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EXISTENCE AND UNIQUENESS OF GENERALIZED SOLUTION OF SINE-GORDON EQUATION BY USING FIXED POINT IN COLOMBEAU ALGEBRA

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Abstract. In this work, we study the existence and uniqueness of solution to the Sine-Gordon equation within the framework of Colombeau algebra. Using the concept of fixed point and exploiting the well-known fixed point theorem of J.A.Marti [6]. Additionally, we provide an example to illustrate our results.

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1. Introduction

The Sine-Gordon equation is a nonlinear integrable partial differential equation in the space-time coordinate. It is a nonlinear hyperbolic partial differential equation for a function φ dependent on two variables typically denoted x and t, involving the wave operator and the sine of φ, note that it is known in the 19th century during the study of various problems of differential geometry. The equation has numerous applications in physics [3,7]. The equation can be written simply as $\xi_{tt} - \xi_{yy} + \sin(\xi) = 0$, where $\xi = \xi(y,t)$. In the case of mechanical trasmission line, $\xi(y,t)$ describes an angle of rotation of the pendulums including applications in relativistic field theory. In the early eighties of the last century, Colombeau introduced an algebra $\mathcal{G}$ of generalized functions to deal with the multiplication problem of distributions, see [1, 2]. This algebra $\mathcal{G}$ is differential which contains the space $\mathcal{D}'$ of Schwartz distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $\mathcal{G}$. Therefore this algebra is a very convenient one to find and study solutions of nonlinear differential equations with singular data and coefficients. This algebra plays a crucial role in order to give a sense of multiplication of distributions [4,9]. As a nonlinear extension of distribution theory to deal with nonlinearities and singularities of data and coefficients in PDE theory [5,9,11,12]. These

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algebras contain the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. Elements of $\mathcal{G}$ are classes of nets of smooth functions called moderates functions with respect to a set of negligibles functions. The reason for introducing this regularity is the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order. Fixed point theory has fascinated many researchers since 1922 with Banach's famous fixed point theorem. There is a vast literature on the subject and it is a very active area of research at present. Fixed point theory is very important tools for proving the existence and uniqueness of the solutions to various mathematical models: integral and partial differential equations, variational inequalities, etc. This theory has been studied by many researchers, but it is rare to find a paper that presented the fixed point theory in Colombeau algebra. We will rely on the work of J.Martin in [6] and we will use the topology of locally convex spaces to make sense of the concept of a fixed point in a class of Colombeau algebra compatible with our study of the sine-Gordon equation. In this paper, we investigate the existence and uniqueness of solutions to the problem given by

$$\begin{cases} (\partial_t^2 - \partial_y^2)\xi = 2\sin(\xi), \quad y \in \mathbb{R},\ t \in \mathbb{R}_+, \\ \quad \xi(0,y) = a(y), \quad y \in \mathbb{R}, \\ \quad \partial_t \xi(0,y) = b(y), \quad y \in \mathbb{R}, \end{cases}$$

where a and b are two given distributions.
The organization of the paper is as follows. In section 2, we recall some fundamentals properties of the generalized functions theory. The new notion of generalized semigroup take place in section 3. Section 4 is consecrated for the proof of the fixed point thoerem in Colombeau algebra. In Section 5 we have introduced an example to illustrate our work.

2. Preliminairies

In this section, we recall a few basics properties from the theory of Colombeau generalized functions. The regularization methods of Colombeau-type is to model nonsmooth objects by approximating nets of any smooth functions, which has a moderate asymptotics bounds and to identify regularizing nets whose differences compared to the moderateness scale are negligible. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations nets, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ϵ. Therefore, for any set X, the family of sequences $(\xi_\epsilon)_{\epsilon\in(0,1)}$ of elements of a set X will be denoted by $X^{(0,1)}$, such sequences will also be called nets and simply written as ξ_ϵ. Let $n \in \mathbb{N}^*$, as in [4], we define the set

$$\mathcal{E}(\mathbb{R}^n) = \left(\mathcal{C}^\infty\left(\mathbb{R}^n\right)\right)^{(0,1)}.$$

The set of moderate functions is given as follows

$$\mathcal{E}_M(\mathbb{R}^n) = \Big\{(\xi_\epsilon)_{\epsilon>0} \subset \mathcal{E}(\mathbb{R}^n) : \forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N} \ / \\ \sup_{y\in K} |\partial^\alpha \xi_\epsilon(y)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\}.$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(\xi_\epsilon)_{\epsilon>0} \subset \mathcal{E}(\mathbb{R}^n)/\forall K \subset\subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \forall p \in \mathbb{N} \ / \\ \sup_{y\in K} |\partial^\alpha \xi_\epsilon(y)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{p})\Big\}.$$

With operations defined componentwise, e.g., $(\xi_\epsilon) + (v_\epsilon) = (\xi_\epsilon + v_\epsilon)$ etc. The Colombeau algebra is defined as a factor set $\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n)$.

Also we define the following sets

$$\big|\mathcal{E}_M(\mathbb{R}^n)\big| = \Big\{ \left(|\xi_\epsilon|\right)_\epsilon, \ (\xi_\epsilon)_\epsilon \in \mathcal{E}_M(\mathbb{R}^n)\Big\},$$

and

$$\big|\mathcal{N}(\mathbb{R}^n)\big| = \Big\{ \left(|\xi_\epsilon|\right)_\epsilon, \ (\xi_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^n)\Big\}.$$

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}\left(\mathbb{R}\right)/\mathcal{I}\left(\mathbb{R}\right),$$

where

$$\mathcal{E}(\mathbb{R}) = \Big\{(y_\epsilon)_\epsilon \in (\mathbb{R})^{(0,1)}/\exists m \in \mathbb{N}, |y_\epsilon| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-m})\Big\},$$

and

$$\mathcal{I}(\mathbb{R}) = \Big\{(y_\epsilon)_\epsilon \in (\mathbb{R})^{(0,1)}/\forall m \in \mathbb{N}, |y_\epsilon| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{m})\Big\}.$$

We note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families. It is easy to prove that

Proposition 2.1. *The space $\mathcal{E}(\mathbb{R})$ is an algebra, and $\mathcal{I}(\mathbb{R})$ is an ideal of $\mathcal{E}(\mathbb{R})$.*

In the same we define

$$\big|\mathcal{E}(\mathbb{R})\big| = \Big\{ \left(|r_\epsilon|\right)_\epsilon, \ r_\epsilon \in \mathcal{E}(\mathbb{R})\Big\},$$

and

$$\big|\mathcal{I}(\mathbb{R})\big| = \Big\{ \left(|r_\epsilon|\right)_\epsilon, \ r_\epsilon \in \mathcal{I}(\mathbb{R})\Big\}.$$

We will closed this section by the Grönwall's inequality given in the following

Lemma 2.2. *Let α, β and u be three functions defined on an a bounded interval $I = (a, b)$. Suppose that β and u are continuous, moreover, we assume that α is locally integrable on I.*

(1) *If β is non-negative and if u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \ \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s) \exp\big(\int_s^t \beta(r)dr\big)ds, \ t \in I.$$

(2) *If, in addition, the function α is non-decreasing, then*

$$u(t) \leq \alpha(t) \exp\big(\int_a^t \beta(s)ds\big), \ t \in I.$$

3. Generalized Fixed Points

3.1. **Locally convex spaces.** In this subsection, we present the notion of locally convex spaces and the notions of completeness in this type of space.

Definition 3.1. Let Y be a vector space indowed with a familly $(N_i)_{i\in I}$ of seminorms. For all $i \in I$, we denote τ_i the topology induced by the seminorm N_i, and τ the topology generated by the classe of the all union sets τ_i . The pair (Y, τ) is said to be locally convex space.

The set of all balls of the form

$$B(i, r) = \Big\{y \in Y / \quad N_i(y) < r\Big\}, \quad \forall i \in I \text{ and } r > 0.$$

is called a basis of 0-neighbourhood where $(N_i)_i$ is a family of seminorms. Then, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if.

$$\big(\forall \epsilon > 0\big)\big(\forall i \in I\big)\big(\exists n_0 \in \mathbb{N}\big)\big(\forall n, p \in \mathbb{N} \quad n \geq n_0 \text{ implies } N_i(y_{n+p} - y_n) < \epsilon\big),$$

and Y is sequentially complete if any Cauchy sequence converges to an element of Y.

3.2. **Contraction in locally convex and complete spaces.** The contraction map in type Colombeau algebra is discussed in this subsection this contraction type is inspired by that in the classic case in locally convex spaces Y.

Definition 3.2. We recall that a map $H_\epsilon : Y \longrightarrow Y$ is called contraction if for all $i \in I$ there exits $k_i < 1$ such that

$$\forall (y_\epsilon, z_\epsilon) \in X \times X, \ N_i(H_\epsilon y_\epsilon - H_\epsilon z_\epsilon) \leq k_i \, N_i(y_\epsilon - z_\epsilon).$$

We have the following result

Theorem 3.3. [6] *Any contraction $H_\epsilon : Y \longrightarrow Y$ has a fixed point. Moreover if Y is Hausdorff, this fixed point is unique.*

3.3. **Contraction operator in $\tilde{Y}$.** We will give a notion of contraction map in type Colombeau algebra.

Definition 3.4. Consider a locally convex space Y endowed with a familly of seminorms $(N_i)_{i\in I}$. A class of moderate functions compatible with properties of the space X is defined by

$$\mathcal{E}_M(Y) = \Big\{(y_\epsilon)_\epsilon \in (Y)^{(0,1)} / \exists m \in \mathbb{N}, \forall i \in I, N_i(y_\epsilon) = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-m})\Big\}.$$

The corresponding class of negligible functions is given as follows

$$\mathcal{N}(Y) = \Big\{(y_\epsilon)_\epsilon \in (Y)^{(0,1)} / \forall m \in \mathbb{N}, \forall i \in I, \quad N_i(y_\epsilon) = \mathcal{O}_{\epsilon\to 0}(\epsilon^{m})\Big\}.$$

The Colombeau algebra type is given in this case by

$$\tilde{Y} = \mathcal{E}_M(Y)/\mathcal{N}^s(Y).$$

First, we will see if it's possible to give a definition of a map $H : \tilde{Y} \longrightarrow \tilde{Y}$ by the data of a family $(H_\epsilon)_{\epsilon\in(0,1)}$ of maps $H_\epsilon : X \longrightarrow X$ where H_ϵ is a linear and continuous operator on X. The general idea is given in the following result

Lemma 3.5. [10] *Let $(H_\epsilon)_{\epsilon\in(0,1)}$ be a given family of maps $H_\epsilon : Y \longrightarrow Y$. For each $(y_\epsilon)_\epsilon \in \mathcal{E}_M(Y)$ and $(y_\epsilon)_\epsilon \in \mathcal{N}(Y)$, suppose that*

(1) $\Big(H_\epsilon y_\epsilon\Big)_\epsilon \in \mathcal{E}_M(Y)$,
(2) $\Big(H_\epsilon(y_\epsilon + y_\epsilon)\Big)_\epsilon - \Big(H_\epsilon y_\epsilon\Big)_\epsilon \in \mathcal{N}(Y)$.

Then

$$H : \begin{cases} \tilde{Y} \longrightarrow \tilde{Y} \\ y = \Big[y_\epsilon\Big] \longmapsto Hy = \Big[H_\epsilon y_\epsilon\Big], \end{cases}$$

is well defined.

Definition 3.6. [6] A map $H : \tilde{Y} \to \tilde{Y}$ is called a contraction if only if

a) $(y_\epsilon)_\epsilon \in \mathcal{E}_M(Y)$, implies $(H_\epsilon y_\epsilon)_\epsilon \in \mathcal{E}_M(Y)$ for all $\epsilon \in (0,1)$.

b) H_ϵ is a contraction in $\big(Y, \tau_\epsilon\big)$ endowed with the family $M_\epsilon = (M_{\epsilon,i})_{i\in I}$ and the corresponding contraction constants are denoted by $l_{\epsilon,i} < 1$.

c) For every $i \in I$ and $\epsilon \in (0,1]$, $\exists \alpha_{\epsilon,i} > 0$ and $\beta_{\epsilon,i} > 0$, such that

$$\alpha_{\epsilon,i}\, N_i \leq M_{\epsilon,i} \leq \beta_{\epsilon,i}\, N_i.$$

d) For each $i \in I$, $\forall \epsilon \in (0,1]$, $(\frac{\beta_{\epsilon,i}}{\alpha_{\epsilon,i}})_\epsilon$ and $(\frac{1}{1-l_{\epsilon,i}})_\epsilon \in |\mathcal{E}_M(\mathbb{R})|$.

The essential result given in the next theorem which has been proven in [10]

Theorem 3.7. *With the same previous notations, any contraction $H : \tilde{Y} \longrightarrow \tilde{Y}$ has a fixed point in $\tilde{Y}$.*

4. Main Results

In order to study the existence and uniqueness of Colombeau generalized solutions of Sine-Gordon equation with initial data are distributions, one introduces the algebra of generalized functions suitable to this context. Let $\Omega_0 = [-k, k]$ be a compact interval. For $0 \leq t \leq s \leq k$, define an interval I_t and a trapezoidal region Ω_s are defined by

$$I_t = \{y \in \mathbb{R}, \ / \ |x| \leq k - t\},$$

$$\Omega_s = \{(t, y) \in \mathbb{R}^+ \times \mathbb{R} \ / \ 0 \leq t \leq s, \ , y \in I_t\}.$$

We define the simplified algebra of global generalized functions, which must be compatible with the study of the Sine-Gordon equation denoted $\mathcal{G}([0, \infty) \times \mathbb{R})$ by the quotient algebra

$$\mathcal{G}([0, \infty) \times \mathbb{R}) = \mathcal{E}^s_M([0, \infty) \times \mathbb{R}) / \mathcal{N}^s([0, \infty) \times \mathbb{R}),$$

where

$$\mathcal{E}^s_M([0, \infty) \times \mathbb{R}) = \Bigg\{ (\xi_\epsilon)_\epsilon \in (\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}))^{(0,1)} / \exists m \in \mathbb{N}, \forall T > 0, \sup_{t \in [0,T]} \{\| \xi_\epsilon(t, .) \|_{L^\infty(\Omega_t)}\} = \mathcal{O}_{\epsilon \to 0}(\epsilon^{-m}) \Bigg\},$$

and

$$\mathcal{N}^s([0, \infty) \times \mathbb{R}) = \Bigg\{ (\xi_\epsilon)_\epsilon \in (\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}))^{(0,1)} / \forall m \in \mathbb{N}, \forall T > 0, \sup_{t \in [0,T]} \{\| \xi_\epsilon(t, .) \|_{L^\infty(\Omega_t)}\} = \mathcal{O}_{\epsilon \to 0}(\epsilon^{m}) \Bigg\}.$$

Let us consider the Sine-Gordon equation with initial data are distributions,

$$\begin{cases} (\partial_t^2 - \partial_y^2)\xi = 2\sin(\xi), \quad y \in \mathbb{R}, \ \ t \in \mathbb{R}^+ \\ \xi(0, y) = a(y), \quad y \in \mathbb{R}, \\ \partial_t \xi(0, y) = b(y), \quad y \in \mathbb{R}. \end{cases} \tag{4.1}$$

Theorem 4.1. *The problem (4.1) has unique solution in $\mathcal{G}([0, \infty) \times \mathbb{R})$.*

The problem is equivalent to finding a fixed point of the map

$$H : \begin{cases} \mathcal{G} \longrightarrow \mathcal{G} \\ \xi \mapsto H(\xi), \end{cases}$$

where

$$H(\xi(t,y)) = \frac{a(y-t)+a(y+t)}{2} + \frac{1}{2}\int_{y-t}^{y+t} b(y)dy + \int_0^t \int_{y+s-t}^{y+t-s} \sin\big(\xi(s,z)\big)dzds,$$

for all $(t,y) \in \mathbb{R}^+ \times \mathbb{R}$. We will check the four conditions of the Definition 3.6.

a) Let

$$H_\epsilon(\xi_\epsilon(t,y)) = \frac{1}{2}\big(a_\epsilon(y-t)+a_\epsilon(y+t)\big) + \frac{1}{2}\int_{y-t}^{y+t} b_\epsilon(y)dy + \int_0^t \int_{y+s-t}^{y+t-s} \sin\big(\xi_\epsilon(s,z)\big)dzds \tag{4.2}$$

It is clear that H_ϵ is defined from $\mathcal{C}^\infty\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big)$ into $\mathcal{C}^\infty\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big)$. Endowed with the topology τ given by the family of norms $(N_T)_{T\in[0,k]}$, $\mathcal{C}^\infty\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big)$ is a topological space, such that

$$N_T(\xi_\epsilon) = \sup_{t\in[0,T]} \{ \parallel \xi_\epsilon(t,.) \parallel_{L^\infty(\Omega_t)} \}.$$

Let $(\xi_\epsilon)_\epsilon \in \mathcal{E}^s_M\big(\mathbb{R}^+ \times \mathbb{R}\big)$. We can obtain

$$\parallel H_\epsilon\big(\xi_\epsilon(t,.)\big) \parallel_{L^\infty(\Omega_t)} \leq \parallel a_\epsilon \parallel_{L^\infty(I_0)} + \parallel b_\epsilon \parallel_{L^\infty(I_0)} + 2t\int_0^t \parallel \sin\big(\xi_\epsilon(s,.)\big) \parallel_{L^\infty(\Omega_s)} ds.$$

Since $|\sin(t)| \leq |t|, \quad \forall t \in \mathbb{R}$, then

$$\parallel H_\epsilon\big(\xi_\epsilon(t,.)\big) \parallel_{L^\infty(\Omega_t)} \leq \parallel a_\epsilon \parallel_{L^\infty(I_0)} + \parallel b_\epsilon \parallel_{L^\infty(I_0)} + 2t\int_0^t \parallel \xi_\epsilon(s,.) \parallel_{L^\infty(\Omega_s)} ds,$$

which implies that

$$\sup_{t\in[0,T]} \parallel H_\epsilon\big(\xi_\epsilon(t,.)\big) \parallel_{L^\infty(\Omega_t)} \leq \parallel a_\epsilon \parallel_{L^\infty(I_0)} + \parallel b_\epsilon \parallel_{L^\infty(I_0)} + 2T^2 \sup_{t\in[0,T]} \parallel \xi_\epsilon(s,.) \parallel_{L^\infty(\Omega_t)}$$

then,

$$N_T\big(H_\epsilon(\xi_\epsilon)\big) \leq N_T(a_\epsilon) + N_T(b_\epsilon) + 2T^2 p_T(\xi_\epsilon),$$

it follows that

$$N_T\big(H_\epsilon(\xi_\epsilon)\big) \in |\mathcal{E}^s_M| \Rightarrow \big(H_\epsilon(\xi_\epsilon)\big)_\epsilon \in \mathcal{E}^s_M\big(\mathbb{R}^+ \times \mathbb{R}\big).$$

• Now, let $(v_\epsilon)_\epsilon \in \mathcal{N}^s\big(\mathbb{R}^+ \times \mathbb{R}\big)$. We have

$$H_\epsilon\big(\xi_\epsilon(t,y) + v_\epsilon(t,y)\big) - H_\epsilon(\xi_\epsilon)(t,y)$$
$$= \int_0^t \int_{y+s-t}^{y+t-s} \big(\sin\left(\xi_\epsilon(s,z) + v_\epsilon(s,z)\right)$$
$$- \sin\left(\xi_\epsilon(s,y)\right)\big) dz ds$$

which implies

$$\| \ H_\epsilon\big(\xi_\epsilon(t,.) + v_\epsilon(t,y)\big) - H_\epsilon(\xi_\epsilon)(t,.) \ \|_{L^\infty(\Omega_t)}$$
$$\leq 2T \int_0^t \| \sin\big(\xi_\epsilon(s,.) + v_\epsilon(s,.)\big),$$
$$- \sin\big(\xi_\epsilon(s,.)\big) \ \|_{L^\infty(\Omega_t)} \ ds,$$

thus,

$$\| \ H_\epsilon\big(\xi_\epsilon(t,.) + v_\epsilon(t,y)\big) - H_\epsilon(\xi_\epsilon)(t,.) \ \|_{L^\infty(\Omega_t)}$$
$$\leq 2T \int_0^t \| \ \xi_\epsilon(s,.) + v_\epsilon(s,.)$$
$$- \xi_\epsilon(s,.) \ \|_{L^\infty(\Omega_t)} \ ds,$$

taking the sup in the last inequality, it follows

$$\sup_{0\leq t\leq T} \{\| \ H_\epsilon(\xi_\epsilon(t,.) + v_\epsilon(t,y)) - H_\epsilon(\xi_\epsilon)(t,.) \ \|_{L^\infty(\Omega_t)}\}$$
$$\leq 2T \int_0^t \sup_{0\leq t\leq T} \{\| \ \xi_\epsilon(s,.) + v_\epsilon(s,.) - \xi_\epsilon(s,.) \ \|_{L^\infty(\Omega_t)}\} ds$$

hence

$$\sup_{0\leq t\leq T} \{\| \ H_\epsilon\big(\xi_\epsilon(t,.) + v_\epsilon(t,y)\big) - H_\epsilon(\xi_\epsilon)(t,.) \ \|_{L^\infty(\Omega_t)}\}$$
$$\leq 2T^2 \sup_{0\leq t\leq T} \{\| \ v_\epsilon(s,.) \ \|_{L^\infty(\Omega_t)}\} ds$$

so,

$$H_\epsilon(\xi_\epsilon + v_\epsilon) - H_\epsilon(\xi_\epsilon) \in \mathcal{N}^s\big(\mathbb{R}^+ \times \mathbb{R}\big).$$

According to the Definition 3.6, we can see that the map H is well defined.

b) The problem (4.1) can be written in term of representatives as follows

$$\begin{cases} (\partial_t^2 - \partial_y^2)\xi_\epsilon = 2\sin(\xi_\epsilon), \quad y \in \mathbb{R}, \ t \in \mathbb{R}^+ \\ \xi_\epsilon(y,0) = a_\epsilon(y), \quad y \in \mathbb{R}, \\ \partial_t \xi_\epsilon(0,y) = b_\epsilon(y), \quad y \in \mathbb{R}. \end{cases}$$

H_ϵ is well defined from $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}))$ into $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. Endowed with the topology τ_ϵ given by the family of norms $(M_{T,\epsilon})_{T\in\mathbb{R}^+}$, $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ is a topological space, such that for all $\xi_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$.

$$M_{T,\epsilon}(y_\epsilon) = \sup_{t\in[0,T]} \{e^{-2tT} \parallel \xi_\epsilon(t) \parallel_{L^\infty(\Omega_t)}\}.$$

Let $(\xi_\epsilon)_\epsilon, (v_\epsilon)_\epsilon \in \mathcal{E}^s_M(\mathbb{R}^+ \times \mathbb{R})$, we have

$$H_\epsilon\big(\xi_\epsilon(t,y)\big) - H_\epsilon\big(v_\epsilon(t,y)\big) = \int_0^t \int_{y+s-t}^{y+t-s} \big(\sin\left(\xi_\epsilon(s,z)\right) - \sin\left(v_\epsilon(s,y)\right)\big) dz ds,$$

which implies

$$\begin{aligned} &\parallel H_\epsilon\big(\xi_\epsilon(t,.)\big) - H_\epsilon\big(v_\epsilon(t,.)\big) \parallel_{L^\infty(\Omega_t)} \\ &\qquad \le 2t \int_0^t \parallel \xi_\epsilon(s,.) - v_\epsilon(s,.) \parallel_{L^\infty(\Omega_s)} ds, \\ &\qquad \le 2T \int_0^t \parallel \xi_\epsilon(s,.) - v_\epsilon(s,.) \parallel_{L^\infty(\Omega_s)} ds. \end{aligned} \tag{4.3}$$

Multipling the both sides of the inequality by e^{-2tT}, we can write

$$\begin{aligned} &e^{-2tT} \parallel H_\epsilon(\xi_\epsilon(t,.)) - H_\epsilon(v_\epsilon(t,.)) \parallel_{L^\infty(\Omega_t)} \\ &\qquad \le 2Te^{-2tT} \int_0^t \parallel \xi_\epsilon(s,.) - v_\epsilon(s,.) \parallel_{L^\infty(\Omega_s)} ds, \end{aligned}$$

then

$$\begin{aligned} &2Te^{-2tT} \int_0^t \parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)} ds \\ &\qquad = 2Te^{-2tT} \times \\ &\qquad\quad \int_0^t e^{-2sT} \parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)} e^{2sT} ds, \end{aligned}$$

which gives

$$\begin{aligned} &2t \int_0^t \parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)} ds \\ &\qquad \le \sup_{t\in(0,T]} \{\parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)}\} \int_0^t 2Te^{2sT} ds \\ &\qquad \le \sup_{t\in(0,T]} \{\parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)}\} \times \left(e^{2tT} - 1\right) \\ &\qquad \le \sup_{t\in(0,T]} \{\parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)}\} \left(1 - e^{-2tT}\right). \end{aligned}$$

Hence

$$\begin{aligned} &\sup_{t\in(0,T]} \{e^{-2tT} \parallel (H_\epsilon\xi_\epsilon - H_\epsilon v_\epsilon)(t,.)\big) \parallel_{L^\infty(\Omega_t)}\} \le \\ &\qquad \sup_{t\in(0,T]} \{e^{-2tT} \parallel (\xi_\epsilon - v_\epsilon)(s,.) \parallel_{L^\infty(\Omega_s)}\} \left(1 - e^{-2tT}\right), \end{aligned}$$

that is

$$M_{T,\epsilon}(H_\epsilon(\xi_\epsilon) - H_\epsilon(v_\epsilon)) \leq M_{T,\epsilon}(\xi_\epsilon - v_\epsilon)(1 - e^{-2tT}),$$

but $1 - e^{-2tT} < 1$, it follows H_ϵ is a contraction in $\left(\mathcal{C}^\infty\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big), \tau_\epsilon\right)$.

c) For all $T \in \mathbb{R}^+$ and $\xi_\epsilon \in C^\infty\big(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\big)$, we have

$$\begin{aligned} e^{-2T^2} \sup_{t\in[0,T]} \{\| \xi_\epsilon(t) \|_{L^\infty(\Omega_t)}\} &\leq \sup_{t\in[0,T]} \{\| \xi_\epsilon(t) \|_{L^\infty(\Omega_t)} \, e^{-2tT}\} \\ &\leq \sup_{t\in[0,T]} \| \xi_\epsilon(t) \|_{L^\infty(\Omega_t)}, \end{aligned}$$

then

$$e^{-2T^2} N_T \leq M_{T,\epsilon} \leq N_T.$$

d) For each $T \in \mathbb{R}^+$ we have $e^{2T^2} \in |\mathcal{E}_M^s|$ and $1/(1 - e^{-2T^2}) \in |\mathcal{E}_M^s|$.

Finally, according to the Definition 3.6 the map

$$H : \begin{cases} \mathcal{G} \longrightarrow \mathcal{G} \\ \xi(t,y)) = \big[\xi_\epsilon(t,y)\big] \longmapsto H(\xi(t,y) = \Big[H_\epsilon\big(\xi_\epsilon\big)(t,y)\big)\Big], \end{cases}$$

is a contraction. From Theorem 3.3, H possesses a fixed point $\xi = [(\xi_\epsilon)_\epsilon]$, where ξ_ϵ is a fixed point of H_ϵ. Hence ξ is a solution of (4.1).

Uniqueness: Suppose that $v = [v_\epsilon]$ is another solution of the problem (4.1), we set

$$v_\epsilon = H_\epsilon(v_\epsilon) + n_\epsilon,$$

where $n_\epsilon \in \mathcal{N}^s(\mathbb{R}^+ \times \mathbb{R})$. We can write

$$\xi_\epsilon(t,y) - v_\epsilon(t,y) = \int_0^t \int_{y+s-t}^{y+t-s} (\sin(\xi_\epsilon(s,z)) - sin(v_\epsilon(s,y)))dzds + n_\epsilon(t,y).$$

We can obtain

$$\begin{aligned} &\| (\xi_\epsilon - v_\epsilon)(t,.) \|_{L^\infty(\Omega_t)} \\ &\qquad \leq 2t \int_0^t \| (\xi_\epsilon - v_\epsilon)(s,.) \|_{L^\infty(\Omega_t)} ds + \| n_\epsilon(t,.) \|_{L^\infty(\Omega_t)}, \end{aligned}$$

which implies

$$\begin{aligned} &\| (\xi_\epsilon - v_\epsilon)(t,.) \|_{L^\infty(\Omega_t)} \\ &\qquad \leq 2T \int_0^t \| (\xi_\epsilon - v_\epsilon)(s,.) \|_{L^\infty(\Omega_t)} ds + \| n_\epsilon(t,.) \|_{L^\infty(\Omega_t)} . \end{aligned}$$

By Gronwall's lemma, we have

$$\| \xi_\epsilon(t,.) - v_\epsilon(t,.) \|_{L^\infty(\Omega_t)} \leq e^{2T} \| n_\epsilon(t,.) \|_{L^\infty(\Omega_t)} .$$

And thus

$$\sup_{t\in[0,T]}\{\| \xi_\epsilon(t,.)-v_\epsilon(t,.)\|_{L^\infty(\Omega_t)}\}\le e^{2T}\sup_{t\in[0,T]}\{\| n_\epsilon(t,.)\|_{L^\infty(\Omega_t)}\}.$$

Since $(n_\epsilon)_\epsilon\in\mathcal{N}^s\big(\mathbb{R}^+\times\mathbb{R}\big)$, then $(\xi_\epsilon-v_\epsilon)_\epsilon\in\mathcal{N}^s\big(\mathbb{R}^+\times\mathbb{R}\big)$, it follows that

$$\xi=v.$$

Thus, the solution is unique in $\mathcal{G}\big(\mathbb{R}^+\times\mathbb{R}\big)$.

5. Example

Let us consider the following problem

$$\begin{cases}(\partial_t^2-\partial_y^2)\xi=2\sin(\xi), & y\in\mathbb{R},\ t\in\mathbb{R}^+\\ \xi(0,y)=\delta(y), & y\in\mathbb{R},\\ \partial_t\xi(0,y)=\delta'(y), & y\in\mathbb{R}.\end{cases}\tag{5.1}$$

Where $\delta=[(\delta_\epsilon)_\epsilon]$ is the embedding of the Dirac measure in $\mathcal{G}(\mathbb{R})$ and

$$\delta_\epsilon(y)=\delta*\psi_\epsilon(y)=\psi_\epsilon(y)=\frac{1}{\epsilon}\psi(\frac{x}{\epsilon}),\ \ y\in\mathbb{R},\quad \text{for all } \epsilon\in(0,1).\tag{5.2}$$

$\delta'=[(\delta'_\epsilon)_\epsilon]$ is the embedding of the Dirac measure in $\mathcal{G}(\mathbb{R})$ where

$$\delta'_\epsilon(y)=\delta'*\psi_\epsilon(y)=-\psi'_\epsilon(y)=-\frac{1}{\epsilon^2}\psi'(\frac{x}{\epsilon}),\ \ y\in\mathbb{R},\quad \text{for all } \epsilon\in(0,1).\tag{5.3}$$

where ψ is a test function such that $\psi\in\mathcal{C}^\infty(\mathbb{R}),\ \ \int_{\mathbb{R}}\psi(y)dy=1,\ \ \ \psi(y)\ge 0$. The solution of (5.1) is given by

$$\xi(t,y)=[(\xi_\epsilon(t,y))_\epsilon],\quad \text{for all } \epsilon\in(0,1).$$

where

$$\xi_\epsilon(t,y)=\psi_\epsilon(y-t)+\int_0^t\int_{y+s-t}^{y+t-s}\sin\big(\xi_\epsilon(s,z)\big)dzds,\quad (t,y)\in\mathbb{R}^+\times\mathbb{R}.$$

We have

$$\| \xi_\epsilon(t,.)\|_{L^\infty(\Omega_t)}\le\ \ \| \psi_\epsilon\|_{L^\infty(I_0)}+2t\ \textstyle\int_0^t\|\sin\big(\xi_\epsilon(s,.)\big)\|_{L^\infty(\Omega_s)}\,ds.$$

Since $|\sin(t)|\le|t|,\quad \forall t\in\mathbb{R}$, then

$$\| \xi_\epsilon(t,.)\|_{L^\infty(\Omega_t)}\le\| \psi_\epsilon\|_{L^\infty(I_0)}+2t\int_0^t\| \xi_\epsilon(s,.)\|_{L^\infty(\Omega_s)}\,ds.$$

ψ_ϵ is of order $\mathcal{O}(\epsilon^{-m})$ for some m. From gronwall's inegality, this family (ξ_ϵ) is moderates. Then $u\in\mathcal{G}([0,\infty)\times\mathbb{R})$.

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