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INVARIANT GEOMETRIC CHARACTERISTICS UNDER THE DUAL MAPPING OF AN ISOTROPIC SPACE

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Abstract. In the paper, we give the definitions, basic concepts, and formulas of the basic geometric characteristics associated with the surface of an isotropic space R_3^2. The geometric sense of the dual mapping is revealed. Using the geometric interpretation of the dual mapping, we prove that the dual asymptotic mapping is asymptotic. Some properties of shortest curves on a surface are defined and proved. The geodesicity and conformality of the dual mapping is proved.

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1. Introduction

Singularities associated with isotropic space vectors arise when studying the geometry of pseudo-Euclidean spaces. Often these singularities form a whole subspace. One of such subspaces is the isotropic space R_3^2. An isotropic space is the subspace $M(x, y, z, z) \in^1 R_4$. Despite this, the geometry of such subspaces has been little studied. In this paper, we show that an asymptotic direction on a surface under a dual mapping corresponds to an asymptotic direction on the dual surface.

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Differential geometry of isotropic spaces was first investigated by K. Strubecker [21]. The works by E.M. Aydin [11], M.S. Lone and M.K. Karacan [18] are devoted to the reconstruction of a surface with constant total curvature. In the paper [4], we solve the question of the existence of a surface with a given total and mean curvature in the isotropic space R_3^2. The work by H.Sachs [20] is devoted to groups of the motion of an isotropic space; Yoon, D.W., Lee J.W. [22] study helicoidal surfaces in the three-dimensional isotropic space R_3^2 and construct helicoidal surfaces satisfying a linear equation in terms of the Gaussian curvature and the mean curvature of the surface; M. Dede [12] investigate the recovery problem in Galilean space. [15] in the paper, they satisfy some algebraic equations in terms of coordinate functions and Laplace operators with respect to the first and second fundamental forms of a three-dimensional simple isotropic space I_3^1. And they also give explicit shapes to those surfaces.

We determine and study the properties of a shortest curve on a surface and in its dual image. It is proved that the dual mapping is geodesic and also conformal.

2. Preliminaries results

2.1. **Basic concepts of an isotropic space R_3^2.** Consider an affine space A_3 with the coordinate system *Oxyz*. Let $\vec{X}(x_1, y_1, z_1)$ and $\vec{Y}(x_2, y_2, z_2)$ be vectors of A_3.

Definition 1. If the scalar product of the vectors $\vec{X}$and $\vec{Y}$ is defined by the formula

$$(1)\qquad \begin{cases} (X,Y)_1 = x_1x_2 + y_1y_2 & \text{if} \quad x_1x_2 + y_1y_2 \neq 0, \\ (X,Y)_2 = z_1z_2 & \text{if} \quad x_1x_2 + y_1y_2 = 0, \end{cases}$$

then A_3 is said to be an isotropic space R_3^2. [2,8]

As it is known, the norm of a vector $\vec{X}$ is defined by the formula $|\vec{X}| = \sqrt{(\vec{X}, \vec{X})}$, and the distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is defined as the norm of the vector: $|\overrightarrow{AB}| = \sqrt{(\overrightarrow{AB}, \overrightarrow{AB})}$. From here we obtain the definition of the distance *d* between A and B :

$$(2)\qquad d = \begin{cases} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} & \text{if} \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \neq 0, \\ |z_2 - z_1| & \text{if} \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = 0. \end{cases}$$

The geometry in a plane of an isotropic space will be Euclidean if it is not parallel to the axis *Oz*. When a plane is parallel to *Oz*, the geometry on it will be Galilean [7].

Since an isotropic space has an affine structure, there is an affine transformation that preserves the distance determined by formula (2). This motion of an isotropic space is given by the formula [13,20]

$$(3)\qquad \begin{cases} x' = x\cos\alpha - y\sin\alpha + a, \\ y' = x\sin\alpha + y\cos\alpha + b, \\ z' = Ax + By + z + c. \end{cases}$$

The matrix of this transformation has the form:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ A & B & 1 \end{pmatrix}.$$

The sphere of an isotropic space, that is, the set of points in the space equidistant from one point, has the equation

$$(4)\qquad (x - x_0)^2 + (y - y_0)^2 = r^2$$

here (x_0, y_0, z) is the center of the sphere (4), r is its radius. When the center coincides with the origin, the equation of the sphere is $x^2 + y^2 = r^2$.

2.2. **Surface theory in isotropic space, duality.** Let $\gamma : I \subseteq R \to R_3^2$ be an admissible curve parameterized by arc-length $s \in R$. In coordinate form, this can be written as

$$\gamma(s) = \{x(s), y(s), z(s)\},$$

where x, y, and z are smooth functions of one variable. Denote the first derivative with respect to s by a prime, etc. Then the curvature and torsion functions of γ are respectively defined by the formulas

$$(5)\qquad k(s) = x'(s)y''(s) - x''(s)y'(s)$$

and

$$(6)\qquad \sigma(s) = \frac{1}{k(s)} det\left(\gamma'(s), \gamma''(s), \gamma'''(s)\right).$$

Moreover, the associated trihedron of γ is given as

$$\tau = \gamma'(s),$$

$$\nu = \frac{1}{k}\gamma''(s),$$

$$\beta = (0, 0, 1).$$

The Frenet's formulas for such vectors have the form [19]

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau + \sigma\nu, \\ \dot{\beta} = 0. \end{cases} \tag{7}$$

Let a regular surface from the class C^2 be given in an isotropic space by the vector equality

$$r(u,v) = (x(u,v); y(u,v); z(u,v)), \quad (u,v) \in D. \tag{8}$$

We introduce the concept of an isotropic surface normal vector [21] $\overrightarrow{n}_m(0,0,1)$ and a normal to a surface as in Euclidean space: $\overrightarrow{n} = [r_u, r_v]$. In this case, we will use the superimposed space method, that is, the coordinate system $Oxyz$ will be considered as the Euclidean Cartesian coordinate system. Moreover, the surface will be considered as a surface of Euclidean space.

By analogy with an Euclidean space, we define the first and second quadratic forms of the surface (8). The first quadratic form

$$I = ds^2 = Edu^2 + 2Fdudv + Gdv^2, \tag{9}$$

where

$$\begin{cases} E = r_u^2 = x_u^2 + y_u^2, \\ G = r_v^2 = x_v^2 + y_v^2, \\ F = r_u r_v = x_u x_v + y_u y_v. \end{cases} \tag{10}$$

The second quadratic form

$$II = (d^2r, n) = Ldu^2 + 2Mdudv + Ndv^2, \tag{11}$$

where

$$\begin{cases} L = (r_{uu}, n) = \dfrac{(r_{uu}, r_u, r_v)}{\sqrt{EG - F^2}} \\ M = (r_{uv}, n) = \dfrac{(r_{uv}, r_u, r_v)}{\sqrt{EG - F^2}} \\ N = (r_{vv}, n) = \dfrac{(r_{vv}, r_u, r_v)}{\sqrt{EG - F^2}} \end{cases}.$$

Consider also the second quadratic form of the surface (8) with respect to the isotropic normal to the surface

$$II = (d^2r, n_m) = L_m du^2 + 2M_m dudv + N_m dv^2, \tag{12}$$

where

$$\begin{cases} L_m = (r_{uu}, n_m) = z_{uu}, \\ M_m = (r_{uv}, n_m) = z_{uv}, \\ N_m = (r_{vv}, n_m) = z_{vv}. \end{cases}$$

We define the normal section of the surface in the given direction by a plane passing in the given direction and the isotropic normal n_m.

The normal curvature of a curve is defined as the curvature of the curve resulting from the normal section. Since the isotropic normal n_m is directed along the axis Oz, the geometry of the normal plane will be Galilean [1].

The normal curvature of a curve is defined by the formula [8], [3]

$$k_n = \frac{II_m}{I}. \tag{13}$$

There are two kinds of spheres in an isotropic space. The definition of the first of them we gave in the previous paragraph (see (4)).

The second sphere is defined as a surface with the constant normal curvature. This sphere of the unit radius has the equation [10]

$$x^2 + y^2 = 2z, \tag{14}$$

we call it the isotropic sphere.

The total and mean curvature of the surface, defined by analogy with the Euclidean space, respectively, have the following form

$$K = \frac{LN - M^2}{EG - F^2}, \quad \text{and} \quad 2H = \frac{EN - 2FM + GL}{EG - F^2}.$$

When considering surfaces that are uniquely projected onto the plane Oxy, the formulas for the total and mean curvature of the surface are further simplified and have the following form

$$K = LN - M^2, \quad \text{and} \quad 2H = L + N.$$

Definition 2. Minimal surface is a surface whose mean curvature at any point is zero.

By definition, it will be $L + N = 0$.

Let a plane π be given in R_3^2, which is not parallel to the axis Oz of the space. Consider the section of the isotropic sphere by the plane π and denote it by Γ. Since an isotropic sphere is a paraboloid of revolution, the section Γ by a plane will be a closed curve. It was proved in [2] that Γ is an ellipse.

Draw tangent planes to isotropic sphere (14) through points $M \in \Gamma$. Denote the set of tangent planes to points F by $\{\pi\}$.

The following statement holds.

Theorem 1. All planes of the set $\{\pi\}$ intersect at one point. [14]

If a plane π_0 is given by the equation

$$z = Ax + By + C, \tag{15}$$

then the intersection point of the planes of the set $\{\pi\}$ will be $(A, B, -C)$.

Definition 3. The point $(A, B, -C)$ is said to be dual to plane (15) with respect to isotropic sphere (14) [1].

Consider the plane $z = H$ and its section Γ by an isotropic sphere. Let the surface F be given by the equation

$$F : \{z = f(x, y) \,|\, (x, y) \in D'\}, \tag{16}$$

and the edge of the surface be the curve Γ. The surface (16) itself is convex and contained in the inner part of the space bounded by the plane and the isotropic sphere.

Let us draw the tangent plane π_M to the surface F at the point $M(x_0, y_0, z_0)$. Denote by M^* the dual image of the tangent space π_M with respect to the isotropic sphere. When the point $M \in F$ changes on the surface F, its dual image describes a surface F^*.

Definition 4. The surface F^* is said to be the dual surface to the surface F in an isotropic space [1].

When F is given by the equation $z = f(x, y)$, F^* has the equations

$$\begin{cases} x^*(u, v) = f'_u(u, v), \\ y^*(u, v) = f'_v(u, v), \\ z^*(u, v) = u \cdot f'_u(u, v) + v \cdot f'_v(u, v) - f(u, v). \end{cases} \tag{17}$$

Calculating the total curvature K^* and the mean curvature H^* of the dual surface, we obtain

$$K^* = \frac{1}{K}, \quad \text{and} \quad H^* = \frac{H}{K}.$$

3. Main results

In this section, by analogy with the Euclidean space, we study the properties of curves on the surface of an isotropic space. The main types of curves on the surface are defined in the same way as in the Euclidean space.

For example, the principal direction on a surface is the direction in which the normal curvature takes on an extreme value. A curve on a surface is called a line of curvature if its direction is always the same as the principal one.

The direction in which the normal curvature vanishes, is called asymptotic. A curve is called asymptotic if the normal curvature is zero at all its points.

The study of the surface "in the small" differs little from the Euclidean one. We are interested in problems "in the large" on the surface of an isotropic space. Therefore, we compare the surface with its dual surface and determine their properties, which are mainly related to the geometry "in the large".

Let γ be a curve on a surface F. Denote by $\gamma^* \in F^*$ its dual image. It should be noted that for regular surfaces, the dual mapping is unique. Therefore, the image of the curve γ will be a curve on F^*. In the general case, the dual mapping is ambiguous. We study regular surfaces.

Theorem 2. Under the dual mapping, the asymptotic direction of the surface F corresponds to the asymptotic direction of the surface F^*.

Proof. Since the normal curvature of a curve on the surface of an isotropic space is calculated by formula (13), $II = 0$ in the asymptotic direction, that is, the second quadratic form must vanish.

Therefore, we calculate the second quadratic form of the dual surface.

First, calculating the discriminant of the first quadratic form of the dual surface, we obtain:

$$E^*G^* - (F^*)^2 = \left[f''_{uu}f''_{vv} - {f''_{uv}}^2\right]^2.$$

Calculating the coefficients, we get

$$L^* = \frac{\begin{vmatrix} f'''_{uuu} & f'''_{uuv} & f''_{uu} + uf'''_{uuu} + vf'''_{uuv} \\ f''_{uu} & f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix}}{\sqrt{E^*G^* - (F^*)^2}} = \frac{1}{f''_{uu}f''_{vv} - {f''_{uv}}^2}\left\{ f'''_{uuu}\begin{vmatrix} f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix} - \right.$$

$$\left. f'''_{uuv}\begin{vmatrix} f''_{uu} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & uf''_{uv} + vf''_{vv} \end{vmatrix} + \left(f''_{uu} + uf'''_{uuu} + vf'''_{uuv}\right)\begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix} \right\} =$$

$$\frac{1}{f''_{uu}f''_{vv} - {f''_{uv}}^2}\left\{ -uf'''_{uuu}\left(f''_{uu}f''_{vv} - {f''_{uv}}^2\right) - vf'''_{uuv}\left(f''_{uu}f''_{vv} - {f''_{uv}}^2\right) + \right.$$

$$\left. \left(f''_{uu} + uf'''_{uuu} + vf'''_{uuv}\right)\left(f''_{uu}f''_{vv} - {f''_{uv}}^2\right) \right\} = \frac{f''_{uu}\left(f''_{uu}f''_{vv} - {f''_{uv}}^2\right)}{f''_{uu}f''_{vv} - {f''_{uv}}^2} = f''_{uu} = L;$$

$$M^* = \frac{\begin{vmatrix} f'''_{uuv} & f'''_{uvvv} & f''_{uv} + uf'''_{uuv} + vf'''_{uvv} \\ f''_{uu} & f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix}}{\sqrt{E^*G^* - (F^*)^2}} = \frac{1}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} \left\{ f'''_{uuv} \begin{vmatrix} f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix} - \right.$$

$$\left. f'''_{uvv} \begin{vmatrix} f''_{uu} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & uf''_{uv} + vf''_{vv} \end{vmatrix} + (f''_{uv} + uf'''_{uuv} + vf'''_{uvv}) \begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix} \right\} =$$

$$\frac{1}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} \left\{ -uf'''_{uuv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) - vf'''_{uvv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) + \right.$$

$$\left. (f''_{uv} + uf'''_{uuv} + vf'''_{uvv}) \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) \right\} = \frac{f''_{uv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right)}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} = f''_{uv} = M;$$

$$N^* = \frac{\begin{vmatrix} f'''_{uvv} & f'''_{vvv} & f''_{vv} + uf'''_{uvv} + vf'''_{vvv} \\ f''_{uu} & f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix}}{\sqrt{W}} = \frac{1}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} \left\{ f'''_{uvv} \begin{vmatrix} f''_{uv} & uf''_{uu} + vf''_{uv} \\ f''_{vv} & uf''_{uv} + vf''_{vv} \end{vmatrix} - \right.$$

$$\left. f'''_{vvv} \begin{vmatrix} f''_{uu} & uf''_{uu} + vf''_{uv} \\ f''_{uv} & uf''_{uv} + vf''_{vv} \end{vmatrix} + (f''_{vv} + uf'''_{uvv} + vf'''_{vvv}) \begin{vmatrix} f''_{uu} & f''_{uv} \\ f''_{uv} & f''_{vv} \end{vmatrix} \right\} =$$

$$\frac{1}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} \left\{ -uf'''_{uvv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) - vf'''_{vvv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) + \right.$$

$$\left. (f''_{vv} + uf'''_{uvv} + vf'''_{vvv}) \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right) \right\} = \frac{f''_{vv} \left(f''_{uu}f''_{vv} - f''^{\,2}_{uv} \right)}{f''_{uu}f''_{vv} - f''^{\,2}_{uv}} = f''_{vv} = N.$$

Therefore $L^* = L$, $M^* = M$, $N^* = N$.

Hence, the second quadratic form of the dual surface is equal to the second quadratic form of its image. $II = 0$ for asymptotic directions of the surface F. Hence, $II^* = 0$ in the corresponding direction of the surface, too. This proves Theorem 2.

From this theorem, we can conclude that the dual mapping transfers asymptotic lines on the surface F into asymptotic lines on the surface F^*.

Let a point $M \in F$ and $M^* \in F^*$ be the point corresponding in duality. Consider a normal section π at the point M.

Theorem 3. A point of the surface $M \in F$ and its dual image $M^* \in F^*$ belong to the same normal section.

Proof. Draw a tangent plane α to the surface F through the point M. Then the sections of the planes π and α determine the normal direction $(du,\ dv)$. Draw at the point M a tangent line in the direction $(du,\ dv)$.

Denote by P and Q the intersection points of this tangent with the isotropic sphere. Obviously, these points belong to the normal section π. Moreover, the normal section π intersects with the isotropic sphere along a cycle.

Draw at points P and Q tangents to the cycle formed by the section of the plane π with the isotropic sphere. The intersection point of these tangents is the point M^* because these tangents will be generators of the tangent cone of the isotropic sphere. Their intersection point coincides with the top of the cone. Therefore, the normal section passes through the vertex of the cone. This proves Theorem 3.

Corollary 2. The normal curvature of a curve on a dual surface can be calculated by the formula $k_n^* = \dfrac{II}{III}$ where II, III are, respectively, the second and the third quadratic form of the surface F.

Indeed, the third quadratic form of the surface F and the first quadratic form of F^* are related by the equality

$$I^* = (dx^*)^2 + (dy^*)^2 = dp^2 + dq^2 = III.$$

Moreover, $II^* = II$. Therefore, $k_n^* = \dfrac{II^*}{I^*} = \dfrac{II}{III}$.

Theorem 4. If F is a minimal surface, then the dual mapping will be conformal.

Proof. For a mapping to be conformal, the first quadratic form must be [17]

$$I^* = \lambda(x, y)\left(dx^2 + dy^2\right).$$

Let us find the conditions under which this equality holds. Calculate the first quadratic form of the dual surface F^* under the condition that F is the minimal surface:

$$I^* = (dx^*)^2 + (dy^*)^2 = (f_{xx}dx + f_{xy}dy)^2 + (f_{xy}dx + f_{yy}dy)^2 =$$

$$(f_{xx}^2 + f_{xy}^2)dx^2 + 2f_{xy}(f_{xx} + f_{yy})dxdy + (f_{xy}^2 + f_{yy}^2)dy^2.$$

The expression $f_{xx} + f_{yy} = 0$ follows from the definition of a minimal surface F and the mean curvature formula. Hence, if we consider the expression $f_{xx} = -f_{yy}$:

$$I^* = (f_{xx}^2 + f_{xy}^2)dx^2 + (f_{xy}^2 + (-f_{xx})^2)dy^2 = (f_{xx}^2 + f_{xy}^2)\left(dx^2 + dy^2\right).$$

We get from here that $\lambda(x,y) = f_{xx} + f_{xy}$ or $\lambda(x,y) = f_{xy} + f_{yy}$.

This proves that the dual mapping of the minimal surface is conformal.

Corollary 2. If the translation surface satisfies the condition $f_{xx} = f_{yy}$, then its dual mapping will be conformal.

Indeed, $f_{xy} = 0$ always for translation surfaces; when the condition $f_{xx} = f_{yy}$ is satisfied, the first quadratic form of the dual surface has the form $I^* = f_{xx}\left(dx^2 + dy^2\right)$. By Theorem 4, the mapping is conformal.

One of the concepts associated with a curve on a surface is the concept of a shortest curve on a surface. In an isotropic space, the metric is degenerate. Therefore, the concept of a shortest curve in an isotropic space differs significantly from such a concept in the Euclidean space since the distance between points is defined as the length of its projection onto the plane Oxy.

Moreover, when projecting parallel to the axis Oz, the shortest line on the plane Oxy should be used to determine a shortest curve on a surface. The geometry of the plane Oxy is Euclidean. The shortest line on the Euclidean plane is the segment connecting the considered points.

Therefore, the concept of a shortest curve on a surface between the points A and B on a surface is defined in the following way.

Definition 5. A curve on the surface F formed by a section of the surface by a plane g passing through points A and B, parallel to the axis Oz is called the shortest curve on the surface F connecting the points A and B.

The plane g will be the only one because it must pass through A, parallel to the vector $\overrightarrow{AB}$ and the vector $\vec{k}$, which is parallel to the axis Oz. But the shortest curve connecting the points A and B may not be unique. We mainly consider surfaces uniquely projected onto the plane Oxy. For such surfaces, the shortest curve will be unique.

The shortest curve on the surface of an isotropic space has some properties of the Euclidean shortest curve on the surface, but it also has its own characteristics.

For example, the shortest curve on a convex polyhedron can pass through its vertex. It is known that the shortest curve on a polyhedron of the Euclidean space does not pass through the vertex of the polyhedron [9].

This can be seen from the fact that through the vertex of a polyhedral angle, which is uniquely projected onto the plane Oxy, one can always draw a plane parallel to the axis Oz. Obviously, for points on the section lying on different sides of the vertices of the polyhedral angle, the shortest path passes through the vertex.

The condition of non-imposition of shortest curves on surfaces in isotropic space is satisfied.

Definition 6. A geodesic curve on a surface is a continuous curve that is shortest on any sufficiently small segment of itself.

Lemma 1. The curvature of a shortest curve is equal to zero.

Proof of Lemma follows from the fact that the curvature of a segment on the Euclidean plane is equal to zero.

A one-to-one mapping of the surface F_1 onto the surface F_2 is called geodesic if the image $\gamma_2 \in F_2$ of the geodesic curve $\gamma_1 \in F_1$ is also a geodesic curve on the surface F_2 [16].

Theorem 5. If a surface F is convex, then its dual mapping is a geodesic mapping.

Proof. The shortest curve always lies on the plane parallel to the axis Oz. Hence, at the point under consideration, it coincides with the normal section along the corresponding direction of the shortest curve. It is proved in Theorem 3, that a point of a surface and its dual image belong to the same normal section. A normal section always intersects the plane Oxy in a straight line. Therefore, if the considered curve γ is geodesic, then its dual image γ^* is also a geodesic curve because they have a common projection on the plane Oxy. Hence, they have the same, equal to zero, curvature. This shows that the dual image of a geodesic curve is a geodesic curve.

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