Asia Pac. J. Math. 2023 **10**:14

WEAK SOLUTIONS FOR A QUASILINEAR ELLIPTIC AND PARABOLIC PROBLEMS INVOLVING THE $(p(x), q(x))$-LAPLACIAN OPERATOR

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Received Apr. 14, 2023

Abstract. In the present paper, in view of the topological degree methods and the theory of the variable exponent Sobolev spaces, we discuss some quasilinear problems for elliptic and parabolic equations involving the $(p(x), q(x))$-Laplacian operator. Under certain assumptions, we establish the existence of at least one weak solution to these problems. Our results extends some recent work in the literature.

2020 Mathematics Subject Classification. 35J60, 35J70, 35D30, 47H11.

Key words and phrases. $(p(x), q(x))$-Laplacian operators; topological degree methods; variable exponent Sobolev spaces.

1. Introduction and motivation

The study of differential equations with with nonstandard $p(x)$-Laplacian operator ($(p(x), q(x))$-Laplacian operator) is an attractive topic and has been the object of considerable attention in recent years (see [33]). Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents. In the subject of fluid mechanics, for example, Rajagopal and Růžička recently developed a very interesting model for these fluids in [32]. Other applications relate to image processing [1,11], elasticity problems [35], the flow in porous media [4], and problems in the calculus of variations involving variational integrals with nonstandard growth [2,10,16–19].

Here and in the sequel, we will assume that Ω is a bounded domain in $\mathbb{R}^N (N > 1)$, with a Lipschitz boundary denoted by $\partial\Omega$, $T > 0$ is a fixing time, $p(x), \xi(x) \in C_+(\overline{\Omega})$, ω, υ and σ are three real parameters.

DOI: 10.28924/APJM/10-14

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In this paper, we study the existence of the weak solution to the following quasilinear problems:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_{p(x)} u - \Delta_{q(x)} u = \phi(x,t) & \text{in } \Omega_T := \Omega \times (0,T), \\ u(x,t) \ = 0 & \text{in } \Gamma := \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta_{p(x)} u - \Delta_{q(x)} u + \omega |u|^{\xi(x)-2} u = \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\phi \in \mathcal{W}^*$ (that will be defined in Section 2), $u_0 \in L^2(\Omega)$, $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth, and the variables exponents $p, q \in C_+(\overline{\Omega})$ satisfy the assumption (2.1) in Section 2.

Studying this type of problems is both significant and relevant. On the one hand, we have the physical motivation; since the double phase operator has been used to model the steady-state solutions of reaction diffusion problems, that arise in biophysics, plasma-physics and in the study of chemical reactions. On the other hand, these operators provide a useful paradigm for describing the behaviour of strongly anisotropic materials, whose hardening properties are linked to the exponent governing the growth of the gradient change radically with the point, see [5,12,13,20–22] and the references given there.

Several scholars have discussed problems that are similar to problem (1.1), proving independently the existence of at least a weak solution for these problems (see, for example, [6,28]).

Let us recall some known results on problem (1.2). For example, Fan and Zhang [27], based on the theory of the spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, present several sufficient conditions for the existence of solution for the problem (1.2) with $\xi(x) = p(x)$, $\upsilon = 1$ and $\sigma = \omega = 0$ and without the term $-\Delta_{q(x)} u$ (see alsoo [12,13,25]).

R. Alsaedi [3] establishes sufficient conditions for the existence of nontrivial weak solution for the problem (1.2) without the term $-\Delta_{q(x)} u$, when $\sigma = 0$ and $\mathcal{A}(x,u) = |u|^{p(x)-2} u$ (see alsoo [23,24]).

In this work, by using the topological degree methods for operators of the type $\mathcal{T} + \mathcal{S}$, where $\mathcal{T}$ is a linear densely defined maximal monotone map and $\mathcal{S}$ is a bounded demicontinuous map of type (S_+) with respect to a domain of $\mathcal{T}$, we prove the existence of weak solution for the problem (1.1), and we will employ a topological degree for a type of demicontinuous operators of generalized (S_+) type to show the existence of weak solution for the problem (1.2).

The remainder of the paper is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problems. In Section 3, we introduce some classes of operators, as well as the topological degree methods for operators of the type $\mathcal{T}+\mathcal{S}$ and topological degree for a type of demicontinuous operators of generalized (S_+). In Section 4, we will prove the existence of weak solution of the Problem (1.1). Finally, Section 4 is devoted to discussing the existence of weak solution to (1.2).

2. Preliminaries

In this section, we present some results on the basic properties on $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, which we need in the proof of our results. For more details on these spaces, we refer the reader to [14,15,26,30,31].

Let $\Omega \subset \mathbb{R}^N (N > 1)$ be an open with a Lipschitz boundary denoted by $\partial\Omega$. Denote

$$C_+(\overline{\Omega}) = \Big\{p : \overline{\Omega} \longrightarrow [1, +\infty[\text{ continous such that } p(x) > 1\Big\}.$$

We define

$$p^+ := \max\Big\{p(x),\ x \in \overline{\Omega}\Big\} \text{ and } p^- := \min\Big\{p(x),\ x \in \overline{\Omega}\Big\} \text{ for every } p \in C_+(\overline{\Omega}).$$

The variable exponents $p, q \in C_+(\overline{\Omega})$ are assumed to satisfy the following assumption:

$$1 < q^- \leq q \leq q^+ < p^- \leq p \leq p^+ < +\infty. \tag{2.1}$$

We define the Lebesgue space with a variable exponent $p \in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega) = \Big\{f : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |f(x)|^{p(x)}dx < +\infty\Big\}.$$

$L^{p(x)}(\Omega)$ is endowed with the following Luxembourg-type norm

$$|f|_{p(x)} = \inf\Big\{\lambda > 0 : \varrho_{p(x)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\},$$

with

$$\varrho_{p(x)}(f) = \int_\Omega |f(x)|^{p(x)}dx \text{ for all } f \in L^{p(x)}(\Omega).$$

Proposition 2.1. [30] *For any sequence (f_n) and all $f \in L^{p(x)}(\Omega)$, we have*

$$|f|_{p(x)} < 1(resp. = 1; > 1) \Leftrightarrow \varrho_{p(x)}(f) < 1(resp. = 1; > 1), \tag{2.2}$$

$$|f|_{p(x)} > 1 \Rightarrow |f|^{p^-}_{p(x)} \leq \varrho_{p(x)}(f) \leq |f|^{p^+}_{p(x)}, \tag{2.3}$$

$$|f|_{p(x)} < 1 \Rightarrow |f|^{p^+}_{p(x)} \leq \varrho_{p(x)}(f) \leq |u|^{p^-}_{p(x)}, \tag{2.4}$$

$$\lim_{n\to\infty} |f_n - f|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \varrho_{p(x)}(f_n - f) = 0. \tag{2.5}$$

Remark 2.2. From (2.3) and (2.4), we can infer that

$$|f|_{p(x)} \leq \varrho_{p(x)}(f) + 1, \tag{2.6}$$

$$\varrho_{p(x)}(f) \leq |f|^{p^-}_{p(x)} + |f|^{p^+}_{p(x)}. \tag{2.7}$$

Proposition 2.3. [30] *The space* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ *is a separable and reflexive Banach space.*

Proposition 2.4. [30] *Let* $f \in L^{p(x)}(\Omega)$ *and* $g \in L^{p'(x)}(\Omega)$*. Then, we have the following Hölder-type inequality*

$$\Big|\int_\Omega fg\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|f|_{p(x)}|g|_{p'(x)} \le 2|f|_{p(x)}|g|_{p'(x)}. \tag{2.8}$$

Remark 2.5. If $p,\ q \in C_+(\overline{\Omega})$ with $p(x) \le q(x)$ then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Now, we define the Sobolev space with a variable exponent $p \in C_+(\overline{\Omega})$ by

$$W^{1,p(x)}(\Omega) = \Big\{ f \in L^{p(x)}(\Omega)\ :\ |\nabla f| \in (L^{p(x)}(\Omega))^N \Big\},$$

and it is a Banach space under the norm

$$||f|| = |f|_{p(x)} + |\nabla f|_{p(x)}.$$

We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.6. [30] *If the exponent* $p(x)$ *satisfy the log-Hölder continuity condition, i.e. there is a constant* $a > 0$ *such that for every* $x,\ y \in \Omega,\ x \ne y$ *with* $|x - y| \le \dfrac{1}{2}$ *one has*

$$|p(x) - p(y)| \le \frac{a}{-\log|x-y|}, \tag{2.9}$$

then, there exists $C > 0$ *depending only on* Ω *and the function* p *such that*

$$|u|_{p(x)} \le C|\nabla u|_{p(x)} \ \textit{for all}\ \ u \in W_0^{1,p(x)}(\Omega). \tag{2.10}$$

In this article, we shall use the equivalent norm on $W_0^{1,p(x)}(\Omega)$

$$|u|_{1,p(x)} = |\nabla u|_{p(x)}.$$

Proposition 2.7. [30] *The spaces* $(W^{1,p(x)}(\Omega), ||\cdot||)$ *and* $(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)})$ *are separable and reflexive Banach spaces.*

Remark 2.8. The dual space of $W^{1,p(x)}(\Omega)$ is the space $W^{-1,p'(x)}(\Omega)$ defined by

$$W^{-1,p'(x)}(\Omega) := \left\{ u = u_0 - \sum_{i=1}^N D_i u_i \ \text{ with }\ (u_0, u_1, \dots, u_N) \in (L^{p'(x)}(\Omega))^N \right\},$$

equipped with the norm

$$|u|_{-1,p'(x)} = \inf\Big\{ |u_0|_{p'(x)} + \sum_{i=1}^N |u_i|_{p'(x)} \Big\}.$$

We may also consider the generalized Lebesgue space

$$L^{p(x)}(\Omega_T) = \Big\{ u : \Omega_T \to \mathbb{R} \text{ is measurable with } \int_0^T \int_\Omega |u(x,t)|^{p(x)} dxdt < \infty \Big\},$$

endowed with the norm

$$|u|_{L^{p(x)}(\Omega_T)} = \inf \Big\{ \lambda > 0 : \int_0^T \varrho_{p(x)} \Big(\frac{u}{\lambda}\Big) dt \leq 1 \Big\},$$

which, of course, shares the same type of properties as $L^{p(x)}(\Omega)$.
As in [6], we introduce the functional space

$$\mathcal{W} := \Big\{ u \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)) \ : \ |\nabla u| \in L^{p(x)}(\Omega_T)^N \Big\},$$

which is a separable and reflexive Banach space endowed with the norm

$$|u|_{\mathcal{W}} := |u|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + |\nabla u|_{L^{p(x)}(\Omega_T)}.$$

Thanks to poincaré inequality (2.10), the expression

$$|v|_{\mathcal{W}} := |\nabla u|_{L^{p(x)}(\Omega_T)},$$

is a norm defined on $\mathcal{W}$ and is equivalent to the norm $|v|_{\mathcal{W}}$.
Some interesting properties of the space $\mathcal{W}$ are stated in the following lemma.

Lemma 2.9. [6] *Let $\mathcal{W}$ be the space defined as above and $\mathcal{W}^*$ denote its dual space, then:*

(1) *We have the following continuous dense embedding*

$$L^{p^+}(0,T; W_0^{1,p(x)}(\Omega)) \hookrightarrow \mathcal{W} \hookrightarrow L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)) \tag{2.11}$$

(2) *In particular, since $C_0^\infty(\Omega_T)$ is dense in $L^{p^+}(0,T; W_0^{1,p(x)}(\Omega))$, it is dense in $\mathcal{W}$ and for the corresponding dual spaces we have*

$$L^{(p^-)'}(0,\ T; W^{-1,p'(\cdot)}(\Omega)) \hookrightarrow \mathcal{W}^* \hookrightarrow L^{(p^+)'}(0,\ T; W^{-1,p'(\cdot)}(\Omega)). \tag{2.12}$$

(3) *Under the assumption (2.1), we have*

$$\begin{aligned} |u|^{q^-}_{L^{q(\cdot)}(\Omega_T)} - 1 \leq \int_{\Omega_T} |u|^{q(x)} dxdt \leq |u|^{q^+}_{L^{q(\cdot)}(\Omega_T)} + 1 \leq |u|^{p^-}_{L^{p(x)}(\Omega_T)} - 1 \\ \leq \int_{\Omega_T} |u|^{p(x)} dxdt \leq |u|^{p^+}_{L^{p(x)}(\Omega_T)} + 1. \end{aligned} \tag{2.13}$$

3. Topological degree theory

Now, we give some results and properties from the theory of topological degree. The readers can find more information about the history of this theory in [8, 9, 29]. In the rest of this paper, strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

3.1. **Topological degree theory for operators of the type $\mathcal{T}+\mathcal{S}$.**

In what follows, let Y is a real reflexive and separable Banach space with dual Y^* and continuous pairing $\langle .,.\rangle$, and given a nonempty subset Ω of Y, $\partial\Omega$ and $\overline{\Omega}$ represent the boundary and the closure of Ω in Y, respectively.

Definition 3.1. We consider a mapping $\mathcal{T}$ defined from Y to Y^* and its graph is given by

$$G(\mathcal{T}) = \{(u,v) \in Y \times Y^* : v \in \mathcal{T}(u)\}.$$

(1) $\mathcal{T}$ is said to be *monotone* if for all (u_1,v_1), (u_2,v_2) in $G(\mathcal{T})$, we get that $\langle v_1 - v_2, u_1 - u_2\rangle \geq 0$.

(2) $\mathcal{T}$ is said to be *maximal monotone* if it is monotone and maximal in the sense of graph inclusion among monotone mappings from Y to Y^*, or for any $(u_0,v_0) \in Y \times Y^*$ for which $\langle v_0 - v, u_0 - u\rangle \geq 0$, for all $(u,v) \in G(\mathcal{T})$, we have $(u_0,v_0) \in G(\mathcal{T})$.

Definition 3.2. Let Z be a real Banach space. A operator $\mathcal{T} : \Omega \subset Y \to Z$ is said to be

(1) bounded, if it takes any bounded set into a bounded set.

(2) demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies that $\mathcal{T}(u_n) \rightharpoonup \mathcal{T}(u)$.

(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.3. A mapping $\mathcal{S} : D(\mathcal{S}) \subset Y \to Y^*$ is said to be

(1) of type (S_+), if for any $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle \mathcal{S}u_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$.

(2) quasimonotone, if for any sequence $(u_n) \subset D(\mathcal{S})$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty}\langle \mathcal{S}u_n,\ u_n - u\rangle \geq 0$.

In the sequel, let $\mathcal{L}$ be a linear maximal monotone map from $D(\mathcal{L}) \subset Y$ to Y^*, and we consider the following classes of operators for each open and bounded subset G on Y:

$$\begin{aligned}\mathcal{F}_G := \{\mathcal{L}+\mathcal{S} : \overline{G} \cap D(\mathcal{L}) \to Y^* : \mathcal{S} &\text{ is bounded, demicontinuous} \\ &\text{map of type } (S_+) \text{ with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^*\}, \\ \mathcal{H}_G := \{\mathcal{L}+\mathcal{S}(t) : \overline{G} \cap D(\mathcal{L}) \to Y^* : \mathcal{S}(t) &\text{ is a bounded homotopy of type} \\ &\text{map of type } (S_+) \text{ with respect to } D(\mathcal{L}) \text{ from } \overline{G} \text{ to } Y^*\}.\end{aligned}$$

Definition 3.4. Let E be a bounded open subset of a real reflexive Banach space Y, $\mathcal{T} \in \mathcal{F}_1(\overline{E})$ be continuous and let $F, \mathcal{S} \in \mathcal{F}_{\mathcal{T}}(\overline{E})$. The affine homotopy $\Pi : [0,1] \times \overline{E} \to Y$ defined by

$$\Pi(t,u) := (1-t)Fu + t\mathcal{S}u, \quad \text{for all} \qquad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map $\mathcal{T}$.

Remark 3.5. Note that the class $\mathcal{H}_G$ includes all affine homotopies

$$\mathcal{L} + (1-t)\mathcal{S}_1 + t\mathcal{S}_2, \text{ with } (\mathcal{L} + \mathcal{S}_i) \in \mathcal{F}_G, \ i = 1, 2.$$

Now, we introduce the Berkovits and Mustonen topological degree for the class $\mathcal{F}_G$, and see [8,9] for more informations.

Theorem 3.6. *Let $\mathcal{L}$ a linear maximal monotone densely defined map from $D(\mathcal{L}) \subset Y$ to Y^*, and let*

$$\mathcal{E} = \Big\{(F, G, \phi) : F \in \mathcal{F}_G, \ G \text{ an open bounded subset in } Y, \ \phi \notin F(\partial G \cap D(\mathcal{L}))\Big\}.$$

Then, there exists a topological degree function $d : \mathcal{E} \to \mathbb{Z}$ satisfying the following properties:

(1) (*Existence*) *if $d(F, G, \phi) \neq 0$, then the equation $Fu = \phi$ has a solution in $G \cap D(\mathcal{L})$.*

(2) (*Additivity*) *If G_1 and G_2 are two disjoint open subsets of G such that $\phi \notin F[(\overline{G} \backslash (G_1 \cup G_2)) \cap D(\mathcal{L})]$, then we have*

$$d(F, G, \phi) = d(F, G_1, \phi) + d(F, G_2, \phi).$$

(3) (*Homotopy invariance*) *If $F(t) \in \mathcal{H}_G$ and $f(t) \notin F(t)(\partial G \cap D(\mathcal{L}))$ for all $t \in [0, 1]$, where $f(t)$ is a continuous curve in Y^*, then*

$$d(F(t), G, f(t)) = C, \ \forall t \in [0, 1].$$

(4) (*Normalization*) *$\mathcal{L} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of Y into Y^*, that is,*

$$d(\mathcal{L} + \mathcal{J}, G, \phi) = 1, \text{ for all } \phi \in (\mathcal{L} + \mathcal{J})(G \cap D(\mathcal{L})).$$

The following theorem plays an important role in the proof of the existence result of Problem 1.1.

Theorem 3.7. *Let $\mathcal{L} + \mathcal{S} \in \mathcal{F}_Y$ and $\phi \in Y^*$ and assume that there exists a radius $r > 0$ such that*

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u \rangle > 0, \tag{3.1}$$

for all $u \in \partial B_r(0) \cap D(\mathcal{L})$. Then the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a solution u in $D(\mathcal{L})$.

Proof. To show this theorem, it suffices to prove that $(\mathcal{L} + \mathcal{S})(D(\mathcal{L})) = Y^*$.

Let $F_\omega(t, \ u) = \mathcal{L}u + (1-t)\mathcal{J}u + t(\mathcal{S}u + \omega\mathcal{J}u - \phi)$, for all $\omega > 0$ and $t \in [0, 1]$.

From (3.1) and since $0 \in \mathcal{L}(0)$, we obtain

$$\begin{aligned}
\langle F_\omega(t, u), u \rangle &= \langle t(\mathcal{L}u + \mathcal{S}u - \phi, u \rangle + \langle (1-t)\mathcal{L}u + (1-t+\omega)\mathcal{J}u, u \rangle \\
&\geq \langle (1-t)\mathcal{L}u + (1-t+\omega)\mathcal{J}u, u \rangle \\
&= (1-t)\langle \mathcal{L}u, \ u \rangle + (1-t+\omega)\langle \mathcal{J}u, u \rangle \\
&\geq (1-t+\omega)|u|^2 \\
&= (1-t+\omega)r^2 > 0.
\end{aligned}$$

Which implies that $0 \notin F_\omega(t,\ u)$.

Since $\mathcal{J}$ and $\mathcal{S} + \omega\mathcal{J}$ are continuous, bounded and of type (S_+), then $\{F_\omega(t,\ \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Therefore, applying the homotopy invariance and normalisation property of the degree d stated in Theorem 3.6, we obtain

$$d(F_\omega(t,\cdot), B_r(0), 0) = d(\mathcal{L} + \mathcal{J}, B_r(0), 0) = 1 \neq 0.$$

Consequently, by existence property of the degree d there exists a point $u_\omega \in D(\mathcal{L})$ such that $0 \in F_\omega(t,\ \cdot)$. In particular, by setting $\omega \to 0^+$ and $t = 1$, we get $\phi \in (\mathcal{L} + \mathcal{S})(D(\mathcal{L}))$ for some $u \in D(\mathcal{L})$ and that for all $\phi \in Y^*(\phi$ is arbitrary$)$. Which implies that $(\mathcal{L} + \mathcal{S})(D(\mathcal{L})) = Y^*$. $\square$

3.2. Topological degree theory for a class of demicontinuous operators of generalized (S_+).

We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with dual pairing $\langle\cdot, \cdot\rangle$.

Definition 3.8. Let Y be another real Banach space. A operator $F : \Omega \subset X \to Y$ is said to be

(1) bounded, if it takes any bounded set into a bounded set.

(2) demicontinuous, if for any sequence $(u_n) \subset \Omega,\ u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.

(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.9. A mapping $F : \Omega \subset X \to X^*$ is said to be

(1) of type (S_+), if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$.

(2) quasimonotone, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 3.10. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. For any operator $F : \Omega \subset X \to X$, we say that

(1) F of type $(S_+)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\limits_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$.

(2) F has the property $(QM)_T$, if for any sequence $(u_n) \subset \Omega$ with $u_n \rightharpoonup u,\ y_n := Tu_n \rightharpoonup y$, we have $\limsup\limits_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

Consider the different types of operators as follows:

$$\mathcal{F}_1(\Omega) := \Big\{F : \Omega \to X^* : F \text{ is bounded, demicontinuous and of type } (S_+)\Big\},$$

$$\mathcal{F}_T(\Omega) := \Big\{F : \Omega \to X : F \text{ is demicontinuous and of type } (S_+)_T\Big\},$$

$$\mathcal{F}_{T,B}(\Omega) := \Big\{F \in \mathcal{F}_T(\Omega) : F \text{ is bounded }\Big\},$$

for any $\Omega \subset D(F)$, where $D(F)$ denotes the domain of F, and any $T \in \mathcal{F}_1(\Omega)$.

Now, let $\mathcal{O}$ be the collection of all bounded open sets in X and we define

$$\mathcal{F}(X) := \Big\{F \in \mathcal{F}_T(\overline{E}) : E \in \mathcal{O},\ \mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})\Big\},$$

where, $\mathrm{T} \in \mathcal{F}_1(\overline{\mathrm{E}})$ is called an essential inner map to F.

Lemma 3.11. [29, *Lemma 2.3*] *Let $T \in \mathcal{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true :*

(1) *If S is quasimonotone, then $I + S \circ T \in \mathcal{F}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of type (S_+), then $S \circ T \in \mathcal{F}_T(\overline{E})$.*

Definition 3.12. Suppose that E is bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\overline{E})$ is continuous and $F, S \in \mathcal{F}_T(\overline{E})$. Then the affine homotopy $\Lambda : [0,1] \times \overline{E} \to X$ defined by

$$\Lambda(t,u) := (1-t)Fu + tSu, \quad \text{for} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 3.13. [29, Lemma 2.5] *The above affine homotopy is of type $(S_+)_T$.*

Now, we give the topological degree for the class $\mathcal{F}(X)$ (see [29]).

Theorem 3.14. *Let*

$$M = \Big\{(F,E,h) : E \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{E}),\ F \in \mathcal{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\Big\}.$$

Then, there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfy the following properties:

(1) (Normalization) *For any $h \in E$, we have*
$$d(I,E,h) = 1.$$
(2) (Additivity) *Let $F \in \mathcal{F}_{T,B}(\overline{E})$. If E_1 and E_2 are two disjoint open subsets of E such that $h \notin F(\overline{E} \backslash (E_1 \cup E_2))$, then we have*
$$d(F,E,h) = d(F,E_1,h) + d(F,E_2,h).$$
(3) (Homotopy invariance) *If $\Lambda : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \Lambda(t, \partial E)$ for all $t \in [0,1]$, then*
$$d(\Lambda(t,\cdot),E,h(t)) = C \text{ for all } t \in [0,1].$$
(4) (Existence) *If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

(5) (Boundary dependence) *If* $F,\ S \in \mathcal{F}_{\mathrm{T}}(\overline{\mathrm{E}})$ *coincide on* ∂E *and* $h \notin F(\partial E)$, *then*

$$d(F, E, h) = d(S, E, h)$$

Definition 3.15. [29, Definition 3.3] The above degree is defined as follows:

$$d(F, E, h) \ := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [7] *and* E_0 *is any open subset of* E *with* $F^{-1}(h) \subset E_0$ *and* F *is bounded on* $\overline{E}_0$.

4. Quasilinear parabolic problem involving the $(p(x), q(x))$-Laplacian operator

In this section, we will prove the existence of weak solution of the Problem (1.1). First we will state a lemma that will be used later.

Lemma 4.1. *The operator* $\mathcal{S} := -\Delta_{p(x)}u - \Delta_{q(x)}u$ *defined from* $\mathcal{W}$ *into* $\mathcal{W}^*$ *by*

$$\langle \mathcal{S}u, v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_{\Omega_T} \Big(|\nabla u|^{p(x)-2}\nabla u \nabla v + |\nabla u|^{q(x)-2}\nabla u \nabla v \Big) dxdt,$$

is bounded, continuous and of type (S_+).

Proof. Let $t \in]\,0,\ T[$ and denote by $\mathcal{A}$ the operator defined from $W_0^{1,p(x)}(\Omega)$ into $W^{-1,p'(x)}(\Omega)$ by

$$\langle \mathcal{A}u(x,t), v(x,t)\rangle := \langle \mathcal{A}_1 u(x,t), v(x,t)\rangle + \langle \mathcal{A}_2 u(x,t), v(x,t)\rangle,$$

where

$$\langle \mathcal{A}_1 u(x,t), v(x,t)\rangle := \int_{\Omega} \Big(|\nabla u(x,t)|^{p(x)-2}\nabla u(x,t) \nabla v(x,t) \Big) dx,$$

and

$$\langle \mathcal{A}_2 u(x,t), v(x,t)\rangle := \int_{\Omega} \Big(|\nabla u(x,t)|^{q(x)-2}\nabla u(x,t) \nabla v(x,t) \Big) dx,$$

for all $u(\cdot,t),\, v(\cdot,t) \in W_0^{1,p(x)}(\Omega)$, with $\langle \cdot,\cdot\rangle$ is the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Then, we obtain

$$\langle \mathcal{S}u, v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_0^T \langle \mathcal{A}u(x,t), v(x,t)\rangle dt, \ \text{ for all } \ u, v \in \mathcal{W},$$

with $\langle \cdot,\cdot\rangle_{\mathcal{W}^*,\mathcal{W}}$ is the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$.
Next, it follows from [27, Lemma 3.1] that $\mathcal{A}_1$ and $\mathcal{A}_2$ are bounded, continuous and of type (S_+); so the operator $\mathcal{A} := \mathcal{A}_1 + \mathcal{A}_2$ is bounded, continuous and of type (S_+) and consequently the operator $\mathcal{S}$ is bounded, continuous and of type (S_+). □

We are now in the position to get existence result of weak solution for (1.1).

Theorem 4.2. *Let* $\phi \in \mathcal{W}^*$ *and* $u_0 \in L^2(\Omega)$, *then the problem* (1.1) *admits at least one weak solution* $u \in D(\mathcal{L})$, *where* $D(\mathcal{L}) = \{u \in \mathcal{W} : \frac{du}{dt} \in \mathcal{W}^* \ \text{ and } \ u(0) = 0\}$.

Proof. First, let us define the operator $\mathcal{L} := \dfrac{d}{dt}$ with domain $D(\mathcal{L})$ given by

$$D(\mathcal{L}) = \{u \in \mathcal{W} \,:\, \frac{du}{dt} \in \mathcal{W}^* \text{ and } u(0) = 0\},$$

where the time derivative $\dfrac{du}{dt}$ is understood in the sense of vector-valued distributions, i.e.,

$$\langle \mathcal{L}u,\ v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_0^T \langle u'(t), v(t)\rangle dt,\ \forall\, v \in \mathcal{W},$$

with $\langle\cdot,\cdot\rangle_{\mathcal{W}^*,\mathcal{W}}$ the duality pairing between $\mathcal{W}^*$ and $\mathcal{W}$, and $\langle\cdot,\cdot\rangle$ the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$.

Second, we define the operator $\mathcal{S} : \mathcal{W} \to \mathcal{W}^*$ as defined in Lemma 4.1

$$\langle \mathcal{S}u, v\rangle_{\mathcal{W}^*,\mathcal{W}} = \int_{\Omega_T} \Big(|\nabla u|^{p(x)-2}\nabla u\nabla v + |\nabla u|^{q(x)-2}\nabla u\nabla v\Big)dxdt.$$

Consequently, the weak formulation of the problem (1.1) is given by the operator equation

$$u \in D(\mathcal{L}) : \mathcal{L}u + \mathcal{S}u = \phi.$$

Next, it follows from lemma 4.1 that $\mathcal{S}$ is bounded, continuous and of type (S_+), and the operator $\mathcal{L}$ is well known to be closed, densely defined, and maximal monotone [34, Theorem 32.L, pp.897-899].

Let $u \in \mathcal{W}$. Using the monotonicity of $\mathcal{L}$ and the inequality (2.13), we obtain

$$\begin{aligned}
\langle \mathcal{L}u + \mathcal{S}u, u\rangle &\geq \langle \mathcal{S}u, u\rangle \\
&= \int_{\Omega_T} (|\nabla u|^{p(x)} + |\nabla u|^{q(x)})dxdt \\
&\geq 2(|\nabla u|^{p^-}_{L^{p(x)}(\Omega_T)} - 1) \\
&\geq 2(|u|^{p^-}_{\mathcal{W}} - 1).
\end{aligned}$$

Because the right-hand side of the previous inequality approximates to ∞ when $|u|_{\mathcal{W}} \to \infty$, then the operator $\mathcal{L} + \mathcal{S}$ is coercive. Thus for each $\phi \in \mathcal{W}^*$ there is a radius $r = r(\phi) > 0$ such that

$$\langle \mathcal{L}u + \mathcal{S}u - \phi, u\rangle > 0, \quad \text{for each} \quad u \in B_r(0) \cap D(\mathcal{L}).$$

So all the conditions of Theorem 3.7 are satisfied. Consequently, Theorem 3.7 leads us to the conclusion that the equation $\mathcal{L}u + \mathcal{S}u = \phi$ has a weak solution in $D(\mathcal{L})$, which implies that the problem (1.1) has a weak solution in $u \in D(\mathcal{L})$. This completes the proof. □

5. Quasilinear elliptic problem involving the $(p(x), q(x))$-Laplacian operator

In this section, we will discuss the existence of weak solution of (1.2). In the beginning, let us assume that $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.9), $\xi \in C_+(\overline{\Omega})$ with $2 \leq \xi^- \leq \xi(x) \leq \xi^+ < p^- \leq p(x) \leq p^+ < \infty$, $\mathcal{A} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are functions such that:

(A_1)**:** $\mathcal{B}$ is a Carathéodory function.

(A_2)**:** There exists $\alpha_1 > 0$ and $f \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{B}(x, y, z)| \leq \alpha_1(f(x) + |y|^{k(x)-1} + |z|^{k(x)-1}).$$

(A_3)**:** $\mathcal{A}$ is a Carathéodory function.

(A_4)**:** There are $\alpha_2 > 0$ and $g \in L^{p'(x)}(\Omega)$ such that

$$|\mathcal{A}(x, y)| \leq \alpha_2(g(x) + |y|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and all $(y, z) \in \mathbb{R} \times \mathbb{R}^N$, where $q, s \in C_+(\overline{\Omega})$ with $2 \leq k^- \leq k(x) \leq k^+ < p^-$ and $2 \leq s^- \leq s(x) \leq s^+ < p^-$.

- Let $\vartheta \in W_0^{1,p(x)}(\Omega)$, then

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \vartheta + |\nabla u|^{q(x)-2}\nabla u \nabla \vartheta\Big)dx$$

  is well defined (see [27]).
- Let $u \in W_0^{1,p(x)}(\Omega)$, then we have $\omega|u|^{\xi(x)-2}u \in L^{p'(x)}(\Omega)$, $\upsilon\,\mathcal{A}(x, u) \in L^{p'(x)}(\Omega)$ and $\sigma\,\mathcal{B}(x, u, \nabla u) \in L^{p'(x)}(\Omega)$ under the assumptions (A_2) and (A_4) and the given hypotheses about the exponents p, ξ, q and s because: $f \in L^{p'(x)}(\Omega)$, $g \in L^{p'(x)}(\Omega)$, $r(x) = (k(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $r(x) < p(x)$, and $\beta(x) = (\xi(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$ and $\kappa(x) = (s(x) - 1)p'(x) \in C_+(\overline{\Omega})$ with $\kappa(x) < p(x)$.
  Then, using Remark 2.5, we conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$, $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$.
  Therefore, with $\vartheta \in L^{p(x)}(\Omega)$, we have

$$\Big(- \omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x, u) + \sigma\,\mathcal{B}(x, u, \nabla u)\Big)\vartheta \in L^1(\Omega).$$

  This means that

$$\int_\Omega \Big(- \omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x, u) + \sigma\,\mathcal{B}(x, u, \nabla u)\Big)\vartheta dx < \infty.$$

Then, let us introduce the definition of a weak solution for (1.2).

Definition 5.1. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.2), if for any $\vartheta \in W_0^{1,p(x)}(\Omega)$, it satisfy the following:

$$\begin{aligned}\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \vartheta + |\nabla u|^{q(x)-2}\nabla u \nabla \vartheta\Big)dx \\ = \int_\Omega \Big(- \omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x, u) + \sigma\,\mathcal{B}(x, u, \nabla u)\Big)\vartheta dx.\end{aligned}$$

Let us now give some lemmas that will be used later. First, let us consider the following functional:

$$\mathcal{C}(u) := \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{q(x)}|\nabla u|^{q(x)}dx.$$

From [27], it is clear that the derivative operator of the functional $\mathcal{C}$ in the weak sense at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathcal{G}(u) := \mathcal{C}'(u) \in W^{-1,p'(x)}(\Omega)$, given by

$$\langle \mathcal{G}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx,$$

for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$ where $\langle \cdot, \cdot \rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. Furthermore, we have the following properties of the operator $\mathcal{G}$.

Lemma 5.2. [27, *Theorem 3.1.*]*The mapping*

$$\begin{aligned} &\mathcal{G} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{G}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u \nabla \vartheta + |\nabla u|^{q(x)-2} \nabla u \nabla \vartheta \Big) dx, \end{aligned} \tag{5.1}$$

is a continuous, bounded, strictly monotone operator and is of type (S_+).

Lemma 5.3. *If* $(A_1) - (A_2)$ *hold, then the operator*

$$\begin{aligned} &\mathcal{N} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega) \\ &\langle \mathcal{N}u, \vartheta \rangle = -\int_\Omega \Big(-\omega |u|^{\xi(x)-2} u + \upsilon\, \mathcal{A}(x,u) + \sigma\, \mathcal{B}(x,u,\nabla u) \Big) \vartheta dx, \end{aligned} \tag{5.2}$$

is compact.

Proof. We follow four steps to prove this lemma.

Step 1 : Let $\Psi_1 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ be an operator defined by

$$\Psi_1 u(x) := -\upsilon \mathcal{A}(x,u).$$

We wiil prove that the operator Ψ_1 is bounded and continuous. Let $u \in W_0^{1,p(x)}(\Omega)$, bearing (A_4) in mind and using (2.6) and (2.7), we infer

$$\begin{aligned} |\Psi_1 u|_{p'(x)} &\le \varrho_{p'(x)}(\Psi_1 u) + 1 \\ &= \int_\Omega |\upsilon \mathcal{A}(x,u(x))|^{p'(x)} dx + 1 \\ &= \int_\Omega |\upsilon|^{p'(x)} |\mathcal{A}(x,u(x)|^{p'(x)} dx + 1 \\ &\le \Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+} \Big) \int_\Omega |\alpha_2 \Big(g(x) + |u|^{s(x)-1} \Big)|^{p'(x)} dx + 1 \\ &\le C\Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+} \Big) \int_\Omega \Big(|g(x)|^{p'(x)} + |u|^{\kappa(x)} \Big) dx + 1 \\ &\le C\Big(|\upsilon|^{p'^-} + |\upsilon|^{p'^+} \Big) \Big(\varrho_{p'(x)}(g) + \varrho_{\kappa(x)}(u) \Big) + 1 \\ &\le C\Big(|g|_{p(x)}^{p'^+} + |u|_{\kappa(x)}^{\kappa^+} + |u|_{\kappa(x)}^{\kappa^-} \Big) + 1. \end{aligned}$$

Then, we deduce from (2.10) and $L^{p(x)} \hookrightarrow L^{\kappa(x)}$, that

$$|\Psi_1 u|_{p'(x)} \le C\Big(|g|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{\kappa^+} + |u|_{1,p(x)}^{\kappa^-} \Big) + 1,$$

that means Ψ_1 is bounded on $W_0^{1,p(x)}(\Omega)$.

Second, we show that the operator Ψ_1 is continuous. To this purpose let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. We need to show that $\Psi_1 u_n \to \Psi_1 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Hence there exist a subsequence (u_m) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ such that

$$u_m(x) \to u(x) \text{ and } |u_m(x)| \leq \phi(x), \tag{5.3}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, from (A_2) and (5.3), we have

$$|\mathcal{A}(x, u_m(x))| \leq \alpha_2(g(x) + |\phi(x)|^{s(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

On the other hand, thanks to (A_3) and (5.3), we get, as $k \longrightarrow \infty$

$$\mathcal{A}(x, u_m(x)) \to \mathcal{A}(x, u(x)) \text{ a.e. } x \in \Omega.$$

Seeing that

$$g + |\phi|^{s(x)-1} \in L^{p'(x)}(\Omega) \text{ and } \varrho_{p'(x)}(\Psi_1 u_m - \Psi_1 u) = \int_\Omega |\mathcal{A}(x, u_m(x)) - \mathcal{A}(x, u(x))|^{p'(x)} dx,$$

then, from the Lebesgue's theorem and the equivalence (2.5), we have

$$\Psi_1 u_m \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

and consequently

$$\Psi_1 u_n \to \Psi_1 u \text{ in } L^{p'(x)}(\Omega),$$

that is, Ψ_1 is continuous.

Step 2 : We define the operator $\Psi_2 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_2 u(x) := \omega |u(x)|^{\xi(x)-2} u(x).$$

We will prove that Ψ_2 is bounded and continuous.

It is clear that Ψ_2 is continuous. Next we show that Ψ_2 is bounded.

Let $u \in W_0^{1,p(x)}(\Omega)$ and using (2.6) and (2.7), we obtain

$$\begin{aligned}
|\Psi_2 u|_{p'(x)} &\leq \varrho_{p'(x)}(\Psi_2 u) + 1 \\
&= \int_\Omega |\omega |u|^{\xi(x)-2} u|^{p'(x)} dx + 1 \\
&= \int_\Omega |\omega|^{p'(x)} |u|^{(\xi(x)-1)p'(x)} dx + 1 \\
&\leq \left(|\omega|^{p'^-} + |\omega|^{p'^+} \right) \int_\Omega |u|^{\beta(x)} dx + 1
\end{aligned}$$

$$= \left(|\omega|^{p'^-} + |\omega|^{p'^+}\right)\varrho_{\beta(x)}(u) + 1$$
$$\leq \left(|\omega|^{p'^-} + |\omega|^{p'^+}\right)\left(|u|_{\beta(x)}^{\beta^-} + |u|_{\beta(x)}^{\beta^+}\right) + 1.$$

Hence, we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.10) that

$$|\Psi_2 u|_{p'(x)} \leq C\left(|u|_{1,p(x)}^{\beta^-} + |u|_{1,p(x)}^{\beta^+}\right) + 1,$$

and consequently, Ψ_2 is bounded on $W_0^{1,p(x)}(\Omega)$.

Step 3 : Let us define the operator $\Psi_3 : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Psi_3 u(x) := -\sigma \mathcal{B}(x, u(x), \nabla u(x)).$$

We will show that Ψ_3 is bounded and continuous.

Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and the inequalities (2.6) and (2.7), we obtain

$$\begin{aligned}
|\Psi_3 u|_{p'(x)} &\leq \varrho_{p'(x)}(\Psi_3 u) + 1 \\
&= \int_\Omega |\sigma \mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&= \int_\Omega |\sigma|^{p'(x)} |\mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx + 1 \\
&\leq \left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right) \int_\Omega |\alpha_1\left(f(x) + |u|^{k(x)-1} + |\nabla u|^{k(x)-1}\right)|^{p'(x)} dx + 1 \\
&\leq C\left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right) \int_\Omega \left(|f(x)|^{p'(x)} + |u|^{r(x)} + |\nabla u|^{r(x)}\right) dx + 1 \\
&\leq C\left(|\sigma|^{p'^-} + |\sigma|^{p'^+}\right)\left(\varrho_{p'(x)}(f) + \varrho_{r(x)}(u) + \varrho_{r(x)}(\nabla u)\right) + 1 \\
&\leq C\left(|f|_{p(x)}^{p'^+} + |u|_{r(x)}^{r^+} + |u|_{r(x)}^{r^-} + |\nabla u|_{r(x)}^{r^+} + |\nabla u|_{r(x)}^{r^-}\right) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.10), we have then

$$|\Psi_3 u|_{p'(x)} \leq C\left(|f|_{p(x)}^{p'^+} + |u|_{1,p(x)}^{r^+} + |u|_{1,p(x)}^{r^-}\right) + 1,$$

and consequently Ψ_3 is bounded on $W_0^{1,p(x)}(\Omega)$.

It remains to show that Ψ_3 is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Psi_3 u_n \to \Psi_3 u$ in $L^{p'(x)}(\Omega)$. We will apply the Lebesgue's theorem.

Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Hence, there exist a subsequence (u_m) and Ψ_3 in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$u_m(x) \to u(x) \text{ and } \nabla u_m(x) \to \nabla u(x), \tag{5.4}$$

$$|u_m(x)| \leq \phi(x) \text{ and } |\nabla u_m(x)| \leq |\psi(x)|, \tag{5.5}$$

for a.e. $x \in \Omega$ and all $k \in \mathbb{N}$.

Hence, thanks to (A_1) and (5.4), we get, as $k \longrightarrow \infty$

$$\mathcal{B}(x, u_m(x), \nabla u_m(x)) \to \mathcal{B}(x, u(x), \nabla u(x)) \text{ a.e. } x \in \Omega.$$

On the other hand, from (A_2) and (5.5), we can deduce the estimate

$$|\mathcal{B}(x, u_m(x), \nabla u_m(x))| \leq \alpha_1(f(x) + |\phi(x)|^{k(x)-1} + |\psi(x)|^{k(x)-1}),$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.

Seeing that

$$f + |\phi|^{k(x)-1} + |\psi(x)|^{k(x)-1} \in L^{p'(x)}(\Omega),$$

and taking into account the equality

$$\varrho_{p'(x)}(\Psi_3 u_m - \Psi_3 u) = \int_\Omega |\mathcal{B}(x, u_m(x), \nabla u_m(x)) - \mathcal{B}(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then, we conclude from the Lebesgue's theorem and (2.5) that

$$\Psi_3 u_m \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega)$$

and consequently

$$\Psi_3 u_n \to \Psi_3 u \ \text{ in } \ L^{p'(x)}(\Omega),$$

and then Ψ_3 is continuous.

Step 4: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the operator $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$.

We then define

$$I^* \circ \Psi_1 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

$$I^* \circ \Psi_2 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega),$$

and

$$I^* \circ \Psi_3 : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On another side, taking into account that I is compact, then I^* is compact. Thus, the compositions $I^* \circ \Psi_1$, $I^* \circ \Psi_2$ and $I^* \circ \Psi_3$ are compact, that means $\mathcal{N} = I^* \circ \Psi_1 + I^* \circ \Psi_2 + I^* \circ \Psi_3$ is compact. With this last step the proof of Lemma 5.3 is completed. □

We are now in the position to get the existence result of weak solution for (1.2).

Theorem 5.4. *Assume that the assumptions $(A_1) - (A_4)$ hold, then the problem (1.2) possesses at least one weak solution u in $W_0^{1,p(x)}(\Omega)$.*

Proof. The basic idea of our proof is to reduce the problem (1.2) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Subsection 3.2 to show the existence of a weak solution to the state problem.

First, for all $u, \vartheta \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathcal{G}$ and $\mathcal{N}$, as defined in (5.1) and (5.2) respectively,

$$\mathcal{G} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{G}u, \vartheta \rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u \nabla \vartheta + |\nabla u|^{q(x)-2}\nabla u \nabla \vartheta \Big) dx,$$

$$\mathcal{N} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$$
$$\langle \mathcal{N}u, \vartheta \rangle = -\int_\Omega \Big(-\omega |u|^{\xi(x)-2}u + \upsilon \mathcal{A}(x,u) + \sigma \mathcal{B}(x,u,\nabla u) \Big) \vartheta dx.$$

Consequently, the problem (1.2) is equivalent to the equation

$$\mathcal{G}u = -\mathcal{N}u, \quad u \in W_0^{1,p(x)}(\Omega). \tag{5.6}$$

Taking into account that, by Lemma 5.2, the operator $\mathcal{G}$ is a continuous, bounded, strictly monotone and of type (S_+), then, by [34, Theorem 26 A], the inverse operator

$$\mathcal{M} := \mathcal{G}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of type (S_+).
On another side, according to Lemma 5.3, we have that the operator $\mathcal{N}$ is bounded, continuous and quasimonotone.
Consequently, following Zeidler's terminology [34], the equation (5.6) is equivalent to the following abstract Hammerstein equation

$$u = \mathcal{M}\vartheta \ \text{ and } \ \vartheta + \mathcal{N} \circ \mathcal{M}\vartheta = 0, \ \ u \in W_0^{1,p(x)}(\Omega) \text{ and } \vartheta \in W^{-1,p'(x)}(\Omega). \tag{5.7}$$

Seeing that (5.6) is equivalent to (5.7), then to solve (5.6) it is thus enough to solve (5.7). In order to solve (5.7), we will apply the Berkovits topological degree introduced in Section 3.
First, let us set

$$\mathcal{R} := \Big\{ \vartheta \in W^{-1,p'(x)}(\Omega) \ \text{ such that there exists } \ t \in [0,1] \ \text{ such that } \ \vartheta + t\mathcal{N} \circ \mathcal{M}\vartheta = 0 \Big\}.$$

Next, we show that $\mathcal{R}$ is bounded in $W^{-1,p'(x)}(\Omega)$.
Let us put $u := \mathcal{M}\vartheta$ for all $\vartheta \in \mathcal{R}$. Taking into account that $|\mathcal{M}\vartheta|_{1,p(x)} = |\nabla u|_{p(x)}$, then we have the following two cases:
First case : If $|\nabla u|_{p(x)} \leq 1$.
Then $|\mathcal{M}\vartheta|_{1,p(x)} \leq 1$, that means $\Big\{ \mathcal{M}\vartheta : \vartheta \in \mathcal{R} \Big\}$ is bounded.
Second case : If $|\nabla u|_{p(x)} > 1$.
Then, we deduce from (2.3), (A_2) and (A_4), the inequalities (2.8) and (2.7) and the Young's inequality

that

$$\begin{aligned}
|\mathcal{M}\vartheta|_{1,p(x)}^{p^-} &= |\nabla u|_{p(x)}^{p-} \\
&\leq \varrho_{p(x)}(\nabla u) \\
&\leq \langle \mathcal{G}u,\ u\rangle \\
&= \langle \vartheta,\ \mathcal{M}\vartheta\rangle \\
&= -t\langle \mathcal{N}\circ\mathcal{M}\vartheta,\ \mathcal{M}\vartheta\rangle \\
&= t\int_\Omega \Big(-\omega|u|^{\xi(x)-2}u + \upsilon\,\mathcal{A}(x,u) + \sigma\,\mathcal{B}(x,u,\nabla u)\Big)u dx \\
&\leq t\max(|\omega|, \alpha_2|\upsilon|, \alpha_1|\sigma|)\Big(\int_\Omega |u|^{\xi(x)}dx + \int_\Omega |g(x)u(x)|dx + \int_\Omega |u(x)|^{s(x)}dx \\
&\qquad + \int_\Omega |f(x)u(x)|dx + \int_\Omega |u(x)|^{k(x)}dx + \int_\Omega |\nabla u|^{k(x)-1}|u|dx\Big) \\
&= t\max(|\omega|, \alpha_2|\upsilon|, \alpha_1|\sigma|)\Big(\varrho_{\xi(x)}(u) + \int_\Omega |g(x)u(x)|dx + \int_\Omega |f(x)u(x)|dx \\
&\qquad + \varrho_{s(x)}(u) + \varrho_{k(x)}(u) + \int_\Omega |\nabla u|^{k(x)-1}|u|dx\Big) \\
&\leq C\Big(|u|_{\xi(x)}^{\xi^-} + |u|_{\xi(x)}^{\xi^+} + |g|_{p'(x)}|u|_{p(x)} + |f|_{p'(x)}|u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} \\
&\qquad + |u|_{k(x)}^{k^+} + |u|_{k(x)}^{k^-} + \frac{1}{k'^-}\varrho_{k(x)}(\nabla u) + \frac{1}{k^-}\varrho_{k(x)}(u)\Big) \\
&\leq C\Big(|u|_{\xi(x)}^{\xi^-} + |u|_{\xi(x)}^{\xi^+} + |u|_{p(x)} + |u|_{s(x)}^{s^+} + |u|_{s(x)}^{s^-} + |u|_{k(x)}^{k^+} + |u|_{k(x)}^{k^-} + |\nabla u|_{k(x)}^{k^+}\Big).
\end{aligned}$$

Then, according to $L^{p(x)} \hookrightarrow L^{\xi(x)}$, $L^{p(x)} \hookrightarrow L^{s(x)}$ and $L^{p(x)} \hookrightarrow L^{k(x)}$, we get

$$|\mathcal{M}\vartheta|_{1,p(x)}^{p^-} \leq C\Big(|\mathcal{M}\vartheta|_{1,p(x)}^{\xi^+} + |\mathcal{M}\vartheta|_{1,p(x)} + |\mathcal{M}\vartheta|_{1,p(x)}^{s^+} + |\mathcal{M}\vartheta|_{1,p(x)}^{k^+}\Big),$$

what implies that $\Big\{\mathcal{M}\vartheta : \vartheta \in \mathcal{R}\Big\}$ is bounded.

On the other hand, we have that the operator is $\mathcal{N}$ is bounded, then $\mathcal{N}\circ\mathcal{M}\vartheta$ is bounded. Thus, thanks to (5.7), we have that $\mathcal{R}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, there exists $r > 0$ such that

$$|\vartheta|_{-1,p'(x)} < r \ \text{ for all } \ \vartheta \in \mathcal{R},$$

which leads to

$$\vartheta + t\mathcal{N}\circ\mathcal{M}\vartheta \neq 0,\ \ \vartheta \in \partial\mathcal{R}_r(0) \ \text{ and } \ t \in [0,1],$$

where $\mathcal{R}_r(0)$ is the ball of center 0 and radius r in $W^{-1,p'(x)}(\Omega)$.

Moreover, by Lemma 3.11, we conclude that

$$I + \mathcal{N}\circ\mathcal{M} \in \mathcal{F}_{\mathcal{M}}(\overline{\mathcal{R}_r(0)}) \ \text{ and } \ I = \mathcal{G}\circ\mathcal{M} \in \mathcal{F}_{\mathcal{M}}(\overline{\mathcal{R}_r(0)}).$$

On another side, taking into account that $I, \mathcal{N}$ and $\mathcal{M}$ are bounded, then $I+\mathcal{N}\circ\mathcal{M}$ is bounded. Hence, we infer that

$$I+\mathcal{N}\circ\mathcal{M} \in \mathcal{F}_{\mathcal{M},B}(\overline{\mathcal{R}_r(0)}) \text{ and } I=\mathcal{G}\circ\mathcal{M} \in \mathcal{F}_{\mathcal{M},B}(\overline{\mathcal{R}_r(0)}).$$

Next, we define the homotopy

$$\begin{aligned} \mathcal{H} : [0,1]\times\overline{\mathcal{R}_r(0)} &\to W^{-1,p'(x)}(\Omega) \\ (t,\vartheta) &\mapsto \mathcal{H}(t,\vartheta) := \vartheta + t\mathcal{N}\circ\mathcal{M}\vartheta. \end{aligned}$$

Hence, thanks to the properties of the degree d seen in Theorem 3.14, we obtain

$$d(I+\mathcal{N}\circ\mathcal{M}, \mathcal{R}_r(0), 0) = d(I, \mathcal{R}_r(0),\ 0) = 1 \neq 0,$$

what implies that there exists $\vartheta \in \mathcal{R}_r(0)$ which verifies

$$\vartheta + \mathcal{N}\circ\mathcal{M}\vartheta = 0.$$

Finally, we infer that $u = \mathcal{M}\vartheta$ is a weak solution of (1.2). The proof is completed. □

Acknowledgment

The authors would like to thank the anonymous reviewers for their helpful and constructive comments that greatly contributed to improving the final version of the paper.

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