A NOVEL NUMERICAL ALGORITHM FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS AND CONVERGENCE ANALYSIS USING MODIFIED HAAR WAVELET METHOD

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Abstract. Authors describe the multi resolution analysis based on standard Haar wavelet method for solving singularly perturbed boundary value problems. After that for the fast calculation of the Haar functions, a numerical algorithm is presented based on lemma and theorems. The purpose of this paper is to obtain the highly accurate numerical solution of singularly perturbed boundary value problem using modified Haar wavelet method for given collocation points and remove the singular behavior from the solution. The present method removes the singular behavior of the problems and provides the high precision numerical solution in the large effective region of convergence in comparison to the other existing methods, as shown in the tested examples. Method capture the solutions in the layer region of the domain as the solution profile depends on the perturbation parameter and their convergence is also highly dependent on the value of the perturbation parameter ϵ. Numerical examples demonstrated the efficiency of the present scheme.

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1. Introduction

Singularly perturbed boundary value problems frequently arise in numerous mathematical sciences, such as problems based on heat transfer with a large quantity of Paclet number, modeling of Navier Stokes flow with a huge Reynolds number [1] [2], control theory [3] [4], nuclear physics, semiconductor devices, fluid mechanics, quantum mechanics, solid mechanics, and mathematical models, such as modeling the motion of fluids with small viscosity and the human papillary light reflex, also arise in transport processes in chemistry and biology [5] that has been studied by several researchers [6] [7],

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[8] [9].

Based on the Haar wavelet, a new novel development is carried out for solving singularly perturbed boundary value problems that arise in fluid mechanics and quantum mechanics when considered in a real interval together with given boundary conditions.

$$Ly(x) \equiv -\epsilon\, y''(x) + f(x)y(x) = g(x),\ f(x) \geq 0,\ x \in [0,1] \tag{1}$$

$$y(0) = \alpha_0,\ y(1) = \alpha_1 \tag{2}$$

where ϵ is a small perturbation parameter $(0 < \epsilon < 1)$ which multiplying the higher order derivative occurs in differential operator and f(x) and g(x) are sufficiently continuously differentiable functions and $y(x)$ satisfy y(x)$\geqslant y_\star > 0\, \forall\, x \in [0,1]$ for some real constant $y_\star$ and the boundary values $\alpha_0, \alpha_1 \in \mathbb{R}$ are real constants.

Moreover, Roos et al., [10] and Howes et al., [11] discussed the existence and uniqueness of the problem (1-2) at boundary layer point $x = 0$ and at $x = 1$ and employed the unique solution y. The small perturbation parameter ϵ, which occurs in the highest order derivative category of these kinds of problems, and they are stiff and there is always either an interior layer or boundary layer where the solution alters promptly. Therefore, these kinds of problems that are considered in this research are a source of difficulty in the computational process due to the presence of interior and boundary layers and also because of various changes in the width of the layers for the small perturbation parameter epsilon $\epsilon(0 < \epsilon < 1)$. To overcome the weak and strong variation in the approximate solutions in the numerical treatment of singularly perturbed problems requires a more efficient technique to solve these kinds of problems. Thus, it needs to adopt a scheme that overcomes both the variations and compares both the large-scale and small-scale characteristic of the solution. Hence, this is the reason that a scheme based on wavelets plays an ideal role in it. Since in the problem of perturbation nature, the most widely used classification techniques are known to provide poor approximations when ϵ is small relative to the collocation point. Earlier, various numerical methods were presented in the literature to solve SPBVPs for ordinary differential equations of second order by numerous researchers. Vukoslavecevic et al., [12] used finite element techniques to solve such problems, Kadalbajoo et al., [13] gave finite difference scheme whereas Chun et al., [14] provides the Homotopy perturbation method, Umesh et al., [15] considered the singularly perturbed boundary value problem using Adomian decomposition method, Aziz et al., [16]; Kumar et al., [17]; Rashidinia et al., [18]; Surla et al., [19]; Kadalbajoo et al., [20] gave a concept of B-spline scheme in terms of hyperbolic and trigonometric splines that are different from earlier ones. Reproducing kernel method is disscued by Geng et al., [21] and Bawa et al., [22]; Aziz et al., [23] gave a quin-tic spline computation approach for the self-adjoint SPBVPs and also showed the convergence of the method. The method based on sextic spline is presented by Khan et al., [24] and Lodhi et al., [25]. Also, a patching approach is discussed by Khuri et al., [26] for the numerical

solution of second-order two-point boundary value problems.
The majority of the approaches need an understanding of the position and width of boundary layers which encourages us to seek for an approach that is adaptable. The goal of this study is to take a version of singularly perturbed boundary value problems (SPBVPs) as provided above and numerically approximate it by the Haar wavelet method (HWM).
Wavelets have been discussed in various physical significance in applied mathematics and have appeared as a dynamic computational tool for approximating differential, algebraic, partial differential, fractional delay, and integral-differential equations and equations in various domains of problems for attaining approximate solutions.
The basic fundamental of using the wavelet basis function is that it reorganizes the differential equation into a system of matrix algebraic expressions of finite variables. As a result, the organized structure of the basis function is constructed. Aside from that, the advantage of conceiving the wavelet scheme is that it provides a convenient, user-friendly computational process as well as timely fast convergence due to its appealing features, such as: the potentiality to detect various facts at different pairs of scales and different points in the computational process; additionally, due to its property to locate singularities and flexibility to display a function at various levels of resolution
By considering all advantages and features, this research is devoted to the Haar wavelet method (HWM) for solving singularly perturbed boundary value problems (1-2) with the quadratic rate of convergence by analyzing the data for error estimation. The remaining studies are arranged as follows: as follows: The Multi-Resolution Analysis (MRA) is described in Section 2. Sect.3 provides a construction of the Haar matrix of integration and a brief description of the Haar wavelet method to solve singularly perturbed boundary value problems (SPBVPs) and describes the function approximation in it. Sect. 4 describes the method of solution, and the numerical algorithm of the method is given in Sect. 5. Theorems on error analysis and convergence of the proposed method are presented in Sect. 6. Section 7, deliberate numerical problems for comparison to test the applicability and effectiveness of the Haar wavelet method (HWM). Section 8 ends with the conclusion of this research.

2. Multi resolution analysis (MRA)

The multi-resolution analysis (MRA) is described to understand the Haar wavelet as an orthonormal wavelet. MRA is the best technique to learn deeply about wavelets. Given a real-valued square-integrable function $y(x) \in L^2(\mathbb{R})$, then multi-resolution analysis of y(x) contributes to the cause of generating a sequence of subspace W_k, W_{k+1}, W_{k+2}..... so that the projections of square integrable function towards $L^2(\mathbb{R})$ provide improved approximations of function y(x) as $k \to \infty$. The function y(x) can be approximated at various resolution levels, which correspond to different translations for the subspace ....,W_{k-1}, W_k, W_{k+1}......

To understand multi-resolution analysis (MRA) mathematically, an MRA consists of a family of increasing closed subspaces $W_k \subset L^2(\mathbb{R}), k \in \mathbb{Z}$ together with the following group of properties:

(I) $....W_{-2} \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 \subset ....$

(II) $\bigcup\limits_{k=\infty}^{\infty} W_k$ is dence in $L^2(\mathbb{R})$ i.e. closure $|\bigcup\limits_{k=-\infty}^{\infty} W_k| = L^2(\mathbb{R})$ and $\bigcap_{i=\infty}^{\infty} W_k = \{0\}$.

(III) $f(x) \in W_k \iff f(2x) \in W_{k+1},\ k \in \mathbb{Z}$.

(IV) $f(x) \in W_k \iff f(x-k) \in W_k,\ k \in \mathbb{Z}$.

(V) there is a function $\zeta_{j,k} \in W_k$ such that $\zeta_{j,k} = 2^{\frac{j}{2}}\zeta(2^j\ x - k),\ j,k \in \mathbb{Z}$

(VI) $\exists\, \zeta$ in W_0 such that $\zeta(x-k), k \in \mathbb{Z}$ form an orthogonal set of basis for W_0.

where k is the resolution level, and this integer index set $k \in \mathbb{Z}$ is connected with resolution levels, and W_k denotes approximation spaces of level k. For each W_k, there exists a complement in W_{k+1}.

In the formation of a wavelet function, $\zeta(x)$, a collection of nested sequences $< W_k >$ has been derived that satisfies the properties mentioned above. That is, for the existence of the wavelet function $\zeta(x)$ which yields a multi-resolution analysis (MRA), then these properties are essential. Here, $\zeta(x) \in L^2(\mathbb{R})$ represents the scaling function. Then:

$$W_k = clos_{L^2(\mathbb{R})}(\zeta_{j,k} : k \in \mathbb{Z}),\ j,k \in \mathbb{Z} \tag{3}$$

scaling function

$$\zeta_{j,k}(x) = 2^{\frac{j}{2}}\zeta(2^j\ x - k),\ j,k \in \mathbb{Z} \tag{4}$$

defined in subspace W_k of $L^2(\mathbb{R})$, $\zeta(x-k), k \in \mathbb{Z}$ satisfies the above properties and $\zeta(x-k), k \in \mathbb{Z}$ forms one of the orthonormal sets of basis that consists of a characteristic of linear independence. Hence, the function defined in equation (4) is the collection of functions ($\zeta_{j,k}(x) : j,k \in \mathbb{Z}$) which will forms an orthogonal basis for the space W_k or $L^2(\mathbb{R})$, which is called an orthogonal basis with mother wavelet ζ (see Goswami et al., [28] and Mallat et al., [29]). For interested readers to have a better understanding, Debnath et al., [30] consulted the notion of a monograph

MRA is a tool for developing wavelet theory using translation and scaling features. The basis which is formed by the set of functions for the space. The basis and operational matrices of integration are discussed in the following section.

3. Haar wavelet and Haar operational matrices of integration

In 1910, Hungarian mathematician Alfred Haar (Lepik at.al., [31]; Chen, Hsiao et.al., [32]) established a well-known wavelet which is based on the functions named the Haar wavelet. A Haar wavelet is one of the classes of wavelets, and they are step functions piece-wise constant functions that have only finitely many pieces. Each piece carries only three constant values $1, -1$ and zero on the real line. The Haar wavelet is an uneven rectangular pulse pair and has the simplest orthonormal series together with the property of compact support in the interval $[0, 1]$. This wavelet is very well localized in the time domain but it is not continuous. Due to mathematical simplicity Chen et. al., [33]; Lepik, [34] discussed the Haar wavelet method and how it has become an effective method for solving many differential equation as well as integral equations arising in the modelling of numerous scientific physical issues. The Haar wavelet family over the given interval $x \in [0,1]$ consists the following functions :

$$H_i(x) = \begin{cases} 1, & \text{if } \alpha \leqslant x \ < \ \beta \\ -1, & \text{if } \beta \leqslant x \ < \ \gamma \\ 0, & \text{elsewhere} \end{cases} \tag{5}$$

where:

$\alpha = \frac{k}{m}$, $\beta = \frac{k+\frac{1}{2}}{m}$, $\gamma = \frac{k+1}{m}$

Here, the integer $m = 2^j$, $j = 0,1,2,....,J$ and $k = 0,1,2,...,.m-1$, where j indicates the level of wavelet, k represent the translation parameter and J denotes the maximum level of resolution.
The index number for the Haar function H_i in equation (5) is computed by the formula $i = m+k+1$. In the case of a minimum index of $i = 1$ for the Haar function, select $m = 1$, $k = 0$. The index number i can reach the maximum value of $i = 2M = 2^{j+1}$ when including all levels of wavelet.
For $i = 1$, the function $H_1(x)$ is the scaling function for the family of Haar wavelet and defined as:

$$H_1(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \ < \ 1 \\ 0, & \text{elsewhere} \end{cases} \tag{6}$$

for $i = 2$, the function $H_2(x)$ is the mother wavelet for the family of Haar wavelet and defined as:

$$H_2(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \ < \ \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leqslant x \ < \ 1 \\ 0, & \text{elsewhere} \end{cases} \tag{7}$$

In addition, $H_2(x)$ is the fundamental square wave or the mother Haar wavelet which spans the whole interval $(0,1)$. Akmal et.al., [35]; raza et.al., [36] discussed all other subsequent curves that are generated from Haar wavelet function $H_i(x)$ with two operations translation and dilation. In particular, Haar wavelets are orthogonal functions on $[0,1]$.
In addition, the Haar operational matrix integration formula is presented for determining the solution of the singularly perturbed boundary value issue. numerically.
The operational matrix $P_{i,\xi}$ of order $m \times m$ is derived from integration of Haar wavelet family with the help of following integral as:

$$P_{i,\xi}(x) = \int_A^x \int_A^x ...... \int_A^x H_i(t)dt^{\xi} = \frac{1}{\xi - 1} \int_A^x (x-t)^{\xi-1} H_i(t)\, dt. \tag{8}$$

where,

$\xi = 1, 2, ..., $ and $i = 1, 2, ..., 2m.$

The explicit form of integrals in equation (8) can be written as:

$$P_{i,\xi}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^\xi}{\xi!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{[(x-\alpha)^\xi - 2(x-\beta)^\xi]}{\xi!}, & \text{if } \beta \leqslant x < \gamma \\ \frac{[(x-\alpha)^\xi - 2(x-\beta)^\xi + (x-\gamma)^\xi]}{\xi!} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{9}$$

The following integrals are used to derive the solution of the singularly perturbed boundary value problem using the Haar wavelet method:

$$P_{i,1}(x) = \int_0^x H_i(t)dt \tag{10}$$

$$P_{i,1}(x) = \begin{cases} (x-\alpha) & \text{if } \alpha \leqslant x < \beta \\ (\gamma - x) & \text{if } \beta \leqslant x < \gamma \\ 0 & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{11}$$

$$P_{i,\xi+1}(x) = \int_0^x P_{i,\xi}(t)dt \tag{12}$$

and

$$C_{i,\xi}(x) = \int_0^1 P_{i,\xi}(t)dt, \; \xi = 1, 2, ... \tag{13}$$

for $\xi = 2, 3$ $P_{i,\xi+1}(x)$ consists the following functions as:

$$P_{i,2}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^2}{2!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{1}{4m^2} - \frac{(\gamma-x)^2}{2!} & \text{if } \beta \leqslant x < \gamma \\ \frac{1}{4m^2} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{14}$$

$$P_{i,3}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^3}{3!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{(x-\beta)}{4m^2} - \frac{(\gamma-x)^3}{3!} & \text{if } \beta \leqslant x < \gamma \\ \frac{(x-\beta)}{4m^2} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{15}$$

$$P_{i,4}(x)=\begin{cases}0, & \text{if } 0\leqslant x\ <\ \alpha\\ \frac{(x-\alpha)^4}{4!}, & \text{if } \alpha\ \leqslant x\ <\ \beta\\ \frac{(x-\beta)}{8m^2}-\frac{(\gamma-x)^4}{4!}+\frac{1}{192m^4} & \text{if } \beta\ \leqslant x\ <\ \gamma\\ \frac{(x-\beta)}{8m^2}+\frac{1}{192m^4} & \text{if } \gamma\ \leqslant x\ <\ 1\end{cases} \tag{16}$$

Next, for $\xi=4,5,....$ values of equation (9) can be evaluated. And also keep that,

$P_{1,\xi}(x)=\frac{x^{\xi}}{\xi!},\ C_{1,\xi}(x)=\frac{1}{(\xi+1)!},\xi=1,2,....$

Here, the expression for the Haar operational matrix of integration of finite order has been calculated for the approximation of the solution to the SPBVPs by developing the numerical algorithm

3.1. **Function approximation by Haar Wavelet.** For a better understanding of the Haar wavelet as a type of orthonormal wavelet, this section describes the function approximation by the Haar wavelet.

3.2. **Function with One Variable :** Pandit et al., [37] introduced that any real- valued twice integrable function $y_c(x)\in L^2[0,1]$ can be expressed as a linear combination of Haar wavelet function as below:

$$y_c(x)=\sum_{i=1}^{\infty}a_iH_i(x)\ =\ a_1H_1(x)+a_2H_2(x)+... \tag{17}$$

where the Haar coefficient $a_i, i=1,2,3...$ can be calculated by

$$a_i=<y_c,H_i>=2^j\int_0^1\ y_c(x)\bar{H}_i(x)dx. \tag{18}$$

This series ends with the countable terms if $y_c(x)$ is approximated by piece-wise terms and piece-wise itself during each sub-interval (x_{i-1},x_i). If the function $y_c(x)$ is piece-wise constant then the countable terms of series $y_c(x)$ can be resolved as fixed terms as:

$$y_c(x)=\sum_{i=1}^{2M}a_iH_i(x)=a_{2M}^TH_{2M}(x) \tag{19}$$

equation (19) can be written in matrix form as $y_c=a^TH$. The scalar function a_{2M}^T and Haar wavelet function vector $H_{2M}(x)$ are defined as below:

$$a_{2M}^T=[a_1,a_2,a_3,....,a_{2M}]and\ H_{2M}(x)=[H_1(x),H_2(x),H_3(x),....H_{2M}(x)]^T \tag{20}$$

Here T is the transposition operator, and M is a power of two.
Since the vector a is in discrete form and the Haar matrix H is of order $m=2^{j+1}$, where $j=0,1,2,...J$,

that is

$$H = \begin{bmatrix} h_1(x_1) & h_1(x_2) & ...h_1(x_m) \\ h_2(x_1) & h_2(x_2) & ...h_2(x_m) \\ . & . & . \\ . & . & . \\ . & . & . \\ h_m(x_1) & h_m(x_2) & ...h_m(x_m) \end{bmatrix} \tag{21}$$

In order to find the Haar coeffient, the collocation points $x_l = \frac{l-\frac{1}{2}}{2m}, l = 1, 2, ...2m$ are obtained by discretizing Haar function $H_i(x)$ by dividing the interval $[0, 1]$ into $2m$ parts of equal length $\Delta t = \frac{1}{2m}$ to get the coefficient matrix H. For different value of m, For $m = 8$ the H square Haar matrix would be as below:

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{22}$$

Moreover, by the definition of Haar wavelet H_8, it is orthogonal.
By the definition of Haar wavelet H_8, it is orthogonal. equation (21) has been used to construct the Haar matrix of any order which developed the matrix $p_{i,1}, p_{i,2}$ i.e. first and second order integration of Haar matrix. Further, these developed matrices are required for approximation of the problem.

4. The Haar wavelet method for solving singularly perturbed boundary value problems:

The solution to the singularly perturbed boundary value problem issue is given below using the Haar operational matrix of integration established in the previous section:

$$Ly(x) \equiv -\epsilon\, y''(x) + f(x)y(x) = g(x),\ f(x) \geq 0,\ x \in [0, 1] \tag{23}$$

$$y(0) = \alpha_0,\ y(1) = \alpha_1 \tag{24}$$

where f(x) and g(x) are sufficiently smooth function in the interval $[0, 1]$.

Begin with the highest order differential operator y" that can be expressed by the Haar wavelet series as given below:

$$y"(x) = \sum_{i=1}^{2M} a_i H_i(x), \tag{25}$$

Next, on integration equation (25) from 0 to x and using the given boundary conditions, then

$$y'(x) = \sum_{i=1}^{2M} a_i P_{i,1}(x) + y'(0), \tag{26}$$

In addition, integrate equation (26) and put in the boundary conditions, then

$$y(x) - y(0) = \sum_{i=1}^{2M} a_i P_{i,2}(x) + xy'(0), \tag{27}$$

In the proceeding equation (27) , at boundary point $x = 1$, gets $y'(0)$ by using equation (13) i.e.

$$y'(0) = y(1) - y(0) - \sum_{i=1}^{2M} a_i C_{i,1}. \tag{28}$$

Consequently, the equation (27) is enhanced as follows:

$$y(x) = \sum_{i=1}^{2M} a_i P_{i,2}(x) - \sum_{i=1}^{2M} a_i x C_{i,1}(x) + x[y(1) - y(0)] + y(0). \tag{29}$$

where $P_{i,1}, P_{i,2}$ are defined in equation (11) and equation (14)

using equation (25), equation (26) and equation (29) in equation (23) , then as a result, the following set of equations obtained.

$$-\epsilon \left[\sum_{i=1}^{2M} a_i H_i(x) \right] + f(x) \left[\sum_{i=1}^{2M} a_i P_{i,2}(x) - \sum_{i=1}^{2M} a_i x C_{i,1}(x) + x[y(1) - y(0)] + y(0) \right] = g(x). \tag{30}$$

with the boundary conditions provided in equation (24), The Haar wavelet coefficient, a_i's are calculated by solving the system of $2M$ linear equations by using the Gauss elimination approach. Further substituting the coefficients in equation (29) to obtain the Haar wavelet solution of singularly perturbed differential equation directly without using Taylor series expansion

5. Numerical algorithm of the method

The procedure to approximate the solution of a singularly perturbed boundary value problem is described below:

$$Ly(x) \equiv -\epsilon\, y''(x) + f(x)y(x) = g(x),\ f(x) \geq 0,\ x \in [0, 1] \tag{31}$$

$$y(0) = \alpha_0,\ y(1) = \alpha_1 \tag{32}$$

The solution to the above problem is obtained through the developed numerical algorithm and is described in the key steps as follows:

Input:

Step (1) Compute the Haar wavelet matrix $H_i((x))$ from equation (5).

Step (2) Compute the first and second order matrix of integration of Haar wavelet $p_{i,1}$ and $p_{i,2}$ from equation (11) and equation (14) respectively.

Step (3) Construct the expression given by equation (25) to equation (30) .

Step (4) By using the step (3), construct the left hand side matrix of equation (30) .

Step (5) Compute the unknown vectors (Haar coefficients) a_i's by solving the system of linear equation (30) .

Step (6) Substitute the vector a_i's in equation (29) .

Output: Obtained the Haar approximate solution $y(x_j)$, for each $j = 1, 2, ...2M$.

where,

$$x_j = \frac{j - \frac{1}{2}}{2M}, \; j = 1, 2, ..., 2M. \tag{33}$$

A novel numerical algorithm for the user-friendly computational process has been developed to approximate the numerical solution of a singularly perturbed boundary value problem using the Haar wavelet method (HWM) described in Sect.4 and the Haar operational matrices of integration described in Sect.3.

For a better understanding of the numerical algorithm, a diagrammatic depiction of the sequence of steps for the computation has been carried out in the flow chart given below. The computation of the algorithm has been done in MATLAB R2019b. A predefined set of inputs has been used in the algorithm. An algorithm creates output values from a given set of inputs for each set of the input expression. The output values represent an approximation of the current problem for a different level of resolution J and collocation points.

6. Convergence and error analysis of Haar wavelet

Lemma 6.1 Let us assume a real-valued continuous function y(x) that satisfies the Lipschitz condition on $x \in [0, 1]$ and its first derivative is bounded. i.e

$$\forall\, x \in [0, 1], \exists\, \text{M}: \mid \tfrac{dy}{dx} \mid \leqslant \text{M}, \text{M} \geqslant 2$$

Then the Haar wavelet method based on method proposed in n Saleem et.al., [38]; Majak et. al., [39] i.e., error vanishes as goes to infinity, the convergence is of order two.

and,

$$||E_r||_2 = \text{O}\left[\left(\tfrac{1}{2^{J+1}}\right)^2\right]$$

It will also be convergent, for the sake of convergence $\mid E_r(x) \mid \to 0$ as $J \to \infty$. Moreover, the order of convergence is 2:

where,

$$E_r \;=\; y(x) - y_{app}(x), y_{app}(x) = \textstyle\sum_{i=1}^{2M} a_i H_i(x).$$

Proof: For the proof of lemma 6.1, see Saleem et.al., [38]; Majak et. al., [39].

Lemma 6.2 Let series form of Haar wavelet $y(x) = \sum_{i=1}^{2M} a_i H_i(x)$ and the real valued function $u \in L^2([0,1])$ such that $\mid u'(x) \mid \leqslant K$ then, the Haar wavelet coefficient a_i satisfies the inequality given below:

$$||a_i||^2 \leqslant \frac{K}{2^{\frac{3j-2}{2}}},$$

where K be any real valued constant in interval (0,1).

Proof: For the proof of lemma (6.2), see the reference Kumar, et. al., [40].

Theorem 1. Let y(x) be a square integrable function such that $\mid y'(x) \mid \leqslant K$ on interval (0,1) and the Haar wavelet approximation $y_j(x)$ of $y(x)$, then the modulus of error norm at Jth level of resolution serve the inequality given below:

$$||E_r|| \leqslant 2M\sqrt{d}\left(\frac{2^{-2J-2}}{3}\right)^2,$$

where d is positive constant on (0,1).

Proof: For the proof of theorem, see Shah at.al., [41], Islam et. al., [42], Pandit et. al., [43].
In case of direct Haar wavelet method, then maximum absolute error is measured by the formula given below for singularly perturbed boundary value problem,

$$E_a = maximum \mid \epsilon y"_J(x_j) + f(x_j)y(x_j) - g(x_j) \mid,$$

Here the value of $y"(x_j), y(x_j), g(x_j)$ are given in equation (25), equation (29) and equation (30), $j = 0, 1, 2....2M, 2M = 2^{J+1}$
Lemma 6.1 refers to the method by which we deliberately say the singularly perturbed boundary value problem converges to some finite value for the sake of the quadratic rate of convergence. Moreover, Lemma 6.2 states the boundedness of the Haar coefficients, and Theorem 1 is based on measuring the maximum absolute error while approximating the numerical solution based on the developed algorithm for different levels of resolution J and collocation points.

7. Numerical applications

In this section, applied the Haar wavelet method described in Section 3 to some singularly perturbed boundary value problems (SPBVP's), and the results are tabulated and a comparison is made with some other existing schemes in the available literature. The numerical computations for these examples have been done in MATLAB R2019b with the AMD Ryzen-5 processor.

Problem 1: Consider the following Singularly perturbed boundary value problem.

$$-\epsilon y''(x) + y(x) = x, \ x \in [0,1] \tag{34}$$

with boundary conditions

$$y(0) = 1,\ y(1) = 1 + exp\left(\frac{1}{\sqrt{\epsilon}}\right). \tag{35}$$

Problem set (34-35) have an exact solution (See Rashidinia et al., [18], khan et al., [24] and, Shah et al., [44]).

$$y(x) = x + exp\left(\frac{-x}{\sqrt{e}}\right). \tag{36}$$

The Haar wavelet approach stated in Section 4 and the numerical algorithm given in Section 5 are used to tackle this problem. Applying to the current research, by theorem 1, the obtained maximum absolute error of problem 1 is shown and compared to the existing schemes in the table given below for ϵ and $2M$ collocation points. Table 1 indicates that the Haar wavelet method performs better than the other methods given in the literature. Apart from that, table 1 shows that, on increasing the collocation point in the method and decreasing the value of epsilon, the maximum absolute error gets reduced. Lemma 6.2 verifies the boundedness of collocation points. The approximated results for $2M = 64$ collocation points and different values of epsilon are presented in this study and the graph of maximum absolute error for different levels of resolution $J = 4, J = 5,$ and $J = 6$ is plotted below for $\epsilon = \frac{1}{128}$. Furthermore, Lemma 6.1 has been validated in terms of solution convergence, resulting in a more accurate approximation solution for the quadratic rate of convergence.

TABLE 1. Comparison of the absolute errors for different levels of resolution and different perturbation parameters ϵ for Problem 1.

	Present Method			
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{16}$	2.0045$e-$5	2.5545$e-$6	3.2264$e-$7	4.0548$e-$8
$\frac{1}{32}$	1.6054$e-$5	1.2734$e-$6	2.5657$e-$7	3.2229$e-$8
$\frac{1}{64}$	6.3942$e-$6	8.0567$e-$7	1.0143$e-$7	3.2229$e-$8
$\frac{1}{128}$	9.5318$e-$7	1.1974$e-$7	1.5057$e-$8	1.8896$e-$9

	Shah et al., [44]			
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{16}$	5.0862$e-$5	2.4680$e-$5	6.1100$e-$7	6.1100$e-$9
$\frac{1}{32}$	2.4680$e-$5	1.2244$e-$6	2.6471$e-$7	1.3139$e-$9
$\frac{1}{64}$	1.2244$e-$5	6.1100$e-$6	2.7082$e-$7	1.3161$e-$8
$\frac{1}{128}$	6.1100$e-$6	2.7082$e-$7	1.1038$e-$8	4.9550$e-$9

Khan et al., [24]				
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{16}$	7.376$e-5$	4.938$e-6$	3.147$e-7$	1.977$e-8$
$\frac{1}{32}$	2.771$e-4$	1.947$e-5$	1.260$e-6$	7.959$e-8$
$\frac{1}{64}$	9.787$e-4$	7.448$e-5$	4.982$e-6$	3.174$e-7$
$\frac{1}{128}$	3.645$e-3$	2.773$e-4$	1.948$e-5$	1.260$e-6$

Rashidinia et al., [18]				
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{16}$	2.96$e-6$	1.18$e-5$	1.15$e-8$	7.24$e-10$
$\frac{1}{32}$	1.18$e-5$	7.54$e-7$	4.67$e-8$	2.96$e-9$
$\frac{1}{64}$	4.74$e-5$	2.96$e-6$	1.863$e-7$	1.16$e-8$
$\frac{1}{128}$	1.78$e-4$	1.18$e-5$	7.46$e-7$	4.67$e-8$

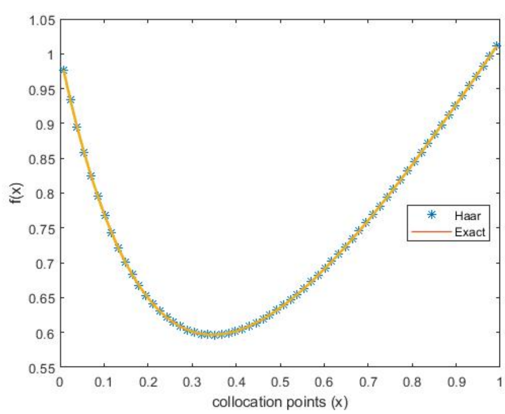

FIG. 1. **Comparison of Approximate solution and exact solution for the case** $2M = 64$ **and** $e = \frac{1}{16}$ **of Problem 1.**

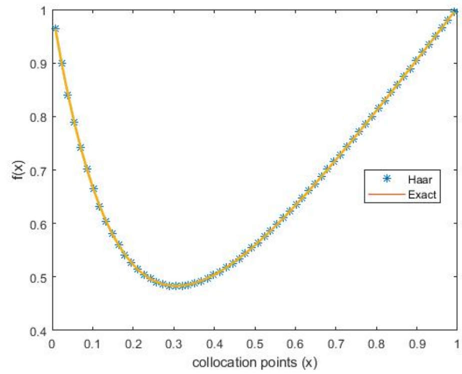

FIG. 2. **Comparison of Approximate solution and exact solution for the case** $2M = 64$ **and** $e = \frac{1}{32}$ **of Problem 1.**

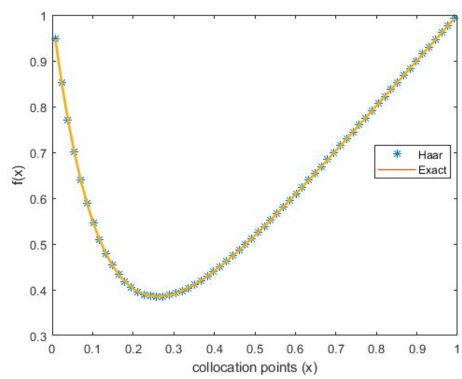

FIG. 3. **Comparison of Approximate solution and exact solution for the case** $2M = 64$ **and** $e = \frac{1}{64}$ **of Problem 1.**

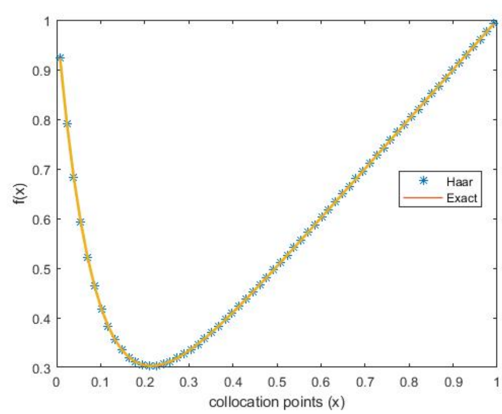

FIG. 4. **Comparison of Approximate solution and exact solution for the case** $2M = 64$ **and** $e = \frac{1}{128}$ **of Problem 1.**

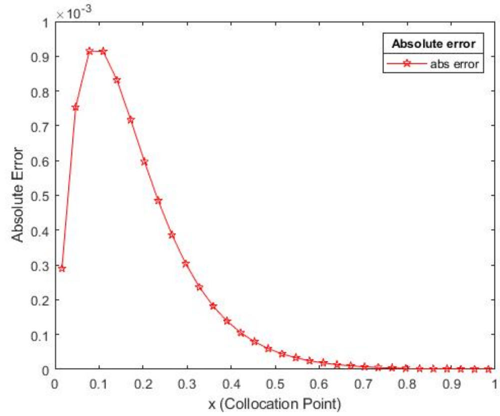

FIG. 5. **Graph of the maximum absolute error for the problem 1 for $J = 4$ and $\epsilon = \frac{1}{128}$.**

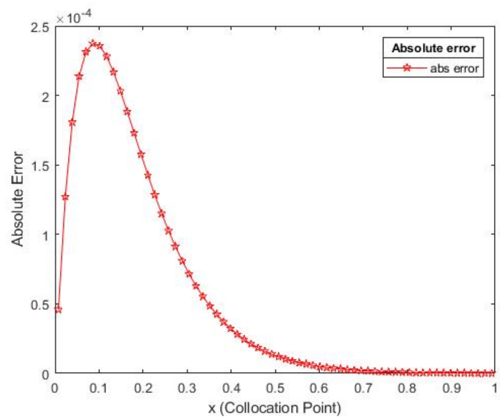

FIG. 6. **Graph of the maximum absolute error for the problem 1 for $J = 5$ and $\epsilon = \frac{1}{128}$.**

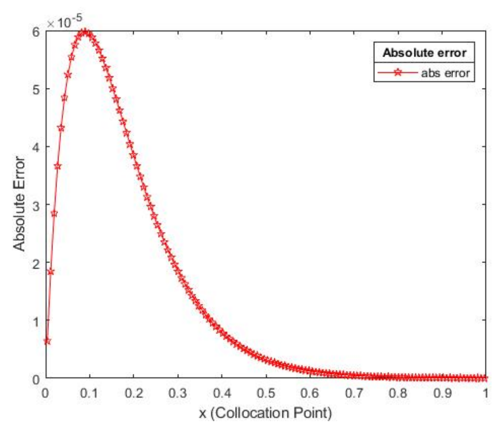

FIG. 7. **Graph of the maximum absolute error for the problem 1 for $J = 6$ and $\epsilon = \frac{1}{128}$.**

Figure 1-4 compares the exact solution to the Haar solution at the level of resolution $J = 6$ for the value of $\epsilon = \frac{1}{16}, \epsilon = \frac{1}{32}, \epsilon = \frac{1}{64}, \epsilon = \frac{1}{128}$ that demonstrating the improved convergence of the Haar wavelet method. Next, figure 5-7 shown above shows the maximum absolute error for different levels of resolution, such as $J = 4, J = 5, J = 6$ is plotted below for $e = \frac{1}{128}$. So, as increase the collocation points in the method and decrease the value of ϵ our present method produces a better approximate solution with less error.

Problem 2: Consider the Singularly perturbed boundary value problem.

$$-\epsilon y''(x) + y(x) = (x-1) - x\, exp\left(\frac{-1}{\epsilon}\right),\ x \in [0,1] \tag{37}$$

with boundary conditions

$$y(0) = 0,\ y(1) = 0. \tag{38}$$

The exact solution to this problem set (37- 38) (see Shah et al., [44])

$$y(x) = (x-1) - xexp\left(\frac{-1}{\epsilon}\right) + exp\left(\frac{-x}{\epsilon}\right). \tag{39}$$

In the present study, the Haar wavelet method introduced in Section 4 and the algorithm described in Section 5 are used to solve this problem and then compare our approximate computational solution with the finite element method. For each collocation point, determine the maximum absolute errors based on the theorem 1 and lamma 6.2, which compares our results to other existing approaches. Table 2 given below indicates that the Haar wavelet method (HWM) produces a more accurate approximate solution than the finite element method. Moreover, lemma 6.1 has been verified concerning the convergence of the solution, so that the approximate solution will be more accurate for the quadratic rate of convergence Schatz and Wahlbin [45].

TABLE 2. Comparison of the absolute errors for different levels of resolution and different perturbation parameters ϵ for Problem 2.

	Present Method			
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{5}$	$1.7371e{-5}$	$6.9485e{-5}$	$2.7784e{-6}$	$1.100e{-8}$
$\frac{1}{5^2}$	$3.4747e{-6}$	$1.3898e{-4}$	$5.5590e{-4}$	$2.2229e{-5}$
$\frac{1}{5^3}$	$6.9493e{-7}$	$2.7797e{-4}$	$1.1119e{-5}$	$4.4472e{-6}$
$\frac{1}{5^4}$	$1.3899e{-7}$	$5.5595e{-4}$	$2.2238e{-5}$	$8.8950e{-5}$
$\frac{1}{5^5}$	$2.7797e{-8}$	$1.1119e{-6}$	$4.4476e{-6}$	$1.7790e{-8}$
$\frac{1}{5^6}$	$5.5595e{-9}$	$2.2238e{-7}$	$8.8952e{-8}$	$3.5581e{-9}$

Schatz and Wahlbin [45]				
ϵ	$J_4(x_j)$	$J_5(x_j)$	$J_6(x_j)$	$J_7(x_j)$
$\frac{1}{5}$	$3.8988e{-3}$	$1.1005e{-3}$	$5.8736e{-4}$	$2.8643e{-5}$
$\frac{1}{5^2}$	$3.8710e{-2}$	$1.7634e{-3}$	$6.8954e{-4}$	$1.2733e{-4}$
$\frac{1}{5^3}$	$5.9764e{-3}$	$5.9534e{-4}$	$2.8733e{-4}$	$7.8823e{-5}$
$\frac{1}{5^4}$	$4.7681e{-5}$	$1.127e{-5}$	$7.8754e{-6}$	$2.9087e{-6}$
$\frac{1}{5^5}$	$2.7612e{-3}$	$3.8751e{-5}$	$1.7643e{-5}$	$4.8965e{-6}$
$\frac{1}{5^6}$	$7.7896e{-6}$	$4.8964e{-6}$	$3.6754e{-7}$	$1.9971e{-7}$

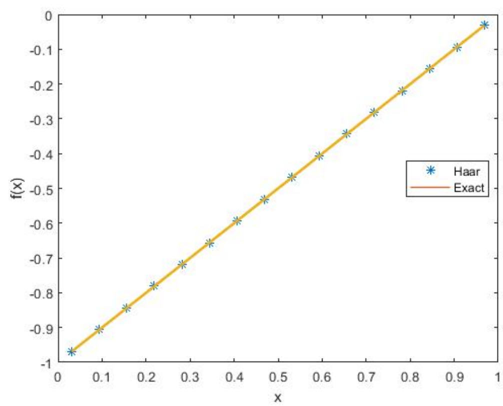

FIG. 8. **Comparison of an approximate solution with the exact solution for the case** $2M = 16$ **and** $\epsilon = \frac{1}{5^2}$.

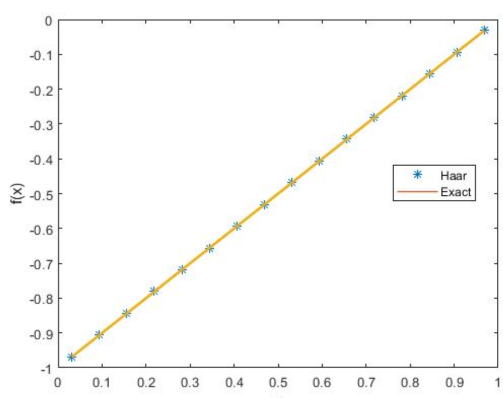

FIG. 9. **Comparison of an approximate solution with the exact solution for the case** $2M = 16$ **and** $\epsilon = \frac{1}{5^4}$.

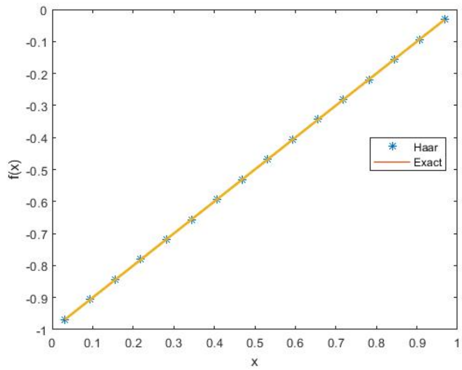

FIG. 10. **Comparison of an approximate solution with the exact solution for the case** $2M = 16$ **and** $\epsilon = \frac{1}{5^6}$.

Here the graph below shows the maximum absolute error for $2M = 16$ and $\epsilon = \frac{1}{5^2}$, $\epsilon = \frac{1}{5^4}$ and $\epsilon = \frac{1}{5^6}$. As a result, conclude that lowering the value of epsilon reduces the error while increasing the collocation points in the solution. The Figures (11-13) below shows the maximum absolute error for some epsilon values $\epsilon = \frac{1}{5^2}$, $\epsilon = \frac{1}{5^4}$ and $\epsilon = \frac{1}{5^6}$.

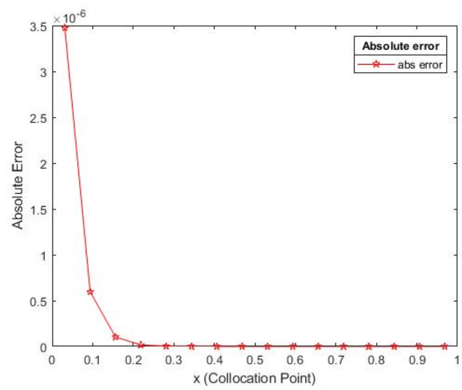

FIG. 11. **Maximum absolute error in problem 2 for** $J = 4$ **and** $\epsilon = \frac{1}{5^2}$.

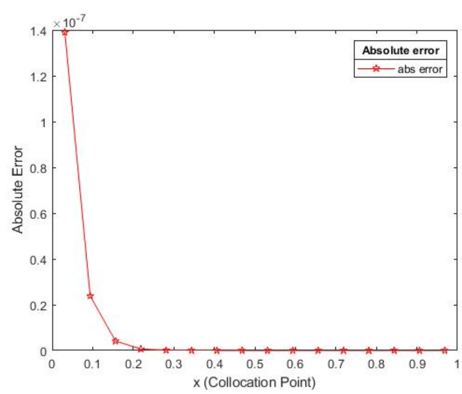

FIG. 12. **Maximum absolute error in problem 2 for** $J = 4$ **and** $\epsilon = \frac{1}{5^4}$.

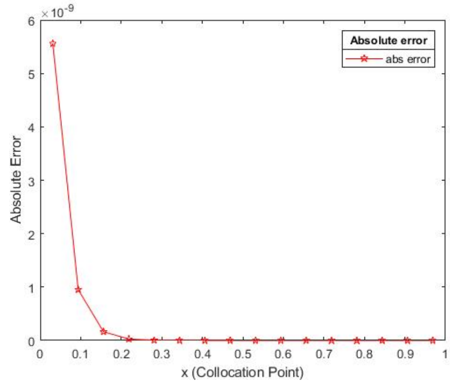

FIG. 13. **Maximum absolute error in problem 2 for** $J = 4$ **and** $\epsilon = \frac{1}{5^6}$.

The maximum absolute error for two different SPBVPs with homogeneous and non-homogeneous boundary conditions is estimated by employing various degrees of resolution. It is observed that the more accurate results are obtained by the wavelet method when the degree of resolution is increased.

8. Conclusion

The present method, the Haar wavelet Method (HWM), is used in this study for the numerical solution of the singularly perturbed boundary value problem. A numerical algorithm for the Haar wavelet has been developed to solve singularly perturbed boundary value issues that arise in fluid mechanics and quantum mechanics. Building up the collocation point system and solving them with the help of the Haar functions makes a another way of solving the SPBVPs. Additionally, calculate the maximum absolute error based on Theorem 1 and tabulate it in Tables 1 and 2 and plot the graphs for the exact solution, approximate solution, and maximum absolute errors for problems 1 and 2. These plotted graphs indicate that the present method produces more accurate results as compared to the method available in the literature with the same number of collocation points. Furthermore, the theorem based on error and the lemma for convergence of the method is verified. In comparison to previous literature, the findings in tables and figures (1–13) indicate that the current procedure produces accurate outcomes in matters of fast convergence and precision. Moreover, the current method has been frequently applied for the solution of a wide range of similar problems. A particular numerical problem shows the effectiveness of this method.

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Data Availability Statement: Data sharing not applicable to this paper as no data sets were created or processed during the present study.
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