Q_T and $\mathcal{M}_0(Q_T)$ designates

$$\mathcal{M}_0(Q_T) = \Big\{\mu \in \mathcal{M}_b(Q_T) : \mu(E) = 0 \text{ for every } E \subset Q_T \text{ such that } cap_{p(\cdot)}(E) = 0\Big\}.$$

To better specify the nature of a measure in $\mathcal{M}_0(Q_T)$, we need then to detail the structure of the dual space $(W_{p(\cdot)}(0,T))'$. We will be interested in a specific type of positive bump functions C_c^∞ known as "cut-off" functions during the proof of our principal result $\varphi_\gamma : \mathbb{R}^{N+1} \to \mathbb{R}$ satisfy

$$\begin{cases} \varphi_\gamma(r) \equiv 1 \qquad if \qquad r \in K_\gamma, \\ \\ \varphi_\gamma(r) = 0 \qquad if \qquad r \in Q_T \backslash K_\gamma, \\ \\ 0 \leq \varphi_\gamma \leq 1, \qquad \forall r \in Q_T. \end{cases}$$

let us define, for every $0 < q(x) < \infty$, the Marcinkiewicz space $\mathcal{M}^{q(x)}(Q_T)$ as the space of every measurable function g such that

$$\exists C > 0 \ \text{ with } \ meas\big\{(t,x) \in Q_T \quad |g(t,x)| \geq h\big\} \leq \frac{C}{h^{q^-}}.$$

for every positive k, endowed with the semi-norm

$$\|g\|_{\mathcal{M}^{q(x)}(Q)} = \inf\Big\{C > 0 : meas\big\{(t,x) : |g(t,x)| \geq h\big\} \leq \Big(\frac{C}{k}\Big)^{q(x)}\Big\}.$$

Note that, if $q(x) \geq q^- > 1$, then we obtain the following continuous embedding

$$L^{q(x)}(Q_T) \hookrightarrow \mathcal{M}^{q(x)}(Q_T) \hookrightarrow L^{q(x)-\varepsilon}(Q_T), \ \forall \epsilon \in (0, q(x)-1].$$

Proposition 1. *Any weak solution of $(\mathcal{P})$ with initial datum $u_0 \in L^1(\Omega)$ satisfies the following estimates*

$$\|u\|_{\mathcal{M}_0^{p(x)-1+\frac{p(x)}{N}}(Q_T)} \leq C_1, \quad \|\nabla u\|_{\mathcal{M}_0^{p(x)-\frac{N}{N+1}}(Q_T)} \leq C_2, \tag{4}$$

where $C_j,\ j = 1,\ 2$, are positive constants only depending on $u_0,\ \mu,\ N,\ T$ and $p^- < p(x) < p^+$.

Proof. The proof is similar to [23, Proof of Theorem 1.7]. □

Lemma 2.1. *Let $0 \leq \Lambda \in \mathcal{M}_b(Q_T)$ be concentrated on a set E such that $cap_{p(x)}(E) = 0$. Then, for each $\gamma > 0$, there exist $\varphi_\gamma \in C_c^\infty(Q_T)$ and $K_\gamma \subset E$ a compact subset such that*

$$(5) \qquad \begin{cases} 0 \leq \varphi_\gamma \leq 1 \ in \ Q_T, \quad \varphi \equiv 1 \ in \ K_\gamma, \ \Lambda(E \backslash K_\gamma) < \gamma, \\ \\ \lim_{\gamma \to 0} ||\varphi_\gamma||_{\mathcal{V}} = 0, \qquad \int_{Q_T} (1 - \varphi_\gamma) d\Lambda = \varpi(\gamma), \end{cases}$$

and, in particular, a decomposition $[(\varphi_\gamma)_t^1, \ (\varphi_\gamma)_t^2]$ such that

$$(6) \qquad \begin{cases} \varphi_\gamma \xrightarrow[\gamma \to 0]{} 0 \ weakly\text{-}^* \ in \ L^\infty(Q_T), \qquad a.e.in \ Q_T \ and \ in \ L^1(Q_T) \\ \\ \| \ (\varphi_\gamma)_t^1 \ \|_{L^{(p')^-}(0,T,W^{-1,p'(x)}(\Omega))} \leq \dfrac{\gamma}{3}, \quad \| \ (\varphi_\gamma)_t^2 \ \|_{L^1(Q_t))} \leq \dfrac{\gamma}{3}. \end{cases}$$

Remark 2.2. Let us recall the following function of $\omega_\Lambda(r) = re^{\Lambda r^2}$ which had this useful property:

$$(7) \qquad a\omega'_\Lambda(r) - b|\omega_\Lambda(r)| \geq 1, \ \forall r \in \mathbb{R}, \ \forall a, \ b > 0, \ \forall \Lambda > \frac{b^2}{8a^2}.$$

The truncation function and the following functions will be used in the sequel:

$$T_k(r) = \max\{-k, min(k, r)\}, \qquad \Theta_k(r) = T_1(r - T_k(r)).$$

3. Assumptions and Technical results

3.1. Assumption and Lemmas.

The following assumptions are assumed throughout this work. We consider the following problem

$$(8) \qquad \begin{cases} u_t - div\Big[\phi(t, x, u, \nabla u)\Big] = \mu \quad in \ Q = (0, T) \times \Omega, \\ \\ u(t, x) = 0 \quad on \ (0, T) \times \partial\Omega, \ u(0, x) = u_0(x) \quad in \ \Omega, \end{cases}$$

where $u_0 \in L^1(\Omega)$, μ is a bounded Radon measure on Q_T and $\phi : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, satisfying the following condition, there exist $k \in L^{p(.)}(Q_T)$ and $\alpha > 0, \ \beta > 0$ such that, for each $(t, x) \in Q_T$ all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

$$(9) \qquad \phi(t, x, u, \zeta) \cdot \zeta \geq L(|u|)|\zeta|^{p(x)},$$

$$(10) \qquad |\phi(t, x, u, \zeta)| \leq \beta[k(t, x) + L(|u|)|\zeta|^{p(x)-1}],$$

$$(11)\qquad [\phi(x,t,u,\xi)-\phi(x,t,u,\eta)](\xi-\eta)>0 \quad \forall\, \xi\neq\eta.$$

Moreover, the function L satisfies

$$(12)\qquad L(|u|)\geq \alpha>0,\ for\ all\ u\in\mathbb{R}.$$

Here

$$(13)\qquad \mu\in\mathcal{M}_b(Q_T).$$

This part introduces several fundamental technical concepts and results that will be used throughout this article. For some details concerning their related contents, the reader can consult (see [22]).

Lemma 3.1. [6] *Suppose that (9)-(13) are satisfied and let (u_m) be a sequence in $L^{p^-}(0,T;L^{p(x)}(\Omega))$ such that $u_m\rightharpoonup u$ in $L^{p^-}(0,T,L^{p(x)}(\Omega))$ and*

$$\int_{Q_T}\Big(\phi(x,t,u_m),\nabla u_m)-\phi(x,t,u_m,\nabla u)\Big)\nabla(u_m-u)dx\to 0.$$

Then, $u_m\to u$ strongly in $L^{p^-}(0,T;L^{p(.)}(\Omega))$.

In general, we will work with measurable functions and truncations in the energy space $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. To do this, we now define $\mathcal{T}_0^{1,p(x)}(Q_T)$ as the set of measurable functions $u:Q_T\to 0$ such that $T_k(u)$ belongs to $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ for each $k>0$ which we will use in the following .

Lemma 3.2. [6] *For every $u\in\mathcal{T}_0^{1,p(x)}(Q_T)$, there exists a unique measurable function $v:Q_T\mapsto\mathbb{R}^N$ such that, $\nabla T_k(u)=v\chi_{\{|u|\leq k\}}$, a.e. in Q_T for each $k>0$, where E is the characteristic function of the measurable set E. Moreover, if*

$$\int_{Q_T}|\nabla T_k(u)|^{p(x)}dxdt\leq C(k+1),$$

then, v coincides with the classical gradient of u and is denoted by $\nabla u=v$. with u is $cap_{p(x)}$- a.e. finite, i.e. $cap_{p(x)}\{(t,x)\in Q_T\ :\ |u(t,x)|=+\infty\}=0$, and there exists a $cap_{p(x)}-q.c.r.$ of u, namely a function $\tilde{u}$ such that $\tilde{u}=u$ a.e. in Q_T and $\tilde{u}$ is $cap_{p(x)}$-quasi continuous.

Definition 3.3. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$. For each $\mu\in\mathcal{M}_b(Q_T)$, we define a "weak" solution to the problem $(\mathcal{P})$ as a measurable function $u\in C([0,T];L^1(\Omega))$ such that $\phi(t,x,u,\nabla u)\in L^1(Q_T)^N,\ T_k(u)\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, and it verifies

$$\int_0^T\langle u_t,\varphi\rangle dt+\int_{Q_T}\phi(t,x,u)\Big(1+|u_\varepsilon|\Big)^{s(x)}\times|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\cdot\nabla\varphi dxdt=\int_{Q_T}f_\varepsilon\varphi d\mu, \tag{14}$$

$\forall\,\varphi\in C_c^\infty(Q_T)$.

3.2. **Technical results.**

We need to provide another key tool in our reasoning before proving our primary findings. This result is a generalized existence result that applies the finding from [26] to the case of measure data. To do this, we must first establish an approximation issue for $\varepsilon\in\mathbb{N}$ and its associated properties. Recall that μ and u can be reasonably approximated by a series of smooth functions (μ_ε) and (u_0^ε) created by convolution. We demonstrate this bellow, we'll investigate the behaviour of the sequence (u_ε) solutions of the following problems

$$(\mathcal{P}_\varepsilon)\quad\begin{cases}(u_\varepsilon)_t-div[\phi(t,x,u_\varepsilon,\nabla u_\varepsilon)]=\mu_\varepsilon\quad in\ (0,T)\times\Omega,\\[2ex] u_\varepsilon(t,x)=0\quad on\ (0,T)\times\partial\Omega,\quad u_\varepsilon(0,x)=u_0(x)\ in\ \Omega.\end{cases}$$

Proposition 2. *Suppose that $\phi(t,x,s,\xi)$ verifies (9)-(12) and let $1<p(x)<N$, $\mu\in\mathcal{M}_b(Q_T)$. Then, there exists a function u in $\mathcal{T}_0^{1,p(x)}(\Omega)$ where $\phi(t,x,u,\nabla u)\in L^{q(x)}(Q_T)$ for each $q(x)<p(x)-\dfrac{N}{N+1}$ and u verifies (14) within the meaning of the definition (3.3).*

Proof. Let μ_ε be a sequence of $C_c^\infty(Q_T)$-functions such that

$$\begin{cases}\mu_\varepsilon\to\ \mu\ tightly\ in\ \mathcal{M}_0(Q_T),\\[2ex] \|\mu_\varepsilon\|_{L^1(Q_T)}\le\|\mu\|_{\mathcal{M}_0(Q_T)},\end{cases}$$

and (u_0^ε) a sequence of $C_c^\infty(\Omega)$-functions such that

$$\begin{cases}u_0^\varepsilon\to u_0\ in\ L^1(\Omega),\\[2ex] \|u_0^\varepsilon\|_{L^1(\Omega)}\le\|u_0\|_{L^1(\Omega)}.\end{cases}$$

In addition, let u_ε be a weak solution of the problem $(\mathcal{P}_\varepsilon)$.

Observe that, according to the results of [20], there is one weak solution for $(\mathcal{P}_\varepsilon)$, i.e., a function $u_\varepsilon\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ such that $(u_\varepsilon)_t\in L^{p^-}(0,T;W^{-1,p(x)}(\Omega))\cap L^\infty(Q_T)$ an the following identity holds true

$$\int_0^t \langle (u_\varepsilon)_t, \varphi\rangle dt + \int_0^t \int_\Omega \phi(s,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\varphi dxds = \int_0^t \int_\Omega \varphi d\mu_\varepsilon, \tag{15}$$

by taking $\varphi = T_k(u_\varepsilon)$, for each $\varphi \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q_T)$ in (15) and integrating in $]0,T[$ we have

$$\int_\Omega \Theta_k(u_\varepsilon)(t)dx + \int_0^t \int_\Omega \phi(s,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla T_k(u)dxds = \int_0^t \int_\Omega T_k(u_\varepsilon)d\mu_\varepsilon + \int_\Omega \Theta_k(u_0^\varepsilon)dx,$$

which implies, from (9) and as $\|u_0^\varepsilon\|_{L^1(\Omega)}$ and $\|\mu_\varepsilon\|_{L^1(Q_T)}$ are bounded, that

$$\int_\Omega \Theta_k(u_\varepsilon)(t)dx + \alpha\int_0^t \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}dxds \leq k(\|\mu\|_{\mathcal{M}\prime(Q_T)} + \|u_0^\varepsilon\|_{L^1(\Omega)}) = Ck.$$

As $\Theta_k(l) \geq 0$ and $|\Theta_1(l)| \geq |l| - 1$, we obtain

$$\int_\Omega |u_\varepsilon|(t)dx + \alpha\int_0^t \int_\Omega |\nabla T_k(u_\varepsilon)|^{p(x)}dxdt \leq C(k+1), \text{ for all } k > 0, \ \forall t \in [0,T],$$

choosing the supremum on $(0,T)$, we have

$$\int_\Omega |u_\varepsilon|(t)dx \leq C, \ for\ all\ t \in [0,T], \tag{16}$$

which gives an estimate of u_ε in $L^\infty(0,T;L^1(\Omega))$ and also

$$\int_{Q_T} |\nabla T_k(u_\varepsilon)|^{p(x)}dxdt \leq C(k+1), \tag{17}$$

This means that for each $k > 0$, $T_k(u)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. As a result, there exists a function $u \in \mathcal{T}_0^{1,p(x)}(\Omega)$ such that, up to subsequences,

$$\begin{cases} u_\varepsilon \to u \ a.\ e. \text{ in } Q_T, \\ T_k(u_\varepsilon) \to T_k(u) \text{ weakly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)), \\ \text{strongly in } L^{p(x)}(Q_T) \text{ and a.e. in } Q_T. \end{cases} \tag{18}$$

Let us choose the test function in the weak formulation of (15) for each $p(x) > 1$, $\varphi = T_k(B(u_\varepsilon))$ with $B(l) = \int_0^l b(|\sigma|)^{\frac{1}{p(x)-1}}d\sigma$ (which is an eligible choice because $T_k(B(u_\varepsilon)) \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$). According to the definition of $T_k(l)$, we obtain

$$\int_0^T \langle (u_\varepsilon)_t, T_k(B(u_\varepsilon))\rangle dt + \int_{Q_T} \phi(t,x,u_\varepsilon, \ \nabla u_\varepsilon) \times \nabla u_\varepsilon b(|u_\varepsilon|)^{\frac{1}{p(x)-1}}dxdt \leq \int_Q T_k(B(u_\varepsilon))d\mu_\varepsilon,$$

applying (9), we get (by fixing $\Phi(l) = \int_0^l T_k(B(\sigma))d\sigma$) that

$$\int_\Omega \Phi(u_\varepsilon)(t)dx \;+\; \int_{\{|B(u_\varepsilon)|\leq k\}} |b(|u_\varepsilon|)|^{p'(x)}|\nabla u_\varepsilon|^{p(x)}dxdt \quad\leq\quad k\big[\|\mu_\varepsilon\|_{\mathcal{M}_0(Q_T)} \;+\; \int_\Omega \Phi(u_0^\varepsilon)dx\big],$$

and since μ_ε is bounded in $L^1(Q_T)$ and u_0^ε is bounded in $L^1(\Omega)$ we obtain

$$\int_\Omega \Phi(u_\varepsilon(t))dx \leq C,\;\; for\; all\; t \in [0,T],$$

which means that $|\nabla B(u_\varepsilon)|$ is bounded in the Marcinkiewicz space $\mathcal{M}_0^{p(x)-1+\frac{p(x)}{N}}(Q_T)$. Hence $b(|u_\varepsilon|)|\nabla u_\varepsilon|^{p(x)-1}$ is bounded in the Marcinkiewicz space $\mathcal{M}_0^{1+\frac{p(x)N-N+p(x)}{N(p(x)-1)}}(Q_T)$ and as

$$\Big\{(t,x) \;:\; |\phi(t,x,u_\varepsilon,\nabla u_\varepsilon)| \;>\; k\Big\} \;\subset\; \Big\{(t,x) \;:\; \beta(d(t,x) \;+\; b(|u_\varepsilon|)|\nabla u_\varepsilon|^{p(x)-1}) \;>\; k\Big\},$$

then, $\phi(t,x,u_\varepsilon,\nabla u_\varepsilon)$ is bounded in $L^{q(x)}(Q_T)$, but we can not yet prove that its weak limit is $\phi(t,x,u,\nabla u)$; this will be accomplished by demonstrating that ∇u_ε converges to u almost everywhere. For this, we will employ the technique used in [25], with minor alterations related to the hypothesis (10).

For $m,\; k > 0$, we choose $T_m(u_\varepsilon - T_k(u))$ as the test function in the weak formulation of $(\mathcal{P}_\varepsilon)$. Since, $\|\mu_\varepsilon\|_{L^1(Q_T)} \leq C_0$ (we will designate from now on by C_i positive constants independent of ε and m), we obtain

$$\begin{aligned}
&\int_{Q_T} \phi(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla T_m(u_\varepsilon - T_k(u))dxdt\\
&\qquad\geq \int_{Q_T} \phi(t,x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))\cdot\nabla T_m(T_k(u_\varepsilon) - T_k(u))dxdt\\
&\qquad- \int_{\{|\mathrm{u}_n|>k\}} |\phi(t,x,T_{k+m}(u_\varepsilon),\;\nabla T_{k+m}(u_\varepsilon))||\nabla T_k(u)|dxdt
\end{aligned} \tag{19}$$

Using (18), $|\nabla T_k(u)|\chi_{|u_\varepsilon|>k}$ converges strongly to zero in $L^{p(x)}(Q_T)$, by tending ε to infinity, the last term goes to zero for each $m > 0$ fixed. This means, by (19), that

$$\int_{Q_T} \phi(t,x,T_k(u_\varepsilon)),\;\; \nabla T_k(u_\varepsilon)) \;\cdot\; \nabla T_m(T_k(u_\varepsilon) \;-\; T_k(u_\varepsilon))dxdt \quad\leq\quad m\;\; C_0 \;+\; \varpi_m(\varepsilon). \tag{20}$$

On the other hand, let's $0 < \lambda(x) < 1$ and $E_k^m = \{(t,x) \in Q_T \;:\; |T_k(u_\varepsilon) - T_k(u)| > m\}$, we obtain

$$\int_{Q_T}\Big[[\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u)]$$
$$\times\nabla(T_k(u_\varepsilon)-T_k(u))\Big]^{\lambda(x)}dxdt$$
$$=\int_{Q_T}\Big[[\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))]$$
$$\times\nabla T_m(T_k(u_n)-T_k(u))\Big]^{\lambda(x)}dxdt$$
$$+\int_{E_k^m}\Big[[\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))]$$
$$\times\nabla(T_k(u_\varepsilon)-T_k(u))\Big]^{\lambda(x)}dxdt,$$

Hence, combined (20) with Hölder's inequality of exponent $\dfrac{1}{\lambda(x)}$, we get

$$\int_{Q_T}\Big[[\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))]$$
$$\times\nabla(T_k(u_\varepsilon)-T_k(u))\Big]^{\lambda^-}dxdt$$
$$\text{(21)}\qquad\leq\ \text{meas }(Q_T)^{1-\lambda^+}[mc_0+\varpi_m(\varepsilon)-\int_{Q_T}\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u))$$
$$\times\nabla T_m(T_k(u_\varepsilon)-T_k(u))]^{\lambda(x)}dxdt$$
$$+\int_{E_k^m}\Big[[\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))]$$
$$\times\nabla(T_k(u_\varepsilon)-T_k(u))\Big]^{\lambda^-}dxdt,$$

As $T_k(u_\varepsilon)$ converges to $T_k(u)$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ By applying the hypothesis $(\mathcal{P}_\varepsilon)$, it is simple to show that the first term on the right hand side of (21) becomes zero when ε approaches infinity. As a result, the sequence $\Big[\big|\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))\big|\Big]_\varepsilon$ is bounded in $L^{p(x)}(Q_T)$for each $m>0$, and then by Hölder's inequality, we obtain

$$\int_{Q_T}\Big[[\phi(t,\varepsilon,x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\ \nabla T_k(u))\Big]$$
$$\times\nabla\Big[T_k(u_\varepsilon)-T_k(u)\Big]^{\lambda^-}dxdt$$
$$\leq\ \text{meas }(Q_T)^{1-\lambda^+}[mc_0+2\varpi_m(\varepsilon)]^{\lambda^-}$$
$$+c_1[\ \text{meas }\{(t,x)\in Q_T\ :\ |T_k(u_\varepsilon)-T_k(u)|>m\}]^{1-\lambda^+}.$$

By tending ε to infinity and then m to zero, and since $T_k(u_\varepsilon)$ converges in measure to $T_k(u)$, we state that

$$\lim_{\varepsilon\to\infty}\int_{Q_T}\Big[[(\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u_\varepsilon))-\phi(t,x,T_k(u_\varepsilon),\nabla T_k(u))]\times\nabla(T_k(u_\varepsilon)-T_k(u))\Big]^{\lambda^-}dxdt=0,$$

we conclude, by reasoning as in [25], that $\nabla T_k(u_\varepsilon)$ a.e. converges to $\nabla T_k(u)$ for all $k>0$; in fact, ∇u_ε converges to ∇u a.e. in Q_T, which proves that

$$\phi(t,x,u_\varepsilon,\nabla u_\varepsilon)\quad\to\quad\phi(t,x,u,\nabla u)\text{ strongly in }L^{q(x)}(Q_T),\quad\forall q(x)\quad<\quad p(x)\quad-\quad\frac{N}{N+1}.$$

Finally, by passing to the limit, tending u to infinity, in the weak formulation of (15) for each $\varphi\in C_c^\infty([0,T]\times\Omega)$ to conclude that u satisfies (14) in the distributional sense (this means that u is a weak solution of (8) and this concludes the proof of Proposition 2. □

4. Principal result and proof

In this section we shall present the notion of weak solution to problem $(\mathcal{P})$ and we shall give the existence result for such solution.

Theorem 4.1. *Assume that ϕ satisfies (10)-(12) and $\mu\in\mathcal{M}_0(Q_T)$. Let $q(x)>1,\ s(x)\geq 0,\ 2-\frac{1}{N+1}<p(x)<N$ and suppose that there are positive constants L where*

(22) $$b(|m|)\geq L|u|^{s(x)},\qquad \textit{for all } m\in\mathbb{R}:|m|>m_0.$$

Then, $(\mathcal{P})$ has a weak solution u such that

(i) *if $s(x)>1$, then $u\in W_0\cap L^{\zeta(x)}(Q_T)$ for each*
$$\zeta(x)<\frac{(p(x)N+p(x)-N)(s(x)+1)}{N+1};$$

(ii) *if $0\leq s^-\leq s(x)\leq s^+\leq 1$ and $p(x)>2-\frac{1+s(x)(N-1)}{N}$, then u belongs to $L^{q^-}(0,T;W_0^{1,q(x)}(\Omega))$ for every*
$$q(x)<\frac{N(p(x)-1+s(x))}{N-(1-s(x))}.$$

In addition, if μ is a function in $L^{m'(x)}(Q_T)$ where $1<m(x)<(p^\star(x))'$, then $(\mathcal{P})$ has a weak solution u such that

(iii) *if $s(x)>1-\frac{p^\star(x)}{m'(x)}$, then $u\in W\cap L^{\frac{(N(p(x)+1)+p(x))m(x)(s(x)+1)}{N(x)+p(x)-m(x)p(x)}}(Q_T)$;*

(iv) *if* $0 \leq s(x) < 1 - \dfrac{p^{\star}(x)}{m'(x)}$ *and* $p(x) > max\Big\{1, 2 - \dfrac{m(x)(1+s(x)(N-1)) + N(m(x)-1)}{Nm(x)}\Big\}$, *then u belongs to* $L^{q^-}(0,T;W_0^{1,q(x)}(\Omega))$ *such that*

$$q(x) = \frac{Nm(x)(p(x)-1+s(x))}{N-m(x)(1-s(x))}.$$

Proof. To prove the Theorem 4.1, let $\mu \in \mathcal{M}_0(Q_T)$ and $u_0 \in L^1$. Consider the two sequences (u_0^{ε}) of $L^{\infty}(\Omega)$-functions and (f_{ε}) of $L^{p'(x)}(Q_T)$-functions verifying

$$(23) \qquad \begin{cases} f_{\varepsilon} \to \mu \text{ in the weak-}^* \text{ topology of measures and } \|f_{\varepsilon}\|_{L^1(Q_T)} \leq C, \\ u_0^{\varepsilon} \to u_0 \quad in\ L^1(\Omega) \quad and \quad \|u_0^{\varepsilon}\|_{L^1(\Omega)} \leq C. \end{cases}$$

Consider that u_{ε} is the weak solution of $(\mathcal{P}_{\varepsilon})$

$$(\mathcal{P}_{\varepsilon}) \begin{cases} (u_{\varepsilon})_t - \operatorname{div}[\phi(t,x,u_{\varepsilon})(1+|u_{\varepsilon}|)^{s(x)}|\nabla u_{\varepsilon}|^{p(x)-2}\nabla u_{\varepsilon}] = f_{\varepsilon} \\ \qquad\qquad in\ Q_T\ := (0,\ T) \times \Omega, \\ u_{\varepsilon}(0,x) = u_0^{\varepsilon}(x) \quad \text{in } \Omega,\ u_{\varepsilon}(t,x) = 0 \quad \text{on} \quad (0,T) \times \Omega, \end{cases}$$

where μ_{ε} and u_0^{ε} are specified as before. Such a solution is established by well-known results (see [20]) and belongs to $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) \cap C(0,T;L^2(\Omega))$. As $\{f_{\varepsilon}\}$ is bounded in $L^1(Q_T)$, and according to Proposition 2, u_{ε} is bounded in $\mathcal{T}_0^{1,p(x)}(Q_T)$ such that $\phi(t,x,u_{\varepsilon},\nabla u_{\varepsilon}) \in L^{q(x)}(Q_T)$ for each $q(x) < p(x) - \dfrac{N}{N+1}$, and u_{ε} solves $(\mathcal{P}_{\varepsilon})$ in the sense of distributions. Consequently, there is u and a subsequence (still denoted by u_{ε}) such that

$$(24) \qquad \begin{cases} u_{\varepsilon} \rightharpoonup u \text{ a. e. in } Q_T, \\ T_k(u_{\varepsilon}) \rightharpoonup T_k(u) \text{ weakly in } L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)), \\ \qquad \text{strongly in } L^{p(x)}(Q_T) \text{ and a.e.in } Q_T. \end{cases}$$

On the other side, we take $\Psi_k(u_{\varepsilon}) = T_1(u_{\varepsilon} - T_k(u_{\varepsilon}))$, with $k \geq k_0$ where $k_0 \in \mathbb{N}$, as test function in the weak formulation of $(\mathcal{P}_{\varepsilon})$ and given that $\nabla\Psi_k(u_{\varepsilon}) = \nabla u_{\varepsilon}\chi_{\{k \leq |u_{\varepsilon}| < k+1\}}$ and $\Psi_k(u_{\varepsilon}) = 0$ if $|u_{\varepsilon}| \leq k$, we can obtain easily

$$(25) \quad \int_0^T \langle (u_{\varepsilon})_t,\ T_1(u_{\varepsilon} - T_k(u_{\varepsilon}))\rangle dt + \int_{\{k \leq |u_{\varepsilon}| < k+1\}} \phi(t,x,u_{\varepsilon},\nabla u_{\varepsilon})\nabla u_{\varepsilon} dxdt \leq \int_{\{|u_{\varepsilon}| \geq k\}} |f_{\varepsilon}| dxdt,$$

for every $k \geq k_0$.

Using the integration by parts formula of Lemma 3.2 and from (22), it follows that the some subsequence $\{u_{\varepsilon}\}$ verifies

$$(26) \quad \int_{\Omega} \theta_k(u_{\varepsilon}(\tau))dx + Lk^{s^-} \int_{\{k \leq |u_{\varepsilon}| < k+1\}} |\nabla u_{\varepsilon}|^{p(x)} dxdt \leq \int_{\{|u_{\varepsilon}| \leq k\}} |f_{\varepsilon}| dxdt + \int_{\Omega} \theta_k(u_{\varepsilon}(0))dx,$$

where $\theta_k(l)$ defined as follows $\theta_k(l) = \displaystyle\int_0^l \Phi(y)dy$. Let's note that there are two cases that need to be distinguished:

1st **case:** If $s(x) > 1$. For a.e. $t \in (0,T)$, by means (23), (26) and the fact that f_ε is bounded in $L^1(Q_T)$ and $|\theta_k(u_0^\varepsilon)| \leq |u_0^\varepsilon|$ a.e. in Ω, we can write that

$$\int_\Omega \theta_k(u_\varepsilon)(t)dx \leq C, \quad \text{for all } t \text{ belongs to } [0,T]. \tag{27}$$

From now on, we will denote by C any constant that is dependent on particular variables and whose value can vary from one line to the next, implying the estimation of u in $L^\infty(0,T;L^1(\Omega))$ and

$$\begin{aligned}
&\int_{\{k\leq|u_\varepsilon|<k+1\}} |\nabla u_\varepsilon|^{p(x)}dxdt \\
&\leq \int_{Q_T} |\nabla T_{k_0}(u_\varepsilon)|^{p(x)}dxdt + \frac{1}{L}\int_{\{|u_\varepsilon|\geq k\}} \frac{|f_\varepsilon|}{k^{s(x)}}dxdt \\
&\leq \int_{Q_T} |\nabla T_{k_0}(u_\varepsilon)|^{p(x)}dxdt + \frac{[\|u_\varepsilon\|_{L^1(\Omega)} + \|\mu\|_{\mathcal{M}_0(Q_T)}]}{L}\sum_{k=1}^{\infty}\frac{1}{k^{s^-}} \\
&\leq C\Big[\|\mu\|_{L^1(Q_T)} + \|u_0^\varepsilon\|_{L^1(\Omega)}\Big].
\end{aligned} \tag{28}$$

These previous results produce a bound for u_ε in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$. As a result, from (24), we obtain a bound for its weak limit u in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\cap L^\infty(0,T;L^1(\Omega))$. In the same way that the proof of Proposition 2, we get a bound of u in $L^{q(x)}(Q_T)$ for each $q(x) < \dfrac{(p(x)N + p(x) - N)(s(x)+1)}{(N+1)}$, which is verified, as $|u_\varepsilon|^{s(x)+1} \leq C(1 + M(u))$ *and* $M(u) \in L^{q(x)}(Q_T)$ for each $q(x) < p(x) - \dfrac{N}{N+1}$ where $M(l)$ is defined as $M(l) = \displaystyle\int_0^l b(|y|)^{\frac{1}{p(x)-1}}dy$.

2nd **case:** If $0 \leq s(x) \leq 1$.

Let $\Gamma(x) \in \mathbb{R}$ where $\Gamma(x) > 1 - s(x)$ and $1 < q(x) < 2$, then, by applying the inequality of Hölder and (26) for a.e. $t \in (0,T)$, we have

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon(t,x)|^{q(x)}dx &= \int_\Omega \frac{|\nabla u(t,x)|^{q(x)}}{(|u(t,x)|+1)^{\frac{\Gamma(x)q(x)}{2}}}\big(|u_\varepsilon(t,x)|+1\big)^{p(x)}dxdt \\
&\leq \Big[\int_\Omega \frac{|\nabla u_\varepsilon(t,x)|^{q(x)}}{(|u_\varepsilon(t,x)|+1)^{\Gamma(x)}}dxdt\Big]^{\frac{q^-}{p^-}}\Big[\int_\Omega \big(|u_\varepsilon(t,x)|+1\big)^{\frac{\Gamma^- q^-}{p^- - q^+}}dxdt\Big]^{\frac{p^- - q^-}{p^-}}.
\end{aligned}$$

Similar to (28) and integrate over t, if $\Gamma(x) > 1 - s(x)$ and $\Gamma(x) = \dfrac{N(p^-) - q^-)}{N - q^-}$ which leads to $q(x) < \dfrac{N(s^- + 1))}{N - (1 - s^-)}$, we have by means the Sobolev's embedding theorem that

$$\Big(\int_{Q_T}|u_\varepsilon|^{q^\star(x)}dxdt\Big)^{\frac{q^-}{(q^\star)^-}} \le \int_{Q_T}|\nabla u_\varepsilon|^{q(x)}dxdt \le \Big(C+\sum_{k=1}^{\infty}\frac{c}{k^{\Gamma(x)+s(x)}}\Big)^{\frac{q^-}{p^-}}$$
$$\times\Big(\int_{Q_T}\big(|u_\varepsilon(t,x)|+1\big)^{\frac{\Gamma^- q^-}{p^- - q^+}}dxdt\Big)^{\frac{p^- - q^-}{p^-}}$$
$$\le \Big[\int_{Q_T}\big(|u_\varepsilon(t,x)|+1\big)^{\frac{\Gamma^- q^-}{p^- - q^+}}dxdt\Big]^{\frac{p^- - q^-}{p(x)}}$$
$$\le \Big[\int_{Q_T}\big(|u_\varepsilon(t,x)|+1\big)^{\Upsilon^-}dxdt\Big]^{\frac{p^- - q^-}{p^-}}$$

where $\Upsilon^- < \dfrac{p^- q^-}{p^- - q^+}$. As a consequence, one may readily get a priori estimates on u_ε in $L^{q(x)}(0,T;W_0^{1,q(x)}(\Omega))$ for each $q^- < \dfrac{Ns^- + N}{N-1+s^-}$.

We now suppose that the assumptions (10), (11), (12) and (22) are satisfied, and that the datum $\mu = f$ such that $f \in L^{r(x)}(Q_T)$ with $s(x) \ge 1 - \dfrac{(p^\star)^+}{(r^\star)^+}$, then it is possible to use the results of the above calculations to find the solution's summability and its gradient with respect to time and space. Let us consider u_ε the solution of problem (10) with (f_ε) a sequence of regular functions in $L^{r(x)}(Q_T)$ that approximate the datum μ, by (28) we infer that

$$\int_{Q_T}|\nabla u_\varepsilon|^{p(x)}dxdt \le \int_{Q_T}|\nabla T_{k_0}(u_\varepsilon)|^{p(x)}dxdt + \int_{\{|u_\varepsilon|\ge k_0\}}|\nabla u_\varepsilon|^{p(x)}dxdt$$
$$\le C+\sum_{h=1}^{\infty}\int_{\{|u_\varepsilon|\ge h\}}\frac{|f_\varepsilon|}{h^{s^-}}dxdt$$
$$\le C+\sum_{h=1}^{\infty}\sum_{j=h}^{\infty}\int_{\{j\le u_\varepsilon<j+1\}}\frac{|f_\varepsilon|}{h^{1-\frac{(p^\star)^+}{(r^\star)^+}}}dxdt$$
$$\le C+\sum_{j=0}^{\infty}\int_{\{j\le u_\varepsilon<j+1\}}|f_\varepsilon|\sum_{h=0}^{j}\frac{1}{(1+h)^{1-\frac{(p^\star)^+}{(r^\star)^+}}}dxdt.$$

As $\displaystyle\sum_{h=0}^{j}\frac{1}{(1+h)^{s^-}} \le C(1+j)^{1-s^+}$ with $0 < s(x) < 1$, by Holder's inequality and the Sobolev embedding theorem, we can easily determine that

$$\Big(\int_{Q_T}|u_\varepsilon|^{p^\star(x)}dxdt\Big)^{\frac{p^-}{(p^\star)^-}} \le \int_{Q_T}|\nabla u_\varepsilon|^{p(x)}dxdt$$
$$\le C\Big[1+\int_{Q_T}|f_\varepsilon|\Big(1+|u_\varepsilon|\Big)^{\frac{p^\star(x)}{(r^\star)^-}}dxdt\Big]$$
$$\le C\Big[\Big(\int_{Q_T}(|u_\varepsilon+1|)^{p^\star(x)}dxdt\Big)^{\frac{1}{(r^\star)^-}}+1\Big]$$

where $\frac{p^-}{(p^\star)^-} > \frac{1}{(r^\star)^-}$. Then, we simply find an a priori estimate of u_ε in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$.

Step 1 : If $1-\frac{p^\star(x)}{r^\star(x)} < s(x)$. Using $\Psi_k\big(|u_\varepsilon|^{s(x)}u_\varepsilon\big) = T_1\Big(|u_\varepsilon|^{s(x)}u_\varepsilon - T_k\big(|u_\varepsilon|^{s(x)}u_\varepsilon\big)\Big)$ as test function in the weak formulation of $(\mathcal{P}_\varepsilon)$ and reminding that $s(x) > 1-\frac{p^\star(x)}{r^\star(x)}$, we get, for a. e. $t\in[0,T]$, that

$$\int_0^T \big\langle (u_\varepsilon)_t, T_1(|u_\varepsilon|^{s(x)}u_\varepsilon - T_k(|u_\varepsilon|^{s(x)}u_\varepsilon))\big\rangle dt + (s^-+1)\int_{\{k\le |u_\varepsilon|^{s(x)+1}<k+1\}} |u_\varepsilon|^{s(x)}\Phi(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dxdt \le \int_{\{|u_\varepsilon|^{s(x)+1}\ge k\}} |f_\varepsilon| dxdt$$

Thus, by setting $\Phi_1^{s(x)}(l) = \int_0^l T_1(|\omega|^{s(x)}\omega - T_k(|\omega|^{s(x)}\omega))d\omega$ and applying the integration by parts formula, we obtain

$$\int_0^T \big\langle (u_\varepsilon)_t,\ T_1(|u_\varepsilon|^{s(x)}u_\varepsilon - T_k(|u_\varepsilon|^{s(x)}u_\varepsilon))\big\rangle dt = \int_\Omega \Phi_1^{s(x)}(u_\varepsilon)(T)dx - \int_\Omega \Phi_1^{s(x)}(u_\varepsilon)(0)dx,$$

As $\Phi_1^{s(x)}(l)\le |l|$ and $\Phi_1^{s(x)}(u_\varepsilon)(T)\ge 0$, then the first member is positive,

$$\big(s^-+1\big)\int_{\{k\le |u_\varepsilon|^{s(x)+1}<k+1\}} |u_\varepsilon|^{s(x)}\Phi(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dxdt \le \int_{\{|u_\varepsilon|^{s(x)+1}\ge k\}} |f_\varepsilon| dxdt + C\int_\Omega \Phi_1^{s(x)}(u_\varepsilon)(0)dx.$$

Hence, from Lemma 3.2 and (22), we get

$$\int_{\{k\le |u_\varepsilon|^{s(x)+1}<k+1\}} |\nabla(|u_\varepsilon|^{s(x)}u_\varepsilon)|^{p(x)}dxdt \le C\int_{\{|u_\varepsilon|^{s(x)+1}\ge k\}} |f_\varepsilon|dxdt + C\int_{\{|u_\varepsilon|^{s(x)+1}\ge k\}} |u_0|dx \ \ for\ all\ k\ge k_1 = m_0^{s^-+1}.$$

Therefore, we get, as in [12, Theorem 3], the estimate of $|u_\varepsilon|^{s(x)+1}$ in $\frac{(N(p(x)-1)+p(x))r(x)}{N+p(x)-r(x)p(x)}$ and the desired (higher) summability of u_ε in $L^{\frac{(N(p(x)+1)+p(x))r(x)(s(x)+1)}{N+p(x)-r(x)p(x)}}(Q_T)$.

Step 2 : If $0<s(x)<1-\frac{p^\star(x)}{r^\star(x)}$.

Let $q(x)<2,\ t\in[0,T]$ and $\Gamma(x)$ be a function such that $\Gamma(x)<1-s(x)$. Using the previous procedure, and taking the supremum for t in $(0,T)$, we obtain

$$
\begin{aligned}
&C\|u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \int_{Q_T} |\nabla u_\varepsilon|^{q(x)} dxdt \\
&\leq \Big(\sum_{h=1}^{\infty} \sum_{j=h}^{\infty} \int_{\{j \leq |u_\varepsilon| < j+1\}} \frac{|f_\varepsilon|}{h^{\Gamma(x)+s(x)}} dxdt \Big)^{\frac{q^-}{p^-}} \\
&\quad \times \Big(\int_{Q_T} (1+|u_\varepsilon|)^{\frac{\Gamma(x)q(x)}{p(x)-q(x)}} dxdt \Big)^{\frac{p^- - q^-}{p^-}} \\
&\leq C \Big(1 + \int_{Q_T} |f_\varepsilon| \Big(1+|u_\varepsilon|\Big)^{1-(s(x)+\Gamma(x))} dxdt \Big)^{\frac{q^-}{p^-}} \\
&\quad \times \Big(\int_{Q_T} (1+|u_\varepsilon|)^{\frac{\Gamma(x)q(x)}{p(x)-q(x)}} dxdt \Big)^{\frac{p^- - q^-}{p^-}} \\
&\leq C \|f_\varepsilon\|_{L^{r(x)}(Q_T)}^{\frac{q^-}{p^-}} \Big(\int_{Q_T} \Big(1+|u_\varepsilon|\Big)^{\big(1-(s(x)+\Gamma(x))\big) r^\star(x)} dxdt \Big)^{\frac{q^-}{p^-(r^\star)^-}} \\
&\quad \times \Big(\int_{Q_T} (1+|u_\varepsilon|)^{\frac{\Gamma^- q^-}{p^- - q^+}} dxdt \Big)^{\frac{p^- - q^-}{p^-}}.
\end{aligned}
$$

Here, we choose $\Gamma^- = \frac{N(p^- - q^-)}{N - q^+}$ where $q^- = \frac{(N(p^-+1)+p^-)r^-(s^-+1)}{N+p^-(1-r^+)}$, to get $\frac{\Gamma^- q^-}{p^- - q^+} = \frac{Nq^-}{N-q^+} = (q^\star)^-$, which gives that

$$
\|u_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C,
$$

and

$$
\Big(\int_{Q_T} |u_\varepsilon|^{q^*(x)} dxdt \Big)^{\frac{q^-}{(q^\star)^-}} \leq \int_{Q_T} |\nabla u_\varepsilon|^{q(x)} dxdt \leq C \Big(\int_{Q_T} \Big(1 + |u_\varepsilon|\Big)^{q^\star(x)} dxdt \Big)^{\frac{p^- - q^-}{p^-}},
$$

Therefore, we arrive at the desired estimates of u_ε in $L^{q^-}(0,T;W_0^{1,q(x)}(\Omega))$ for each $q(x) < \dfrac{Nr(x)(s(x)+1)}{N - r(x)(1-s(x))}$. □

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