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SOME RELATION-THEORETIC FIXED POINT RESULTS IN FUZZY METRIC SPACES

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Abstract. In this study, using appropriate assumptions on a fuzzy metric space, we establish several relation-theoretic fixed point results involving a class of auxiliary functions and a binary relation. The given results combine, generalize, and enhance numerous prior findings in the literature.

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1. Introduction

Since it includes a wide variety of mathematical tools for addressing many types of issues that arise from several areas of mathematics, fixed point theory is one of the most important and fundamental research domains in nonlinear functional analysis. The central finding of the metric fixed point is the Banach contraction principle, which has since been investigated and improved using many techniques and a wide range of abstract metric spaces.

Another significant and thriving area of fixed point theory is relation-theoretic fixed point results, which initially appeared by Turinici [24] by putting out the idea of an order-theoretic fixed point result. Ran and Reurings [26] offered an extension of their conclusion to matrix equations and gave a natural order-theoretic formulation of the Banach contraction principle. By merging several well-known pertinent order-theoretic theorems with an arbitrary binary relation, Alam and Imdad [10] recently demonstrated a relation-theoretic variant of the Banach contraction principle. Then, a variety of fixed point conclusions with different conceptions of binary relations were presented (e.g [18–20, 23, 25]).

The concept of fuzzy sets has become a vital and significant mathematical tool. The concept of fuzzy metric space was introduced by Kramosil and Michalek [1] by expanding the idea of probabilistic metric space to the fuzzy setting. To obtain a Hausdorff topology, George and Veeramani [4] modified

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Kramosil and Michalek's concept of fuzzy metric space. In recent years, there has been a lot of interest in the study of fixed point theory in fuzzy metric spaces.

L.A. Zadeh [6] established the fuzzy set notion in 1965 as a new mathematical technique to interacting with ambiguity and vagueness in concrete world. It is built on a generalization of the crisp set and characteristic function concepts. The theory of fuzzy sets has emerged into an important and critical modeling tool. By generalizing the idea of probabilistic metric space to the fuzzy framework, Kramosil and Michalek [1] presented fuzzy metric space. In addition, George and Veeramani [4] changed Kramosil and Michalek's notion of fuzzy metric space in order to get a Hausdorff topology. The study of fixed point theory in fuzzy metric spaces has aroused an extensive attention in recent years. In this respect, Gregori and Sapena [3], who obtained specific fixed point findings, developed the idea of fuzzy contractive mappings. Then Mihet [13] proposed the concept of ψ-contractive mappings. The idea of $\mathcal{H}$-contractive mappings was first discussed and studied by Wardowski [14]. A fuzzy metric version of the simulation function approach was recently initiated by Abdelhamid Moussaoui *et al* [30]. Further distinct types of contractions can be viewed in [2,15,17,19,21,27–31]. In this study, using appropriate assumptions on a fuzzy metric space, we establish several relation-theoretic fixed point results involving a class of auxiliary functions and a binary relation. The given results combine, generalize, and enhance numerous prior findings in the literature.

2. Preliminaries

Throughout this article, $\mathbb{N}$ and $\mathbb{R}$ will be used to signify the set of all positive integer numbers and the set of all real numbers, respectively.

Definition 2.1. [7] A continuous binary operation $\curlywedge : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous t-norm if it is commutative, associative and satisfies:

(1) $\hbar_1 \curlywedge 1 = \hbar_1$ for all $\hbar_1 \in [0,1]$,

(2) $\hbar_1 \curlywedge \hbar_2 \leq \hbar_3 \curlywedge \hbar_4$ whenever $\hbar_1 \leq \hbar_3$ and $\hbar_2 \leq \hbar_4$, for all $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in [0,1]$.

Example 2.2. The following are some classic continuous t-norm examples: minimum t-norm, that is, $\hbar_1 \curlywedge_m \hbar_2 = \min\{\hbar_1, \hbar_2\}$, Lukasiewicz t-norm, $\hbar_1 \curlywedge_L \hbar_2 = \max\{\hbar_1 + \hbar_2 - 1, 0\}$ and product t-norm $\hbar_1 \curlywedge_p \hbar_2 = \hbar_1.\hbar_2$, for all $\hbar_1, \hbar_2 \in [0,1]$.

Definition 2.3. [4] Let E be a nonempty set, $\curlywedge$ is a continuous t-norm and $\Xi : E^2 \times (0,\infty) \to [0,1]$ is a fuzzy. An ordered triple $(E, \Xi, \curlywedge)$ is said to be a fuzzy metric space, if

$(\mathcal{M}1)$: $\Xi(s,t,\epsilon) > 0$,

$(\mathcal{M}2)$: $\Xi(s,t,\epsilon) = 1$ if and only if $s = t$,

$(\mathcal{M}3)$: $\Xi(s,t,\epsilon) = \Xi(t,s,\epsilon)$

$(\mathcal{M}4)$: $\Xi(s,t,\epsilon) \curlywedge \Xi(t,r,\zeta) \leq \Xi(s,r,\epsilon+\zeta)$,

($\mathcal{M}5$): $\Xi(s,t,):(0,\infty)\to[0,1]$ is continuous .

for all $s,t,r\in E$ and $\epsilon,\zeta>0$.

Example 2.4. [4] Let $(E,\mathcal{L})$ be a metric space. Define the function $\Xi : E\times E\times(0,\infty)\to[0,1]$ by $\Xi(s,t,\epsilon)=\frac{\epsilon}{\epsilon+\mathcal{L}(s,t)}$, for all $s,t\in E$, $\epsilon>0$. Then $(E,\Xi,\curlywedge_m)$ is a fuzzy metric space.

Example 2.5. [4,5] Let $(E,\mathcal{L})$ be a metric space and $\wp:\mathbb{R}^+\to[0,\infty)$ be an increasing continuous function. Define $\Xi : E\times E\times(0,\infty)\to[0,1]$ by

$$\Xi(s,t,\epsilon)=\exp\left(-\frac{\mathcal{L}(s,t)}{\wp(\epsilon)}\right),\text{ for all } s,t\in E \text{ and } \epsilon>0.$$

Then $(E,\Xi,\curlywedge_p)$ is a fuzzy metric space.

Example 2.6. [4,5] Let E be a nonempty set, $\xi:[0,\infty)\to\mathbb{R}^+$ be an increasing continuous function and $\varphi:E\to\mathbb{R}^+$ be a one-to-one function. For fixed $\aleph,\Im>0$, define $\Xi : E\times E\times(0,\infty)\to[0,1]$ by $\Xi(s,t,\epsilon)=\left(\frac{(\min\{\varphi(s),\varphi(t)\})^{\aleph}+\xi(\varsigma)}{(\max\{\varphi(s),\varphi(t)\})^{\aleph}+\xi(\varsigma)}\right)^{\Im}$, for all $s,t\in E$ and $\epsilon>0$. Then $(E,\Xi,\curlywedge_m)$ is a fuzzy metric space.

Lemma 2.7. [2] *$\Xi(s,t,.)$ is nondecreasing for all s,t in E.*

Definition 2.8. [4] Let $(E,\Xi,\curlywedge)$ be a fuzzy metric space.

(1) A sequence $\{s_n\}\subseteq E$ is said to be convergent to $s\in E$ if an only if $\lim_{n\to\infty}\Xi(s_n,s,\epsilon)=1$ for all $\epsilon>0$.

(2) A sequence $\{s_n\}\subseteq E$ is said to be a Cauchy sequence iff for each $\hbar\in(0,1)$ and $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that $\Xi(s_n,s_m,\epsilon)>1-\hbar$ for all $n,m\geq n_0$.

A complete fuzzy metric space is a fuzzy metric space in which every Cauchy sequence is convergent.

Definition 2.9. [3] Let $(E,\Xi,\curlywedge)$ be a fuzzy metric space. A mapping $\mathcal{F}:E\to E$ is said to be fuzzy contractive mapping if there exists $k\in(0,1)$ such that

$$\frac{1}{\Xi(\mathcal{F}s,\mathcal{F}t,\epsilon)}-1\leq k\left(\frac{1}{\Xi(s,t,\epsilon)}-1\right),$$

for each $s,t\in E$ and $\epsilon>0$.

Definition 2.10. [13] Let $(E,\Xi,\curlywedge)$ be a fuzzy metric space. A mapping $\mathcal{F}:E\to E$ is said to be fuzzy ψ-contractive mapping if

$$\Xi(\mathcal{F}s,\mathcal{F}t,\epsilon)\geq\psi(\Xi(s,t,\epsilon))\text{ for all } s,t\in E,\epsilon>0,$$

where Ψ is the set of all continuous and non-decreasing functions $\psi:(0,1]\to(0,1]$ such that $\psi(s)>s$, for all $s\in(0,1)$.

Definition 2.11. [14] Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space. A mapping $\mathcal{F} : E \to E$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in (0, 1)$ such that

$$\eta(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon)) \leq k\eta((\Xi(s, t, \epsilon)) \text{ for all } s, t \in E, \epsilon > 0,$$

where $\mathcal{H}$ is the family of strictly decreasing mappings $\eta : (0, 1] \longrightarrow [0, \infty)$ such that η transforms $(0, 1]$ onto $[0, \infty)$.

Definition 2.12. ([16], [17]) The function $\Gamma : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ is said to be a $\mathcal{FZ}$-simulation function, if the following conditions hold :

$(\Gamma 1)$: $\Gamma(1, 1) = 0$,

$(\Gamma 2)$: $\Gamma(a, b) < \frac{1}{b} - \frac{1}{a}$ for all $a, b \in (0, 1)$,

$(\Gamma 3)$: if $\{a_q\}, \{b_q\}$ are sequences in $(0, 1]$ such that $\lim_{q\to\infty} a_q = \lim_{q\to\infty} b_q < 1$ then $\lim_{q\to\infty} \sup \Gamma(a_q, b_q) < 0$.

The collection of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 2.13. ([16], [17]) Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space, $\mathcal{F} : E \to E$ a mapping and $\Gamma \in \mathcal{FZ}$. Then $\mathcal{F}$ is called a $\mathcal{FZ}$-contraction with respect to Γ if the following condition is satisfied

$$\Gamma(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon), \Xi(s, t, \epsilon)) \geq 0 \text{ for all } s, t \in E, \epsilon > 0.$$

Example 2.14. ([16], [17]) Let $\Gamma_i : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}, i = 1, 2, 3$ be defined by

(1) $\Gamma_1(a, b) = k\left(\frac{1}{b} - 1\right) - \frac{1}{a} + 1$ for all $a, b \in (0, 1]$, where $k \in (0, 1)$;

(2) $\Gamma_2(a, b) = \frac{1}{\psi(b)} - \frac{1}{a}$ for all $a, b \in (0, 1]$ and $\psi \in \Psi$;

(3) $\Gamma_3(a, b) = \frac{1}{\eta^{-1}(k.\eta(b))} - \frac{1}{a}$ for all $a, b \in (0, 1]$, where $\eta \in \mathcal{H}$.

Γ_i for $i = 1, 2, 3$ are $\mathcal{FZ}$-simulation functions.

To establish our results, we need some basic relation theoretic notions, concepts, and related results, which are outlined below.

Definition 2.15. [8] A subset $\mathcal{B}$ of $E \times E$ is called a binary relation on E. If $(s, t) \in \mathcal{B}$, then we say that s is related to t under $\mathcal{B}$ (or $s\mathcal{B}t$). If either $(s, t) \in \mathcal{B}$ or $(v, u) \in \mathcal{R}$ and we write $[s, t] \in \mathcal{R}$.

Note that, E^2 is a binary relation on E called the universal relation. Trivially, $\emptyset$ is the empty relation on E.

Definition 2.16. [8,9] A binary relation $\mathcal{B}$ on a non-empty set E is called:

(1) reflexive if $s\mathcal{B}s$ for all $s \in E$.

(2) transitive if $s\mathcal{B}t$ and $t\mathcal{B}w$ imply $s\mathcal{B}w$ for all $s, t, w \in E$.

(3) complete if $[s, t] \in \mathcal{B}$ for all $s, t \in E$.

(4) $\mathcal{F}$-closed if

$$(s,t) \in \mathcal{B} \Rightarrow (\mathcal{F}s, \mathcal{F}t) \in \mathcal{B},$$

for all $s, t \in E$, where $\mathcal{F}$ is a self-mapping defined on E.

Definition 2.17. [10] Let E be a non-empty set and $\mathcal{B}$ be a binary relation on E. A sequence $\{s_n\} \subseteq E$ is said to be an $\mathcal{B}$-preserving sequence if $(s_n, s_{n+1}) \in \mathcal{B}$ for all $n \in \mathbb{N}$.

Definition 2.18. [11] A binary relation $\mathcal{B}$ on E is said to be Ξ-self-closed if for any $\mathcal{B}$-preserving sequence $\{s_n\} \subseteq E$ such that

$$s_n \xrightarrow{\Xi} s \text{ as } n \to \infty,$$

there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ with $(s_{n_k}, s) \in \mathcal{B}$.

Definition 2.19. [25] Let $(E, \Xi, *)$ be a fuzzy metric space and $\mathcal{B}$ a binary relation on E. A sequence $\{s_n\} \subseteq E$ is said to be a $\mathcal{B}$-Cauchy sequence if $(s_n \mathcal{B} s_{n+1})$ for all $n \in \mathcal{N}$, $\varepsilon \in (0,1)$ and $\varsigma > 0$, there exists $n_0 \in \mathbb{N}$ such that $\Xi(s_{n+m}, s_n, \epsilon) > 1 - \hbar$ for all $n \geq n_0$ and $m \in \mathbb{N}$.

Remark 2.20. [25] For any arbitrary binary relation $\mathcal{B}$, every Cauchy sequence is an $\mathcal{B}$-Cauchy sequence. $\mathcal{B}$-Cauchyness coincides with Cauchyness if $\mathcal{B}$ is assumed to be the universal relation.

Note that, if every $\mathcal{B}$-Cauchy sequence is convergent in E, a fuzzy metric space $(E, \Xi, *)$ endowed with a binary relation $\mathcal{B}$ is said to be $\mathcal{B}$-complete.

Remark 2.21. [25] For any arbitrary binary relation $\mathcal{B}$, every complete fuzzy metric space is $\mathcal{B}$-complete fuzzy metric space,

Definition 2.22. [12] Let E be a nonempty set and $\mathcal{B}$ a binary relation on E. For $s, t \in E$, a path of length p in $\mathcal{B}$ from s to t is a finite sequence $\{\lambda_0, \lambda_1, \lambda_2, ..., \lambda_l\} \subset E$ satisfying:

$\mathcal{L}1$): $\lambda_0 = s$ and $\lambda_p = t$,

$\mathcal{L}2$): $(\lambda_m, \lambda_{m+1}) \in \mathcal{B}$ for all m $(0 \leq m \leq p-1)$.

Note that, a path of length p involves $p+1$ elements of E.

Let ϑ be a nonempty set, $\mathcal{H} \subseteq E$ and $\mathcal{F} : E \to E$ be a self-mapping. We will use the following notations:

$$E(\mathcal{F}, \mathcal{B}) := \{s \in E : (s, \mathcal{F}s) \in \mathcal{B}\},$$

$$\Upsilon(s, t, \mathcal{B}) := \text{the class of all path in } \mathcal{B} \text{ from } s \text{ to } t.$$

and $\mathcal{B}|_{\mathcal{H}}$ the restriction of $\mathcal{B}$ to $\mathcal{H}$ is the set $\mathcal{B}|_{\mathcal{H}} := \mathcal{B} \cap \mathcal{H}^2$. In fact, $\mathcal{B}|_{\mathcal{H}}$ is a relation on $\mathcal{H}$ induced by $\mathcal{B}$.

Definition 2.23. Let be a fuzzy metric space, $\mathcal{B}$ is a binary relation in E. A mapping $\mathcal{F} : E \to E$ is said to be $\mathcal{B}$-continuous at $s \in E$ if for all $\mathcal{B}$-preserving sequence $\{s_n\}$ such that $s_n \xrightarrow{\Xi} s$, we have $\mathcal{F}s_n \xrightarrow{\Xi} \mathcal{F}s$. $\mathcal{F}$ is said to be $\mathcal{B}$-continuous if it is $\mathcal{B}$-continuous at each point of E.

Remark 2.24. Every continuous mapping is $\mathcal{B}$-continuous, for any binary relation $\mathcal{B}$. Particularly, under the universal relation the notion of $\mathcal{B}$-continuity coincides with usual continuity.

In order to prove our main results, the following lemma will be useful.

Lemma 2.25. [25] *Let $\mathcal{F} : E \to E$ be a self-mapping and $\mathcal{B}$ a transitive binary relation which is $\mathcal{F}$-closed. Assume that there exists $s_0 \in E$ such that $s_0 B \mathcal{F} s_0$ and define $\{s_n\}$ in E by $s_n = \mathcal{F}s_{n-1}$, for all $n \in \mathbb{N}$. Then*

$$s_m \mathcal{B} s_n \text{ for all } n, m \in \mathbb{N} \text{ with } n < m.$$

3. Main results

Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space, $\mathcal{B}$ is a binary relation in E and $\Gamma \in \mathcal{FZ}$, we denote by $\Re(\mathcal{F}, \Gamma, \mathcal{B})$ the family of mappings $\mathcal{F} : E \to E$ satisfying

$$\Gamma(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon), \Xi(s, t, \epsilon)) \geq 0, \tag{1}$$

for all $s, t \in E, \epsilon > 0$ such that $(s, t) \in \mathcal{B}$.

Proposition 3.1. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space, $\mathcal{B}$ is a binary relation in E and $\mathcal{F} : E \to E$ be a mapping belongs to $\Re(\mathcal{F}, \Gamma, \mathcal{B})$ for some $\Gamma \in \mathcal{FZ}$. Then the following are complement to each other*

(i)**:** $\Gamma(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon), \Xi(s, t, \epsilon)) \geq 0$, *for all $s, t \in E$ with $(s, t) \in \mathcal{B}$,*

(ii)**:** $\Gamma(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon), \Xi(s, t, \epsilon)) \geq 0$, *for all $s, t \in E$ with $[s, t] \in \mathcal{B}$.*

Proof. The first implication $(ii) \Rightarrow (i)$ is trivial. Conversely, assume that (i) holds. Consider $s, t \in E$ with $[s, t] \in \mathcal{B}$, then (ii) can be derived directly from (i). Otherwise, if $(s, t) \in \mathcal{B}$, then using the symmetry of the fuzzy metric Ξ and (i), we get

$$\Gamma(\Xi(\mathcal{F}t, \mathcal{F}s, \epsilon), \Xi(t, s, \epsilon)) = \Gamma(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon), \Xi(s, t, \epsilon)) \geq 0,$$

Thus, (i) yields (ii). □

Theorem 3.2. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric equipped with a binary relation $\mathcal{B}$ and $\mathcal{F} : E \to E$ be a self-mapping. Suppose that*

- **(i)** *there exist $\mathcal{H} \subseteq E$, $\mathcal{F}E \subseteq \mathcal{H} \subseteq E$, so that $(\mathcal{H}, \Xi, \curlywedge)$ is $\mathcal{B}$-complete,*
- **(ii)** $E(\mathcal{F}, \mathcal{B}) \neq \emptyset$,
- **(iii)** *$\mathcal{B}$ is $\mathcal{F}$-closed and $\mathcal{B}$ is transitive,*
- **(iv)** *$\mathcal{F} \in \Re(\mathcal{F}, \Gamma, \mathcal{B})$ for some $\Gamma \in \mathcal{FZ}$,*

(**v**) *either $\mathcal{B}|_{\mathcal{H}}$ is Ξ-self-closed or $\mathcal{F}$ is $\mathcal{B}$-continuous.*

Then $\mathcal{F}$ has a fixed point.

Proof. Since $E(\mathcal{F},\mathcal{B}) \neq \emptyset$, let s_0 be an arbitrary point such that $s_0 \in E(\mathcal{F},\mathcal{B})$. Now, define a Picard sequence $\{s_n\}$ by $s_{n+1} = \mathcal{F}s_n$ for all $n \in \mathbb{N}$. Using the fact that $\mathcal{B}$ is $\mathcal{F}$-closed and $(s_0, \mathcal{F}s_0) \in \mathcal{B}$, we have

$$(\mathcal{F}s_0, \mathcal{F}^2 s_0), (\mathcal{F}^2 s_0, \mathcal{F}^3 s_0), ..., (\mathcal{F}^n s_0, \mathcal{F}^{n+1} s_0) \in \mathcal{B}.$$

Hence,

$$(s_n, s_{n+1}) \in \mathcal{B}, \tag{2}$$

and the sequence $\{s_n\}$ is $\mathcal{B}$-preserving. Since $\mathcal{F} \in \Re(\mathcal{F},\Gamma,\mathcal{B})$ for some $\Gamma \in \mathcal{FZ}$, we have

$$\Gamma(\Xi(\mathcal{F}s_{n-1}, \mathcal{F}s_n, \epsilon), \Xi(s_{n-1}, s_n, \epsilon)) \geq 0, \tag{3}$$

From (3) and $(\Gamma 2)$, we have

$$\begin{aligned} 0 &\leq \Gamma(\Xi(\mathcal{F}s_{n-1}, \mathcal{F}s_n, \epsilon), \Xi(s_{n-1}, s_n, \epsilon)) \\ &= \Gamma(\Xi(s_n, s_{n+1}, \epsilon), \Xi(s_{n-1}, s_n, \epsilon))\} \\ &< \frac{1}{\Xi(s_{n-1}, s_n, \epsilon)} - \frac{1}{\Xi(s_n, s_{n+1}, \epsilon)}. \end{aligned}$$

Thus, $\Xi(s_{n-1}, s_n, \epsilon) < \Xi(s_n, s_{n+1}, \epsilon)$, which means that $\{\Xi(s_{n-1}, s_n, \epsilon)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Then, there exists $w(\epsilon) \leq 1$ such that $\lim_{n\to\infty} \Xi(s_n, s_{n-1}, \epsilon) = w(\epsilon)$ for all $\epsilon > 0$. We shall show that $w(\epsilon) = 1$. By contradiction, assume that $w(\epsilon_0) < 1$ for some $\epsilon_0 > 0$. By (2) and applying $(\Gamma 3)$, we get

$$0 \leq \lim_{n\to\infty} \sup \Gamma(\Xi(s_n, s_{n+1}, \epsilon_0), \Xi(s_{n-1}, s_n, \epsilon_0)) < 0,$$

A contradiction. Which gives that $w(\epsilon) = 1$. Then

$$\lim_{n\to\infty} \Xi(s_n, s_{n+1}, \epsilon) = 1 \text{ for all } \epsilon > 0. \tag{4}$$

Next, we prove that the sequence $\{s_n\}$ is Cauchy. Assume that $\{s_n\}$ is not Cauchy. Then, there exists $h \in (0, 1)$, $\epsilon_0 > 0$ and two subsequences $\{s_{n_k}\}$ and $\{s_{m_k}\}$ of $\{s_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\Xi(s_{m_k}, s_{n_k}, \epsilon_0) \leq 1 - h \tag{5}$$

From Lemma 2.7, we get

$$\Xi(s_{m_k}, s_{n_k}, \frac{\epsilon_0}{2}) \leq 1 - h. \tag{6}$$

By choosing m_k as the lowest index satisfying (6), we get

$$\Xi(s_{m_k}, s_{n_k-1}, \frac{\epsilon_0}{2}) > 1 - h. \tag{7}$$

By Lemma 2.25 and (1), we have

$$0 \leq \Gamma(\Xi(s_{m_k}, s_{n_k}, \epsilon_0), \Xi(s_{m_k-1}, s_{n_k-1}, \epsilon)), \tag{8}$$

From (8) and ($\Gamma 2$), we obtain

$$\Xi(s_{m_k-1}, s_{m_k-1}, \epsilon_0) < \Xi(s_{m_k}, s_{n_k}, \epsilon_0). \tag{9}$$

By (5),(7) and the triangular inequality, we get

$$\begin{aligned} 1 - h &\geq \Xi(s_{m_k}, s_{n_k}, \epsilon_0) \\ &> \Xi(s_{m_k-1}, s_{n_k-1}, \epsilon_0) \\ &\geq \Xi(s_{m_k-1}, s_{m_k}, \frac{\epsilon_0}{2}) \curlywedge \Xi(s_{m_k}, s_{n_k-1}, \frac{\epsilon_0}{2}) \\ &> \Xi(s_{n_k-1}, s_{n_k}, \frac{\epsilon_0}{2}) \curlywedge (1 - h). \end{aligned}$$

Taking limit as $k \to \infty$ and using (4), we get

$$\lim_{k\to\infty} \Xi(s_{m_k}, s_{n_k}, \epsilon_0) = \lim_{k\to\infty} \Xi(s_{m_k-1}, s_{n_k-1}, \epsilon_0) = 1 - h. \tag{10}$$

Thus, the sequences $a_k = \Xi(s_{m_k-1}, s_{n_k-1}, \epsilon_0)$ and $b_k = \Xi(s_{m_k}, s_{n_k}, \epsilon_0)$ have the same limit $1 - h < 1$. Now, by considering that $\mathcal{F} \in \Re(\mathcal{F}, \Gamma, \mathcal{B})$ for some $\Gamma \in \mathcal{FZ}$ and ($\Gamma 3$), we get

$$0 \leq \lim_{k\to\infty} \sup \Gamma(\Xi(s_{m_k}, s_{n_k}, \epsilon_0), \Xi(s_{m_k-1}, s_{n_k-1}, \epsilon_0)) < 0.$$

A contradiction. Then, $\{s_n\}$ is a Cauchy sequence. As $\{s_n\} \subseteq \mathcal{F}E \subseteq \mathcal{H}$, therefore $\{s_n\}$ is $\mathcal{B}$-preserving Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is a $\mathcal{B}$-complete, there exists $c \in \mathcal{H}$ such that $s_n \to c$. If $\mathcal{F}$ is $\mathcal{B}$-continuous, then

$$c = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \mathcal{F}s_n = \mathcal{F}c.$$

Thus, c is a fixed point of $\mathcal{F}$.

Now, if $\mathcal{B}$ is Ξ-self-closed. As $\{s_n\}$ is an $\mathcal{B}$-preserving sequence and $s_n \to s$, there exists a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $[s_{n_k}, s] \in \mathcal{B}|_{\mathcal{H}}$ for all $n \in \mathbb{N}$ and Proposition (3.1), $[s_{n_k}, s] \in \mathcal{B}$ and $s_{n_k} \to s$, we get

$$\begin{aligned} 0 &\leq \Gamma(\Xi(\mathcal{F}s_{n_k}, \mathcal{F}s, \epsilon), \Xi(s_{n_k}, s, \epsilon)) \\ &= \Gamma(\Xi(s_{n_k+1}, \mathcal{F}s, \epsilon), \Xi(s_{n_k}, s, \epsilon)) \\ &< \frac{1}{\Xi(s_{n_k}, s, \epsilon)} - \frac{1}{\Xi(s_{n_k+1}, \mathcal{F}s, \epsilon)}. \end{aligned}$$

Then, $\Xi(s_{n_k}, s, \epsilon) < \Xi(s_{n_k+1}, \mathcal{F}s, \epsilon)$, passing to the limit as $k \to \infty$, we derive that $1 \leq \lim_{k\to\infty} \Xi(s_{n_k+1}, \mathcal{F}s, \epsilon) = \Xi(s, \mathcal{F}s, \epsilon)$ Then, $\Xi(s, \mathcal{F}s, \epsilon) = 1$, then, $\mathcal{F}s = s$. Thus, s is a fixed point of $\mathcal{F}$. □

Theorem 3.3. *Adding $\Upsilon(s, t, \mathcal{B}) \neq \emptyset$ to the hypothesises of Theorem 3.2, then $\mathcal{F}$ has a unique fixed point.*

Proof. By contradiction, assume that s and s^* are two distinct fixed points of $\mathcal{F}$. As $\gamma(s, s^*, \mathcal{B}) \neq \emptyset$, there exists a path $\{\xi_0, \xi_1, \xi_2, ..., \xi_n\}$ of some finite length n in $\mathcal{B}$ from s to s^* such that

$$\xi_0 = s, \xi_n = s^*, (\xi_j, \xi_{j+1}) \in \mathcal{B}, j = 0, 1, 2, ..., n-1.$$

As $\mathcal{B}$ is transitive, then $(\xi_0, \xi_n) \in \mathcal{B}$. Since $\mathcal{F} \in \Re(\mathcal{F}, \Gamma, \mathcal{B})$ with respect to $\Gamma \in \mathcal{FZ}$, we have

$$\begin{aligned} 0 \leq &\Gamma(\Xi(\mathcal{F}\xi_0, \mathcal{F}\xi_n, \epsilon), \Xi(\xi_0, \xi_n, \epsilon)) \\ < &\frac{1}{\Xi(\xi_0, \xi_n, \epsilon)} - \frac{1}{\Xi(\mathcal{F}\xi_0, \mathcal{F}\xi_n, \epsilon)} \\ = &\frac{1}{\Xi(u, u^*, \epsilon)} - \frac{1}{\Xi(u, u^*, \epsilon)}. \end{aligned}$$

a contradiction. Thus, the fixed point of $\mathcal{F}$ is unique. □

Corollary 3.4. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{B}$ and $\mathcal{F} : E \to E$ be a self-mapping. Suppose that*

- **(i)** *$(E, \Xi, \curlywedge)$ is complete,*
- **(ii)** *$E(\mathcal{F}, \mathcal{B}) \neq \emptyset$,*
- **(iii)** *$\mathcal{B}$ is $\mathcal{F}$-closed and $\mathcal{B}$ is transitive,*
- **(iv)** *$\mathcal{F} \in \Re(\mathcal{F}, \Gamma, \mathcal{B})$ for some $\xi \in \mathcal{FZ}$,*
- **(v)** *either $\mathcal{B}$ is Ξ-self-closed or $\mathcal{F}$ is continuous.*

Then $\mathcal{F}$ has a fixed point.

Proof. It follows by setting $\mathcal{H} = E$ in Theorem 3.2. □

Corollary 3.5. *If completeness of $\mathcal{H}$ is remplaced by completeness and $\mathcal{B}$-continuity by continuity, Theorem 3.2 remains valide.*

Proof. It follows as natural outcome of Remarks 2.21 and 2.24. □

Corollary 3.6. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{B}$ and $\mathcal{F} : E \to E$ be a self-mapping, Suppose that*

- **(i)** $\frac{1}{\Xi(\mathcal{F}s, \mathcal{F}s, \epsilon)} - 1 \leq k(\frac{1}{\Xi(s,t,\epsilon)} - 1)$, *$s, t \in E, \epsilon > 0$ with $(s, t) \in \mathcal{B}, k \in (0, 1)$,*
- **(ii)** *there exist $\mathcal{H} \subseteq E$, $\mathcal{F}E \subseteq \mathcal{H} \subseteq E$, so that $(\mathcal{H}, \Xi, \curlywedge)$ is $\mathcal{B}$-complete,*
- **(iii)** *$E(\mathcal{F}, \mathcal{B}) \neq \emptyset$,*

(**iv**) *$\mathcal{B}$ is $\mathcal{F}$-closed and $\mathcal{B}$ is transitive,*

(**v**) *either $\mathcal{B}|_{\mathcal{H}}$ is Ξ-self-closed or $\mathcal{F}$ is $\mathcal{B}$-continuous.*

Then $\mathcal{F}$ has a fixed point.

Proof. It follows by setting $\Gamma : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $\Gamma(a,b) = k\left(\frac{1}{b} - 1\right) - \frac{1}{a} + 1$ for all $a, b \in (0,1]$, in Theorem 3.2. □

Corollary 3.7. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{B}$ and $\mathcal{F} : E \to E$ be a self-mapping, Suppose that*

(**i**) $\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon) \geq \psi(\Xi(s,t,\epsilon)),\ s,t \in E, \epsilon > 0$ *with* $(s,t) \in \mathcal{B}$,

(**ii**) *there exist $\mathcal{H} \subseteq E$, $\mathcal{F}E \subseteq \mathcal{H} \subseteq E$, so that $(\mathcal{H}, \Xi, \curlywedge)$ is $\mathcal{B}$-complete,*

(**iii**) $E(\mathcal{F}, \mathcal{B}) \neq \emptyset$,

(**iv**) *$\mathcal{B}$ is $\mathcal{F}$-closed and $\mathcal{B}$ is transitive,*

(**v**) *either $\mathcal{B}|_{\mathcal{H}}$ is Ξ-self-closed or $\mathcal{F}$ is $\mathcal{B}$-continuous.*

Then $\mathcal{F}$ has a fixed point.

Proof. It follows by defining $\Gamma : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $\Gamma(a,b) = \frac{1}{\psi(b)} - \frac{1}{a}$ for all $a, b \in (0,1]$ in Theorem 3.2. □

Corollary 3.8. *Let $(E, \Xi, \curlywedge)$ be a fuzzy metric space endowed with a binary relation $\mathcal{B}$ and $\mathcal{F} : E \to E$ be a self-mapping, Suppose that*

(**i**) $\eta(\Xi(\mathcal{F}s, \mathcal{F}t, \epsilon)) \leq k\eta(\Xi(s,t,\epsilon))$, $s,t \in E$, $\epsilon > 0$ *with* $(s,t) \in \mathcal{B}$, $k \in (0,1)$,

(**ii**) *there exist $\mathcal{H} \subseteq E$, $\mathcal{F}E \subseteq \mathcal{H} \subseteq E$, so that $(\mathcal{H}, \Xi, \curlywedge)$ is $\mathcal{B}$-complete,*

(**iii**) $E(\mathcal{F}, \mathcal{B}) \neq \emptyset$,

(**iv**) *$\mathcal{B}$ is $\mathcal{F}$-closed and $\mathcal{B}$ is transitive,*

(**v**) *either $\mathcal{B}|_{\mathcal{H}}$ is Ξ-self-closed or $\mathcal{F}$ is $\mathcal{B}$-continuous.*

Then $\mathcal{F}$ has a fixed point.

Proof. It follows by setting $\Gamma : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ by $\Gamma(a,b) = \frac{1}{\eta^{-1}(k.\eta(b))} - \frac{1}{a}$ for all $a, b \in (0,1]$, where $\eta \in \mathcal{H}$ in Theorem 3.2. □

Competing interests

The authors declare that they have no competing interests.

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