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EXISTENCE OF ANTI-PERIODIC SOLUTIONS FOR φ-CAPUTO FRACTIONAL p-LAPLACIAN PROBLEMS VIA TOPOLOGICAL DEGREE METHODS

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Abstract. The main crux of this paper is to investigate the existence of solutions for anti-periodic nonlinear φ-Caputo fractional differential equations with p-Laplacian operators. Our main results are proved through the application of topological degree methods of condensing maps and several properties of φ-Caputo fractional calculus paired with measures of noncompactness. To show the practical significance of our theoretical results, we provide an nontrivial example at the end.

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1. Introduction

The development of fractional calculus has given rise to a new branch of applied mathematics known as fractional differential equations theory. This theory has been widely applied in many fields, such as biophysics, control theory, and biology, among others. Several types of fractional derivatives, including Riemann-Liouville, Caputo, Hadamard, and Hilfer, have been studied extensively in the literature. In a prior publication, Almeida [2] examined the existence and uniqueness of solutions to nonlinear fractional differential equations utilizing Caputo fractional derivatives. He introduced the φ-Caputo fractional derivative, a generalized fractional derivative, which has been applied in a variety of contexts, depending on the selection of the function φ. Meanwhile, other researchers have employed different fixed point theorems and measures of noncompactness to establish the existence of solutions to fractional differential equations involving Caputo-type fractional derivatives with respect to other functions, including anti-periodic and integral boundary conditions, see [6,8]. The use of φ-Caputo fractional differential equations that feature p-Laplacian operators has garnered significant interest in

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recent times due to their wide range of practical applications. Numerous studies have been conducted to investigate the existence and uniqueness of solutions for such equations. Readers interested in learning more about this topic can refer to the following articles [3–5,10–12].

Taking inspiration from the studies mentioned above, our current research endeavors to investigate the existence of solutions to a particular type of fractional differential equation with p-Laplacian operator, specifically in the context of the φ-Caputo formulation. We consider the following fractional problem

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\varphi}\Phi_p\left({}^{C}D_{0^+}^{\beta,\varphi}U(x)\right) = G(x,U(x)), \quad x \in \mathcal{I} = [0,X], \\ \\ U(0) = -U(X). \end{cases} \tag{1}$$

Where ${}^{C}D_{0^+}^{\alpha,\varphi}$ and ${}^{C}D_{0^+}^{\beta,\varphi}$ are the $\varphi-$Caputo fractional derivatives of orders α and β in $(0,1)$ such that $1 < \alpha + \beta < 2$ respectively, G is a continuous function and Φ_p is the p-Laplacian operator, i.e $\Phi_p(U) = |U|^{p-2}U$ such that $p > 1$.

To provide a comprehensive understanding of our research, we have structured the remaining work as follows. Firstly, in Section 2, we present fundamental concepts related to φ-fractional integral and φ-Caputo fractional derivative, which are essential for our subsequent analysis. In Section 3, we employ topological degree theory for condensing maps to establish the existence of solutions to the anti-periodic problem (1). In Section 4, we provide a practical example to demonstrate the applicability of our approach, and finally, we summarize our study in Section 5.

2. Preliminaries

This section is devoted to introducing and defining the notations, concepts, and properties related to φ-fractional derivatives and φ-fractional integrals. Interested readers can find further details in the following references [1,7].

Notations

- Let $\mathcal{E}$ be a Banach space, and let $\mathfrak{d}_{\mathcal{E}}$ denote the collection of all non-empty and bounded subsets of $\mathcal{E}$.
- Let Δ a non-empty subset of $\mathfrak{d}_{\mathcal{E}}$. We use $\overline{\Delta}$ to denote the the closure of Δ and $conv(\Delta)$ for the convex hull of Δ .
- Let B_η be a closed ball centered at 0 with a radius of $\eta > 0$.
- The set of continuous functions mapping $\mathcal{I}$ to $\mathbb{R}$ is represented by $C(\mathcal{I},\mathbb{R})$ and is furnished with the norm.

$$\parallel U \parallel = \sup_{x\in[0,X]} \mid U(x) \mid .$$

- The set of Lebesgue integrable functions mapping $\mathcal{I}$ to $\mathbb{R}$ is represented by $\mathcal{L}^1(\mathcal{I},\mathbb{R})$ and is furnished with the norm.

$$\| U \|_{\mathcal{L}^1} = \int_0^X | U(x) | \, dx.$$

Definition 2.1. [2] *Given $\alpha > 0$, $G \in \mathcal{L}^1(\mathcal{I}, \mathbb{R})$ and $\varphi(x) : \mathcal{I} \longrightarrow \mathbb{R}$ such that $\varphi'(x) > 0$, the φ-Riemann-Liouville fractional integral of order α of G at x is defined as:*

$$I_{0^+}^{\alpha,\varphi} G(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \varphi'(t)(\varphi(x) - \varphi(t))^{\alpha-1} G(t) dt,$$

where $\Gamma(.)$ is the Gamma function.

Definition 2.2. [2]*Let $\alpha > 0$, $G \in C^n(\mathcal{I}, \mathbb{R})$ and $\varphi(x) : \mathcal{I} \longrightarrow \mathbb{R}$ such that $\varphi'(x) > 0$, the φ-Caputo fractional derivative of order α of G at x is defined by*

$$^C D_{0^+}^{\alpha,\varphi} G(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \varphi'(t)(\varphi(x) - \varphi(t))^{n-\alpha-1} G_\varphi^{[n]}(t) dt,$$

where

$$G_\varphi^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n G(t),$$

and $n = [\alpha] + 1$, where $[\alpha]$ is the integer part of α.

Remark 2.3. *If $\alpha \in (0, 1)$, then we have the following expressions for the φ-Caputo fractional derivative at order α of a function G at x*

$$^C D_{0^+}^{\alpha,\varphi} G(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (\varphi(x) - \varphi(t))^{\alpha-1} G'(t) dt,$$

Proposition 2.4. [2] *If $\alpha > 0$ and $G \in C^n(\mathcal{I}, \mathbb{R})$, then we have the following properties for the φ-Caputo fractional derivative and the φ-Riemann-Liouville fractional integral*

(1) $^C D_{0^+}^{\alpha,\varphi} I_{0^+}^{\alpha,\varphi} G(x) = G(x)$.

(2) $I_{0^+}^{\alpha,\varphi} \, {}^C D_{0^+}^{\alpha,\varphi} G(x) = G(x) - \displaystyle\sum_{i=0}^{n-1} \frac{G_\varphi^{[i]}(0)}{i!} (\varphi(x) - \varphi(0))^i$.

(3) $I_{0^+}^{\alpha,\varphi}$ *is a linear and bounded operator defined from $\mathcal{C}(\mathcal{I}, \mathbb{R})$ into $\mathcal{C}(\mathcal{I}, \mathbb{R})$.*

Proposition 2.5. [2] *Let $\alpha \geq 0$, $\theta > 0$, and $x > 0$. Then, the following statements hold:*

(1) $I_{0^+}^{\alpha,\varphi} (\varphi(x) - \varphi(0))^{\theta-1} = \dfrac{\Gamma(\theta)}{\Gamma(\theta+\alpha)} (\varphi(x) - \varphi(0))^{\alpha+\theta-1}$.

(2) $D_{0^+}^{\alpha,\varphi} (\varphi(x) - \varphi(0))^{\theta-1} = \dfrac{\Gamma(\theta)}{\Gamma(\theta-\alpha)} (\varphi(x) - \varphi(0))^{\alpha-\theta-1}$.

(3) $D_{0^+}^{\alpha,\varphi} (\varphi(x) - \varphi(0))^m = 0$, *for all $m \in \mathbb{N}$ such that $m < n$.*

Lemma 2.6. [11] *The $p-$Laplacian operator Φ_p verifies the following properties*

(1) *If $1 < p < 2$, $\mathcal{F}\mathcal{G} > 0$ and $|\mathcal{F}|, |\mathcal{G}| \geqslant m > 0$, then*

$$|\Phi_p(\mathcal{F}) - \Phi_p(\mathcal{G})| \leqslant (p-1) m^{p-2} |\mathcal{F} - \mathcal{G}|.$$

(2) *If* $p > 2, \mathcal{F}\mathcal{G} > 0$ *and* $|\mathcal{F}|, |\mathcal{G}| \leqslant M$, *then*

$$|\Phi_p(\mathcal{F}) - \Phi_p(\mathcal{G})| \leqslant (p-1)M^{p-2}|\mathcal{F} - \mathcal{G}|.$$

(3) Φ_p *is invertible with* $\Phi_p^{-1}(\mathcal{F}) = \Phi_{p'}(\mathcal{F})$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.7. [9] *The Kuratowski measure of non-compactness* ρ *is a map defined on* $\mathfrak{d}_{\mathcal{E}}$ *by :*

$$\rho(\Delta) = \inf\{\ \varrho > 0 : \Delta \text{ accepts a finite cover by sets of diameter} \leqslant \varrho\}.$$

Proposition 2.8. [9] *The Kuratowski measure of noncompactness* ρ *satisfies the following properties*

(1) $\forall \mu \in \mathbb{R}$, $\rho(\mu\Delta) = |\mu|\rho(\Delta)$.

(2) $\rho(\Delta_1 + \Delta_2) \leqslant \rho(\Delta_1) + \rho(\Delta_2)$.

(3) *If* $\Delta_1 \subset \Delta_2$, *then* $\rho(\Delta_1) \leqslant \rho(\Delta_2)$.

(4) $\rho(\Delta) = \rho(\overline{\Delta}) = \rho(conv\Delta)$

(5) Δ *relatively compact* $\Leftrightarrow\ \rho(\Delta) = 0$.

Definition 2.9. [9] *A continuous bounded map* $\xi : \Delta \subset \mathcal{E} \to \mathcal{E}$ *is said to be* ρ*-Lipschitz if a value of* $\lambda \geqslant 0$ *exists that satisfies the condition*

$$\rho(\xi(\mathcal{D})) \leqslant \lambda\rho(\mathcal{D}), \quad for\ all \quad \mathcal{D} \subset \Delta.$$

If $\lambda < 1$, *we refer to the function* ξ *as a strict* ρ*-contraction.*

Definition 2.10. [9] *The map* ξ *is said to be* ρ*-condensing if* $\xi(\rho(\mathcal{D})) < \xi(\mathcal{D})$, *for all bounded subset* $\mathcal{D}$ *of* Δ *with* $\rho(\mathcal{D}) > 0$. *In other words,*

$$\rho(\xi(\mathcal{B})) \geqslant \rho(\mathcal{D}) \Rightarrow \rho(\mathcal{D}) = 0.$$

Lemma 2.11. [9] *Let* ξ *and* ζ *be two* ρ*-Lipschitz operators with constants* λ_1 *and* λ_2, *respectively. Then, the operator* $\xi + \zeta$ *is* ρ*-Lipschitz with constant* $\lambda_1 + \lambda_2$.

Lemma 2.12. [9] *If* $\xi : \Delta \to \mathcal{E}$ *is a compact operator, then it is* ρ*-Lipschitz with constant* $\lambda = 0$.

Lemma 2.13. [9] *If* $\xi : \Delta \to \mathcal{E}$ *is Lipschitz with constant of* λ, *then it is also* ρ*-Lipschitz with the constant* λ.

Theorem 2.14. [13] *Suppose* $\xi : \Delta \to \mathcal{E}$ *is a* ρ*-condensing function and let*

$$\mathbb{S}_\varepsilon = \{U \in \mathcal{E} : U = \varepsilon\xi U, \quad for\ some\ \varepsilon \in [0,1]\}.$$

If $\mathbb{S}_\varepsilon$ *is bounded in* $\mathcal{E}$, *then there exists a positive number* η *such that* $\mathbb{S}_\varepsilon$ *is a subset of the ball* B_η, *then*

$$\mathbf{deg}(\mathbb{I} - \varepsilon\,\xi, B_\eta, 0) = 1, \quad \forall\varepsilon \in [0,1].$$

As a result, the operator ξ *has at least one fixed point, and the set of all fixed points of* ξ *is contained in* B_η.

3. Main results

In order to establish the existence result for problem (1), it is necessary to demonstrate the following crucial lemma.

Lemma 3.1. *A function $U \in C(\mathcal{I}, \mathbb{R})$ is a solution to the p-Laplacian problem (1) if and only if it satisfies the following fractional integral equation*

$$U(x) = \int_0^x \frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}}{\Gamma(\beta)} \Phi_{p'}\Big(\int_0^t \frac{\varphi'(r)(\varphi(t)-\varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, U(r))dr \Big) dt \tag{2}$$

$$- \int_0^X \frac{\varphi'(t)(\varphi(X)-\varphi(t))^{\beta-1}}{2\Gamma(\beta)} \Phi_{p'}\Big(\int_0^t \frac{\varphi'(r)(\varphi(t)-\varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, U(r))dr \Big) dt. \tag{3}$$

Proof. Suppose that $U \in C(\mathcal{I}, \mathbb{R})$ is a solution of the problem (1). By applying the φ-fractional integral $I_{0^+}^{\alpha,\varphi}$ to both sides of the fractional differential equation (1) and utilizing Proposition 2.4, we obtain the following result:

$$\Phi_p \left({}^C D_{0^+}^{\beta,\varphi} U(x) \right) = d_0 + I_{0^+}^{\alpha,\varphi} G(x, U(x)),$$

Such that $d_0 \in \mathbb{R}$, it follows that

$$\Phi_p \left({}^C D_{0^+}^{\beta,\varphi} U(x) \right) = d_0 + \int_0^x \frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\alpha-1}}{\Gamma(\alpha)} G(t, U(t))dt, \tag{4}$$

When $x = 0$, we get : $\Phi_p \left({}^C D_{0^+}^{\beta,\varphi} U(0) \right) = d_0$

In other hand ${}^C D_{0^+}^{\beta,\varphi} U(0) = 0$ then $d_0 := 0$. By applying the operator $\Phi_{p'}$ on the equation (4) we get

$${}^C D_{0^+}^{\beta,\varphi} U(x) = \Phi_{p'} \left(I_{0^+}^{\alpha,\varphi} G(x, U(x)) \right), \tag{5}$$

After using the φ-fractional integral $I_{0^+}^{\beta,\varphi}$ on both sides of equation (5), we obtain

$$U(x) = d_1 + I_{0^+}^{\beta,\varphi} \, \Phi_{p'} \left(I_{0^+}^{\alpha,\varphi} G(x, U(x)) \right),$$

Such that $d_1 \in \mathbb{R}$, then

$$U(x) = d_1 + \int_0^x \frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}}{\Gamma(\beta)} \Phi_{p'} \left(I_{0^+}^{\alpha,\varphi} G(t, U(t) \right) dt. \tag{6}$$

since $U(0) = -U(X)$, we can find that

$$d_1 = - \int_0^X \frac{\varphi'(t)(\varphi(X)-\varphi(t))^{\beta-1}}{2\Gamma(\beta)} \Phi_{p'} \Big(I_{0^+}^{\alpha,\varphi} G(t, U(t)) \Big) dt,$$

with

$$I_{0^+}^{\alpha,\varphi} G(t, U(t)) = \int_0^t \frac{\varphi'(r)(\varphi(t)-\varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, U(r))dr.$$

Substituting d_1 in (6), we get the fractional integral equation (2).

The proof of the converse can be finalized by carrying out a straightforward computation.

□

Our primary result, which pertains to the existence of solutions for problem (1), will be presented next. For this purpose, we introduce the following assumptions:
Let $U, V \in C(\mathcal{I}, \mathbb{R})$ and $x \in \mathcal{I}$

(A_1) The function G fulfills the following condition, for some positive constant Π:

$$|G(x,U) - G(x,V)| \leqslant \Pi \mid U - V \mid .$$

(A_2) The function G satisfies the following condition, for some positive constants K and J:

$$|G(x,U)| \leqslant K \mid U \mid^{\gamma} + J.$$

We define two operators $\mathcal{T}_2$ and $\mathcal{T}_1$ as mappings from $C(\mathcal{I}, \mathbb{R})$ to $C(\mathcal{I}, \mathbb{R})$ by the following expressions:

$$\mathcal{T}_1 U(x) = \int_0^x \frac{-\varphi'(t)(\varphi(T) - \varphi(t))^{\beta-1}}{2\Gamma(\beta)} \Phi_{p'}\Big(\int_0^t \frac{\varphi'(r)(\varphi(t) - \varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, U(r)) dr \Big) dt, \tag{7}$$

and

$$\mathcal{T}_2 U(x) = \int_0^x \frac{\varphi'(t)(\varphi(x) - \varphi(t))^{\beta-1}}{\Gamma(\beta)} \Phi_{p'}\Big(\int_0^t \frac{\varphi'(r)(\varphi(t) - \varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, U(r)) dr \Big) dt. \tag{8}$$

We can express the fractional integral equation (3.1) as an operator equation by defining the operator $\mathcal{T}$ from $C(\mathcal{I}, \mathbb{R})$ to $C(\mathcal{I}, \mathbb{R})$ as

$$\mathcal{T} U(x) = \mathcal{T}_1 U(x) + \mathcal{T}_2 U(x), \quad x \in \mathcal{I}. \tag{9}$$

Lemma 3.2. *The Lipschitz constant of the operator $\mathcal{T}_1$ is bounded by $\Pi\omega$. Furthermore, the operator $\mathcal{T}_1$ satisfies the following inequality:*

$$\|\mathcal{T}_1 U\| \leqslant K \|U\|^{\gamma} + J, \ for\ all\ U \in C(\mathcal{I},\ \mathbb{R}). \tag{10}$$

Proof. Let $U, V \in \mathcal{C}(\mathcal{I}, \mathbb{R})$, then we have

$$|\mathcal{T}_1 U(x) - \mathcal{T}_1 V(x)| \leqslant \int_0^x \frac{\varphi'(t)(\varphi(T) - \varphi(t))^{\beta-1}}{2\Gamma(\beta)} \Big| \Phi_{p'}(\int_0^t \frac{\varphi'(r)(\varphi(t) - \varphi(r))^{\alpha-1}}{\Gamma(\alpha)} \times$$
$$G(r, U(r)) dr - \Phi_{p'}(\int_0^t \frac{\varphi'(r)(\varphi(t) - \varphi(r))^{\alpha-1}}{\Gamma(\alpha)} G(r, V(r)) ds) \Big| dt.$$

by using Lemma 2.6 and the the hypothesis (A_1) we get

$$|\mathcal{T}_1 U(x) - \mathcal{T}_1 V(x)| \leqslant \int_0^x \frac{(p'-1) M^{p'-2} \Pi \|U - V\| \varphi'(t)(\varphi(X) - \varphi(t))^{\beta-1}}{2\Gamma(\beta)} \times$$
$$\Big(\int_0^t \frac{\varphi'(r)(\varphi(t) - \varphi(r))^{\alpha-1}}{\Gamma(\alpha)} dr \Big) dt.$$

then

$$|\mathcal{T}_1 U(x) - \mathcal{T}_1 V(x)| \leqslant (p'-1) M^{p'-2} \Pi \|U - V\| \frac{(\varphi(X) - \varphi(0))^{\alpha+\beta}}{2\Gamma(\alpha+\beta+1)}$$

taking supremum over x, we obtain

$$\|\mathcal{T}_1 U - \mathcal{T}_1 V\| \leqslant \Pi\omega\|U - V\|, \text{ with } \omega = \frac{(p'-1)M^{p'-2}(\varphi(X)-\varphi(0))^{\alpha+\beta}}{2\Gamma(\alpha+\beta+1)}$$

We can conclude that $\mathcal{T}_1$ is a Lipschitz operator with constant $\Pi\omega$

We can establish the inequality (10) by assuming that U belongs to $C(\mathcal{I}, \mathbb{R})$ and utilizing Lemma (2.6) along with the condition (A_2).

$$|\mathcal{T}_1 U(x)| \leqslant (p'-1)M^{p'-2}(K\|U\|^\gamma + J)\int_0^X \frac{\varphi'(t)(\varphi(X)-\varphi(t))^{\alpha-1}}{2\Gamma(\alpha)} \times \Big(\int_0^t \frac{\varphi'(r)(\varphi(t)-\varphi(r))^{\beta-1}}{\Gamma(\beta)} dr\Big) dt.$$

Then finally we get

$$\|\mathcal{T}_1 U\| \leqslant \omega(K\|U\|^\gamma + J).$$

□

Lemma 3.3. *The operator $\mathcal{T}_2$ is continuous and fulfills the ensuing inequality:*

$$\|\mathcal{T}_2 U\| \leqslant 2\omega(K\|U\|^\gamma + J), \ for\ all\ U \in C(\mathcal{I},\ \mathbb{R}). \tag{11}$$

Proof. Consider a sequence $U_N \in C(\mathcal{I}, \mathbb{R}) \longrightarrow U \in C(\mathcal{I}, \mathbb{R})$, i.e., $\exists\delta > 0$ such that $\forall N > 0$, $\|U_N\| \leq \delta$, and $\|U\| \leqslant \delta$. By the continuity of the function G, we can obtain:

$$\lim_{n\to\infty} G(x, U_N(x)) = G(x, U(x)).$$

Furthermore, from assumption (A_1), we have:

$$\frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}(\varphi(t)-\varphi(0))^\alpha}{\Gamma(\beta)\Gamma(\alpha+1)}\|G(t, U_n(t)) - G(t, U(t))\| \leqslant (K\delta^\gamma + J)\frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}(\varphi(t)-\varphi(0))^\alpha}{\Gamma(\beta)\Gamma(\alpha+1)},$$

We observe that the function $t \mapsto \dfrac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}(\varphi(t)-\varphi(0))^\alpha}{\Gamma(\beta)\Gamma(\alpha+1)}$ is integrable over $[0, x]$. Therefore, by applying the Lebesgue dominated convergence theorem, we obtain:

$$\lim_{n\mapsto+\infty}\int_0^x \frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}(\varphi(t)-\varphi(0))^\alpha}{\Gamma(\beta)\Gamma(\alpha+1)}\|G(t, U_N(t)) - G(t, U(t))\| dt = 0,$$

it follows that

$$\lim_{N\mapsto+\infty} \| \mathcal{T}_2 U_N - \mathcal{T}_2 U \| = 0.$$

This demonstrates that $\mathcal{T}_2$ is a continuous operator on $C(\mathcal{I}, \mathbb{R})$.

To establish the inequality (11), let $U \in C(\mathcal{I}, \mathbb{R})$. We have the following.

$$|\mathcal{T}_2U(x)| \leqslant \int_0^x \frac{\varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}}{\Gamma(\beta)} \left|\Phi_{p'}\Big(\int_0^t \frac{\varphi'(r)(\varphi(t)-\varphi(r))^{\alpha-1}}{\Gamma(\alpha)}G(r,U(r))dr\Big)\right| dt$$

By utilizing Lemma 2.6 and the assumption (A_2), we can obtain the following.

$$|\mathcal{T}_2U(x)| \leqslant \frac{(p'-1)M^{p'-2}(K\|U\|^{\gamma}+J)}{\Gamma(\beta)\Gamma(\alpha+1)} \int_0^x \varphi'(t)(\varphi(x)-\varphi(t))^{\beta-1}(\varphi(t)-\varphi(0))^{\alpha}dt,$$

That follows that

$$\| \mathcal{T}_2U \| \leqslant 2\omega(K\|U\|^{\gamma}+J).$$

□

Lemma 3.4. *The operator $\mathcal{T}_2$ mapping from $C(\mathcal{I},\mathbb{R})$ to $C(\mathcal{I},\mathbb{R})$ is a compact operator.*

Proof. To demonstrate the compactness of $\mathcal{T}_2$, we must establish that $\mathcal{T}_2(B_\eta)$ is relatively compact in $C(\mathcal{I},\mathbb{R})$. To that end, suppose $U \in B_\eta$. Using the inequality (11), we obtain:

$$\| \mathcal{T}_2U \| \leqslant 2\omega(K\eta^{\gamma}+J) := \Theta.$$

Therefore, we can conclude that $\mathcal{T}_2(B_\eta)$ is uniformly bounded as it is contained in B_Θ.

Let us now demonstrate that $\mathcal{T}_2(B_\eta)$ is equicontinuous on $\mathcal{I}$.

Let $U \in \mathcal{T}_2(B_\eta)$ and $x_1, x_2 \in \mathcal{I}$ such that $x_1 < x_2$, then we have

$$\begin{aligned}
\mathcal{T}_2U(x_2) - \mathcal{T}_2U(x_1) &= \int_0^{x_2} \frac{\varphi'(t)(\varphi(x_2)-\varphi(t))^{\beta-1}}{\Gamma(\beta)}\Phi_{p'}I_{0^+}^{\alpha,\varphi}G(t,U(t))dt \\
&- \int_0^{x_1} \frac{\varphi'(t)(\varphi(x_1)-\varphi(t))^{\beta-1}}{\Gamma(\beta)}\Phi_{p'}I_{0^+}^{\alpha,\varphi}G(t,U(t))dt \\
&= \int_0^{x_1} \frac{\varphi'(t)\Big[(\varphi(x_2)-\varphi(t))^{\beta-1}-(\varphi(x_1)-\varphi(t))^{\beta-1}\Big]}{\Gamma(\beta)}\Phi_{p'}I_{0^+}^{\alpha,\varphi}G(t,U(t))dt \\
&+ \int_{x_1}^{x_2} \frac{\varphi'(t)(\varphi(x_2)-\varphi(t))^{\beta-1}}{\Gamma(\beta)}\Phi_{p'}I_{0^+}^{\alpha,\varphi}G(t,U(t))dt.
\end{aligned}$$

By utilizing Lemma 2.6 and the assumption (A_2), we can obtain the following.

$$\begin{aligned}
|\mathcal{T}_2U(x_2) - \mathcal{T}_2U(x_1)| &\leqslant (p'-1)M^{p'-2}(K \mid U \mid^{\gamma} + J)\frac{(\varphi(X)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \times \\
&\Bigg(\left|\int_0^{x_1} \frac{\varphi'(t)\Big[(\varphi(x_2)-\varphi(t))^{\beta-1}-(\varphi(x_1)-\varphi(t))^{\beta-1}\Big]}{\Gamma(\beta)}\right| + \\
&\left|\int_{x_1}^{x_2} \frac{\varphi'(t)(\varphi(x_2)-\varphi(t))^{\beta-1}}{\Gamma(\beta)}\right|\Bigg) \\
&\leqslant \frac{(p'-1)M^{p'-2}(K\eta^{\gamma}+J)(\varphi(X)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\
&\times \Big((\varphi(x_2)-\varphi(0))^{\beta} - (\varphi(x_1)-\varphi(0))^{\beta}\Big).
\end{aligned}$$

Since φ is a continuous function, $\lim_{x_1 \to x_2} |\mathcal{T}_2 U(x_1) - \mathcal{T}_2 U(x_2)| = 0$. which shows that $\mathcal{T}_2(B_\eta)$ is equicontinuous.

As a result of applying the Arzela-Ascoli Theorem [14], it can be inferred that the set $\mathcal{T}_2(B_\eta)$ is both uniformly bounded and equicontinuous. Therefore, we can conclude that $\mathcal{T}_2(B_\eta)$ is relatively compact, which in turn indicates that the operator $\mathcal{T}_2$ is compact.

□

Corollary 3.5. *The operator $\mathcal{T}_2$ from $C(\mathcal{I}, \mathbb{R})$ to $C(\mathcal{I}, \mathbb{R})$ is ρ-Lipschitz with a constant of zero.*

Proof. By leveraging the compactness of $\mathcal{T}_2$ and Lemma 2.12, it can be concluded that the operator $\mathcal{T}_2$ is ρ-Lipschitz with a constant of zero. □

Theorem 3.6. *If the conditions $(A_1) - (A_2)$ are verified and $\Pi\omega < 1$, then there exists at least one solution $U \in C(\mathcal{I}, \mathbb{R})$ of (1) and the set of all its solutions is bounded in $U \in C(\mathcal{I}, \mathbb{R})$.*

Proof. The operators $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}$ are all continuous and bounded. In other hand according to Lemma 3.2, $\mathcal{T}_1$ is ρ-Lipschitz with the constant $\Pi\omega$, while Corollary 3.5 indicates that $\mathcal{T}_2$ is ρ-Lipschitz with a zero constant. By applying Lemma 2.11, we can conclude that $\mathcal{T}$ is a strict ρ-contraction with the constant $\Pi\omega < 1$. Let us consider the set:

$$\mathbb{S}_\varepsilon = \{U \in C(\mathcal{I},\ \mathbb{R}) : U = \varepsilon \mathcal{T} U \ for\ some\ \varepsilon \in [0, 1]\}.$$

We aim to show that $\mathbb{S}_\varepsilon$ is bounded in $C(I, \mathbb{R})$. To prove this, suppose $U \in \mathbb{S}_\varepsilon$, then $U = \varepsilon \mathcal{T} U = \varepsilon(\mathcal{T}_1 U + \mathcal{T}_2 U)$, which implies that

$$\|U\| = \varepsilon \|\mathcal{T} U\| \leqslant \varepsilon(\|\mathcal{T}_1 U\| + \|\mathcal{T}_2 U\|),$$

by using Lemmas 3.2 and 3.3 we have

$$\|U\| \leqslant 3\omega(K\|U\|^\gamma + J). \tag{12}$$

From (12) the set $\mathcal{S}_\varepsilon$ is bounded in $C(\mathcal{I}, \mathbb{R})$ and by using theorem (2.14)

we conclude that the operator $\mathcal{T}$ possesses at least one fixed point, which is the solution to the $p-$Laplacian problem (1) and the solutions set of $\mathcal{T}$ is bounded in $C(\mathcal{I}, \mathbb{R})$. □

4. An illustrative example

To demonstrate our main result, we will provide an example in this section. Let's examine the following anti-periodic fractional boundary value problem:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2},x}\left(\Phi_{\frac{3}{2}}\left({}^{C}D_{0^+}^{\frac{3}{4},x}U(x)\right)\right) = \frac{e^{-x}}{9+e^{x}} \times \frac{|U(x)|}{1+|U(x)|}, \qquad x \in \mathcal{I} = [0,1], \\ \\ U(0) = -U(1). \end{cases} \tag{13}$$

We have

$$\alpha = \frac{1}{2},\ \beta = \frac{3}{4}, \quad X = 1, \quad \varphi(x) = x,\ p = \frac{3}{2},\ p' = 3,\ M = \frac{1}{10}$$

And the function G is given by

$$G(x,U) = (\frac{e^{-x}}{9+e^{x}})\frac{|U|}{1+|U|}$$

It is evident that G is continuous, and we can observe that: :

$$|\ G(x,U)|\ = \frac{e^{-x}}{9+e^{x}}\left|\frac{|U|}{1+|U|}\right| \leqslant \frac{1}{10}|U|$$

Therefore, with $K = \frac{1}{10}$, $J = 0$, and $\gamma = 1$, condition (A_2) is satisfied. On the other hand, we have:

$$|\ G(x,U) - G(x,V)|\ \leqslant \frac{1}{10}\,|\ U - V|$$

Hence the condition (A_1) holds with $\Pi = \frac{1}{10}$.

On other hand

$$\omega\Pi = \frac{1}{10}(3-1)(\frac{1}{10})^{3-2}\frac{(1-0)^{\frac{1}{2}+\frac{3}{4}}}{\Gamma(\frac{1}{2}+\frac{3}{4}+1)} < 1.$$

By utilizing Theorem 3.6, we can deduce that there is at least one solution $U \in C(\mathcal{I},\mathbb{R})$ to problem (13). Additionally, the set of all solutions to the problem is bounded in $C(\mathcal{I},\mathbb{R})$.

5. Conclusion

In this paper, we have established the existence of solutions for fractional differential equations with anti-periodic conditions, where the fractional derivative is defined in terms of the φ-Caputo operator. The proof of our main result is based on a fixed point theorem by Isaia 2.14, which was derived using coincidence degree theory for condensing maps. Additionally, we provide an example to illustrate the application of our theoretical results.

Conflict of interest

The authors declare that they have no conflict of interest.

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