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THE GENERAL MAXIMAL SOLUTION OF SOME MATRICES OVER THE TROPICAL SEMIRING

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Abstract. Nowadays cryptographical protocols are developed by the use of tropical linear and non linear systems. To attack such cryptographical protocols solutions of tropical linear and non linear systems are essential. In this paper, we present the final general maximal solution of linear systems with some special matrices over the tropical semiring. These matrices include natural arithmetic matrices, γ-diagonal matrices, J-matrices, and circulant matrices. We provide proofs for several theorems based on the general maximal solution of max linear systems, which can be utilized to solve cryptographic algorithms employing the discrepancy method. We also compare the discrepancy method with the normalization method and present results regarding the relationship between these two methods.

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1. Introduction

The initial research publication on tropical geometry was written by Imre Simon, a computer scientist and mathematician hailing from Brazil [1–3]. By the aspects of automata in the tropical semiring, Imre Simon rephrased the finite power property as a Burnside problem in his reasoning [4]. French mathematicians coined the term "tropical" to recognize Simon's efforts in applying min-plus algebra to optimization theory. Since the late 1950s, semirings with an underlying set that is a subset of real number system have been devised and redesign numerous times in various fields of research, where the addition of two elements in a tropical semiring is either the maximum or minimum, similarly the product of two elements in a tropical semiring is the usual addition [5, 6]. There are two tropical semirings that differ depending on the operation being performed. The first is known as the minimum tropical semiring, which addition by taking the minimum of two elements and multiplication through

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adding the elements. This algebraic structure is denoted as the min-plus semiring. Similarly, the maximum tropical semiring finds addition by taking the maximum of two elements and multiplication through adding the elements. This semiring is also known as the max-plus semiring [7–9]. Examples of max-plus semirings are $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, $(\mathbb{W} \cup \{-\infty\}, \oplus, \odot)$. The tropical semiring $(\mathbb{W} \cup \{-\infty\}, \oplus, \odot)$ was introduced by Simons [1]. The max-plus semiring is isomorphic to the min-plus semiring, and both are idempotent semirings [10]. Working with tropical semirings is appealing because of their simplicity and resemblance to algebraic geometry [11,12]. As a result, their ease of use and applicability can be inspiring. The tropical semiring structure is used in a variety of fields including computer science, linear algebra, number theory and automata theory, etc [5]. Tropical semirings are also used in language theory, control theory and operation research [13]. Tropical semirings are playing an important role in linear algebra, especially in solving the linear systems [14–17]. Currently, tropical protocols [18] can be attacked through the application of solutions for tropical linear and non linear systems. This application is inspires us to investigate the resolution of particular tropical linear systems and determine the behavior of their solutions in matrices over the tropical semiring. Since tropical sum is notated as $\oplus$, and the tropical product is notated as $\odot$. The sections of our paper are structured as follows: Section 2 contains the basic definitions needed to understand this paper. Section 3 explains the normalization and discrepancy method with necessary theorems. In Section 4, we have proved some results on the solutions of some special matrices. Section 5 contains the results based on the relationship between the normalization and discrepancy methods.

2. Preliminaries

A semiring S is a set which is non-empty with binary operations, namely addition and multiplication, that satisfy the following axioms:

(1) The addition operation forms a commutative monoid under $(S, +)$, with an identity element denoted as 0.

(2) The multiplication operation forms a monoid under $(S, .)$, with a single identity element denoted as 1.

(3) Multiplication distributes over addition, meaning that for any elements $a, b, c \in S$, we have $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

(4) For any element $a \in S$, we have $a \cdot 0 = 0 \cdot a = 0$.

(5) The element 1 is distinct from the element 0.

These properties define a semiring and are essential to its mathematical structure [2,16,19].

The maximum tropical semiring is defined as the semiring $R = (S \cup (-\infty), \oplus, \odot)$. Here, the operations $\oplus$ and $\odot$ represent maximum tropical addition and maximum tropical multiplication, respectively. The set S is a semiring, and R must satisfy the following properties: commutativity under tropical addition, i.e., $a \oplus b = b \oplus a$ for every $a, b \in R$; associativity under tropical addition and tropical multiplication, i.e., $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ and $(a \odot b) \odot c = a \odot (b \odot c)$ for every $a, b, c \in R$.

In addition, R satisfies the property of multiplication distributes over addition, means that $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ for every $a, b, c \in R$. Furthermore, R has an additive identity denoted by e, where for every $a \in R$, we have $e \oplus a = a \oplus e = a$ (since the additive identity is $-\infty$). However, R does not have an additive inverse [16]. The minimum tropical semiring follows a similar pattern, but it requires choosing the minimum [17].

Let's consider a semiring denoted as S. We define the set of all matrices with m rows and n columns over S as $M_{m\times n}(S)$. In this context, we represent the element at the ij-th position of a matrix $P \in M_{m\times n}(S)$ as p_{ij}, and we denote the transpose of matrix P as P^T. Let $P = (p_{ij}) \in M_{m\times n}(S)$, $Q = (q_{ij}) \in M_{m\times n}(S)$, $T = (t_{ij}) \in M_{n\times l}(S)$ and $\alpha \in S$. Addition of two matrices generally calculated by

$$P + Q = ((p_{ij}) + (q_{ij}))_{m\times n}$$

and similarly product of two matrices can be calculated by,

$$PT = \sum_{i=1}^{n} ((p_{ik})(t_{kj}))_{m\times l}$$

and

$$\alpha P = (\alpha(p_{ij}))_{m\times n}$$

Similarly in the max-plus semiring, addition of two tropical matrices is can be calculated by

$$P \oplus Q = (\max((p_{ij}), (q_{ij})))_{m\times l}$$

and the multiplication of two tropical matrices is calculated by

$$P \odot T = \max\left((p_{ik}) + (t_{kj})\right)_{m\times l}$$

and

$$\alpha \odot P = (\alpha + (p_{ij}))_{m\times n}$$

A system $P \odot x = q$ is said to be a tropical system if all the entries of the system are chosen from the tropical semiring $R = (S \cup \{\pm\infty\}, \oplus, \odot)$. A matrix $P \in M_{m\times n}(S)$ is said to be a tropical matrix if all the elements of the matrix are taken from the tropical semiring $R = (S \cup \{\pm\infty\}, \oplus, \odot)$ [10]. A matrix P is said to be a maximum tropical matrix if all the elements of the matrix are from the maximum tropical semiring $R = (S \cup \{-\infty\}, \oplus, \odot)$. Similarly, a matrix P is said to be a minimum tropical matrix if all the elements of the matrix are from the minimum tropical semiring $R = (S \cup \{\infty\}, \oplus, \odot)$.

Let $S = \mathbb{R}$ be the extended real number system under the max-plus algebra, and let P and Q be $m \times n$ matrices over the extended real numbers under the operation of maximum tropical semirings, where $P = (p_{ij})_{m\times n}$ and $Q = (q_{ij})_{m\times n}$. (p_{ij}) and (q_{ij}) are the ij-th entries of P and Q, respectively. We define $P \leq Q$ as $(p_{ij}) \leq (q_{ij})$ for all i, j [17]. A matrix $P = (p_{ij})$ is said to be regular if $(p_{ij}) \neq \pm\infty$. A vector $b \in S^m$ is said to be a normal vector or regular vector if $b_j \neq -\infty$, for all $j \in m$ [16]. If we consider the min-plus algebra, in a regular vector, we have $b_j \neq \infty$ for all $j \in m$ [17].

A solution of the tropical system $P \odot x = q$ is called the maximal solution x^*, if $x \leq x^*$ for any other solution x [16,17]. A linear system $P \odot x = q$ is said to be a tropical linear system if the elements of the linear system are all from any one of the tropical semirings. Through out this paper we define $\mathbf{T} = (\mathbb{W} \cup \{-\infty\}, \oplus, \odot)$ where $\mathbb{W}$ denoting the set of all whole numbers, $\mathbf{V}$=$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ where $\mathbb{R}$ is a set of all real numbers, $\mathbf{W}$=$(\mathbb{Z} \cup \{-\infty\}, \oplus, \odot)$ where $\mathbb{Z}$ is a set of all integers.

Definition 2.1. A matrix $P \in M_{m\times n}(\mathbf{T})$ is said to be a natural arithmetic matrix if the entries of the matrix P are continuously written with natural numbers following the pattern of the respective row or column. Types of natural arithmetic matrix are,

- Row natural arithmetic matrix
- Column natural arithmetic matrix

Definition 2.2. A matrix $P \in M_{m\times n}(\mathbf{T})$ is said to be a column natural arithmetic matrix if it is in the form of

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & m.n \end{bmatrix}$$

Definition 2.3. A matrix $P \in M_{m\times n}(\mathbf{T})$ is said to be a row natural arithmetic matrix, if it is in the below form,

$$\begin{bmatrix} 1 & m+1 & 2m+1 & \cdots & \cdots \\ 2 & m+2 & 2m+2 & \cdots & \cdots \\ 3 & m+3 & 2m+3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & \cdots & n.m \end{bmatrix}$$

Definition 2.4. Let $P \in M_{m\times n}(\mathbf{V})$, and it is named a J-matrix if all the entries of the matrix P are equal to j.

$$\begin{bmatrix} j & j & j & \cdots & j \\ j & j & j & \cdots & j \\ j & j & j & \cdots & j \\ \vdots & \vdots & \vdots & \ddots & j \\ j & j & j & \cdots & j \end{bmatrix}$$

Definition 2.5. A matrix $P \in M_{m\times m}(\mathbf{V})$ with entries $c_0, c_1, c_2, ..., c_{m-1}$ is said to be a circulant matrix if it is of the form

$$\begin{bmatrix} c_0 & c_{m-1} & c_{m-2} & \cdots & c_1 \\ c_1 & c_0 & c_{m-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \vdots & c_{m-2} \\ \vdots & \vdots & \vdots & \vdots & c_{m-1} \\ c_{m-1} & c_{m-2} & c_2 & \cdots & c_0 \end{bmatrix}$$

3. Normalization and Decrepancy Methods

$P \odot x = q$ is said to be a maximum linear system if the coefficients of the linear systems are from the maximum tropical semirings [20]. We are aware that there are various methods for solving linear equations [14,21,22]. In this paper, we utilized the normalization method [16,17] and the discrepancy method to solve the linear system of equations over the tropical semiring. We will compare both methods and analyze the relationship between them. $P = (p_{ij}) \in M_{m\times n}\ (S/\{-\infty\})$, $Q = (q_{ij}) \in M_{m\times n}(S/\{-\infty\})$, where $(S/\{x\})$ denotes all values of R except x, and $q = (q_j)$ is a regular vector $1 \le j \le m$, and the j-th column of the P matrix is denoted as P_j.

3.1. **Normalization method.**

Let $P \odot x = q$ be a tropical linear system or a linear system over the max plus semiring, where the matrix $P \in M_{m\times n}(S/\{-\infty\})$ and $P_j \in (S/\{-\infty\})^m$ is the j-th normal vector with m rows. The normalized matrix $\tilde{P}$ of the given matrix P is calculated by [16,17].

$$\tilde{P} = \Big[(P_1 - \hat{P}_1) \quad (P_2 - \hat{P}_2) \quad (P_3 - \hat{P}_3) \quad \cdots \quad (P_n - \hat{P}_n)\Big]$$

where P_j denotes the j-th column of the given matrix P

$$\hat{P}_j = \frac{p_{1j} + p_{2j} + \cdots + p_{mj}}{m}, \forall\ j \in n.$$

similarly, normalized vector of the given regular vector $q \in S^m$ is calculated by

$$\tilde{q} = q - \hat{q}$$

where

$$\hat{q} = \frac{q_1 + q_2 + q_3 + \cdots + q_m}{m}$$

we can denote the normalized system of $P \odot x = q$ as $\tilde{P} \odot y = \tilde{q}$, where $y = (\hat{P}_j - \tilde{q}) + x = ((\hat{P}_j - \hat{q} + x_j))$

$P \odot x = q \implies \max(P_1 + x_1, P_2 + x_2, \cdots, P_n + x_n) = q$

$\implies \max((P_1 - \hat{P}_1) + \hat{P}_1 + x_1, \cdots, (P_n - \hat{P}_n) + \hat{P}_n + x_n) = (q - \hat{q}) + \hat{q}$

$\implies \max(\tilde{P}_1 + \hat{P}_1 + x_1), (\tilde{P}_2 + \hat{P}_2 + x_2), \cdots, (\tilde{P}_n + \hat{P}_n + x_n) = \tilde{q} + \hat{q}$

now subtract by $\hat{q}$ on both side

$\implies \max((\tilde{P}_1 + (\hat{P}_1 - \hat{q} + x_1), (\tilde{P}_1 + (\hat{P}_2 - \hat{q} + x_2), \cdots, (\tilde{P}_1 + (\hat{P}_n - \hat{q} + x_n)) = \tilde{q}$

$\implies \max((\tilde{P}_1 + y_1), (\tilde{P}_2 + y_2)), \cdots, (\tilde{P}_n + y_n) = \tilde{q}$

$\implies \tilde{P} \odot y = \tilde{q}$

where $y_j \leq \tilde{q}_i - \tilde{p_{ij}}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Now we discuss the associated normalized matrix $U = (u_{ij}) \in M_{m \times n}(S)$, where $u_{ij} = \tilde{q}_i - \tilde{p_{ij}}$.We take z_j as the element that is the minimum of U_j, where U_j is the j-th column of the U matrix. Now we have to find the maximal solution for the given system. Initially, we have to find the solution y^* for the normalized system. Where y^* is obtained by placing each column's minimum element of U_j as y_j. After finding y^*, we have to find the maximal solution x^* of the given system $P \odot x = q$, with each entry of the maximal solution can be calculated by the formula ${x_j}^* = {y_j}^* - \hat{P}_j + \hat{q}$, where

$$y^* = \begin{bmatrix} {y_1}^* \\ {y_2}^* \\ {y_3}^* \\ \vdots \\ {y_n}^* \end{bmatrix}$$

$$x^* = \begin{bmatrix} {x_1}^* = {y_1}^* - \hat{P}_1 + \hat{q} \\ {x_2}^* = {y_2}^* - \hat{P}_2 + \hat{q} \\ {x_3}^* = {y_3}^* - \hat{P}_3 + \hat{q} \\ \vdots \\ {x_n}^* = {y_n}^* - \hat{P}_n + \hat{q} \end{bmatrix}$$

Theorem 3.1. [16] *The linear system $P \odot x = q$ has a solution if and only if every row of the associated normalized matrix U contains at least one element that is a column minimum.*

3.2. **Discrepancy method.** There is another method to determine the existence of the solution $P \odot x = q$. Where solution is obtained by defining the"discrepancy" and "reducible discrepancy" matrices as follows:

$$D_{Pq} = \begin{bmatrix} q_1 - p_{11} & q_1 - p_{12} & \cdots & q_1 - p_{1n} \\ q_2 - p_{21} & q_2 - p_{22} & \cdots & q_2 - p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ q_m - p_{m1} & q_m - p_{m2} & \cdots & q_m - p_{mn} \end{bmatrix}$$

and

$$RD_{Pq} = \begin{cases} 1 & \text{if } d_{ij} = \text{minimum of the } j\text{-th column} \\ 0 & \text{else} \end{cases}$$

where d_{ij} denotes ij-th entry of the reducible discrepancy matrix RD_{Pq} and note that minimum of each column of D_{Pq} is the element of x^*. If each row of the matrix RD_{Pq} contains only one element with a value 1, then $P \odot x = q$ has a unique solution. In addition, if atleast one row contains an element 1 more than one time, then $P \odot x = q$ has many solutions.

Theorem 3.2. *A system $P \odot x = q$ has a solution if and only if every row of the reducible discrepancy matrix has atleast one element with value 1.*

Proof. Assume that the system has a solution, which implies that for some i-th row, $1 \leq i \leq m$, of the reducible discrepancy matrix, there is no element with the value 1. Since the discrepancy matrix does not have a column minimum element in that corresponding row, we have $x_j < q_i - p_{ij}$ for all $1 \leq j \leq n$. Thus, $\max\{p_{i1} + x_1, p_{i2} + x_2, \cdots, p_{in} + x_n\} < q_i$, which contradicts our assumption that the system has a solution. Every row of the reducible discrepancy matrix should have an element of value 1.

Conversely, if every row of the reducible discrepancy matrix has at least one element with value 1, then $x_j = q_i - p_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. This implies that $\max\{p_{i1}+x_1, p_{i2}+x_2, \cdots, p_{in}+x_n\} = q_i$. Hence, the system has a solution. □

Theorem 3.3. *A system $P \odot x = q$ has many solutions if and only if atleast one row of the reducible discrepancy matrix has more than one element with the value 1*

4. Main Results

In this section, we will discuss the general maximal solution of particular matrices. Let us assume the tropical semirings $\mathbf{T} = (\mathbb{W} \cup \{-\infty\}, \oplus, \odot)$, where $\mathbb{W}$ denotes the set of all whole numbers, $\mathbf{V} = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where $\mathbb{R}$ is the set of all real numbers, and $\mathbf{W} = (\mathbb{Z} \cup \{-\infty\}, \oplus, \odot)$, where $\mathbb{Z}$ is the set of all integers. In this section, we have concluded and written some results on certain matrices using the discrepancy method.

4.1. Analysing the tropical systems over some special matrices with discrepancy method.

Theorem 4.1. *Let $P \in M_{m\times m}(\mathbf{T})$ be a row natural arithmetic matrix and $P \odot x = q$ is a linear system over the tropical semiring $(\mathbf{T})$.*

(1) *If the $m \times 1$ regular vector q is of the form*

$$q = \begin{bmatrix} m^2+1 \\ m^2+2 \\ m^2+3 \\ \vdots \\ m^2+m \end{bmatrix}$$

then the linear system $P \odot x = q$ has many solutions.

(2) *If $q = P_j$ for some $1 \le j \le m$ then the system has many solutions, where P_j denotes the j-th column of P matrix.*

Proof. (1) Assume that P is a row natural arithmetic matrix over the tropical semiring $\mathbf{T}$,

$$\begin{bmatrix} 1 & m+1 & 2m+1 & 3m+1 & \cdots & \cdots & m(m-1)+1 \\ 2 & m+2 & 2m+2 & 3m+2 & \cdots & \cdots & m(m-1)+2 \\ 3 & m+3 & 2m+3 & 3m+3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m & 2m & 3m & 4m & \cdots & m(m-1) & m(m-1)+m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} m^2+1 \\ m^2+2 \\ m^2+3 \\ \vdots \\ m^2+m \end{bmatrix}$$

The discrepancy matrix is obtained as follows,

$$D_{Pq} = \begin{bmatrix} m^2 & m^2-m & m^2-2m & \cdots & m^2-m(m-1) \\ m^2 & m^2-m & m^2-2m & \cdots & m^2-m(m-1) \\ m^2 & m^2-m & m^2-2m & \cdots & m^2-m(m-1) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m^2 & m^2-m & m^2-2m & \cdots & m^2-m(m-1) \end{bmatrix}$$

The reducible discrepancy matrix is obtained as follows,

$$RD_{Pq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & .. & 1 & 1 & 1 \end{bmatrix}$$

Every row of RD_{Pq} matrix contains more than one element of value 1. Since we conclude the system has many solutions

$$x^* = \begin{bmatrix} m^2 \\ m^2 - m \\ m^2 - 2m \\ \vdots \\ m^2 - m(m-1) \end{bmatrix}$$

all other solutions are obtained as follows,

$$x^* = \begin{bmatrix} m^2 \\ v_1^1 \\ v_1^2 \\ \vdots \\ v_1^{m-1} \end{bmatrix}, v_1^1 \leq m^2 - m, v_1^2 \leq m^2 - 2m, ..., v_1^{m-1} \leq m^2 - m(m-1)$$

similarly we can find the solutions till the term

$$x^* = \begin{bmatrix} v_m^1 \\ v_m^2 \\ \vdots \\ v_m^{m-1} \\ m^2 - m(m-1) \end{bmatrix}, v_m^1 \leq m^2, v_m^2 \leq m^2 - m, ..., v_m^{m-1} \leq m^2 - m(m-2)$$

(2) if $q = P_j$, for some $1 \leq j \leq m$ then we have to show that system has many solution. Suppose $q = P_k$, for some $1 \leq k \leq m$ then k-th column of D_{Pq} matrix is a zero vector and $(k-1)$-th column vector is a m vector (Since m vector denotes the vector which has all entries of value m), $(k-2)$-th column vector is a $2m$ vector. Similarly $(k-r)$-th column vector is a rm vector and $(k+1)^{th}$ column is a $(-m)$ vector. Similarly we are getting $(k+h)$-th column is a $(-hm)$ vector.

$$\begin{bmatrix} 1 & m+1 & 2m+1 & 3m+1 & \cdots & \cdots & m(m-1)+1 \\ 2 & m+2 & 2m+2 & 3m+2 & \cdots & \cdots & m(m-1)+2 \\ 3 & m+3 & 2m+3 & 3m+3 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ m & 2m & 3m & 4m & \cdots & m(m-1) & m(m-1)+m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = P_j,$$

The discrepancy matrix is obtained as follows,

$$D_{Pq} = \begin{bmatrix} rm & .. & m & 0 & -m & \cdots & -hm \\ rm & .. & m & 0 & -m & \cdots & -hm \\ rm & .. & m & 0 & -m & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ rm & .. & m & 0 & -m & .. & -hm \end{bmatrix}$$

$$RD_{Pq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & .. & 1 & 1 & 1 \end{bmatrix}$$

Clearly, we can say that system has many solution. Maximal solution is

$$x^* = \begin{bmatrix} rm \\ \vdots \\ 2m \\ m \\ 0 \\ -m \\ \vdots \\ -hm \end{bmatrix}$$

all other solutions are obtained by,

$$x^* = \begin{bmatrix} rm \\ v_1^1 \\ v_1^2 \\ v_1^3 \\ \vdots \\ v_1^k \\ \vdots \\ v_1^{m-1} \end{bmatrix}; \quad \begin{matrix} v_1^1 \leq (r-1)m, v_1^2 \leq (r-2)m, ..., v_1^k \leq 0, ..., v_1^{k+1} \\ \leq -m, ...v_1^h \leq (-hm) \end{matrix}$$

$$x^* = \begin{bmatrix} v_2^1 \\ (r-1)m \\ v_2^2 \\ v_2^3 \\ \vdots \\ v_2^k \\ \vdots \\ v_2^m \end{bmatrix}; \quad v_2^1 \leq rm, v_2^2 \leq (r-2)m, ..., v_2^k \leq 0, ..., v_2^{k+1} \leq (-m), ..., v_2^h \leq (-hm).$$

finally we get solutions till the term as follows,

$$x^* = \begin{bmatrix} v_m^1 \\ v_m^2 \\ v_m^2 \\ v_m^3 \\ \vdots \\ v_m^k \\ \vdots \\ -hm \end{bmatrix}; \quad v_m^1 \leq (rm), v_m^2 \leq (r-1)m, v_m^3 \leq (r-2)m, ..., v_m^k \leq 0, ..., v_m^{k+1} \leq (-m), ..., v_m^{h-1} \leq (-(h-1)m).$$

□

Theorem 4.2. *Let $P \in M_{m\times m}(\mathbf{T})$ be a column natural arithmetic matrix and $P \odot x = q$ is linear system over the tropical semiring $(\mathbf{T})$,*

(1) *If the $m \times 1$ regular vector q is of the form*

$$q = \begin{bmatrix} m^2+m \\ m^2+2m \\ m^2+3m \\ \vdots \\ 2m^2 \end{bmatrix}$$

then the linear system $P \odot x = q$ has a solution.

(2) *If $q = P_j$, for some $1 \leq j \leq m$ then the system has many solutions, where P_j denotes the j-th column of P matrix.*

Proof. (1) Given P matrix is a column natural arithmetic matrix over the tropical semiring $\mathbf{T}$,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & m \\ m+1 & m+2 & m+3 & m+4 & \cdots & 2m \\ 2m+1 & 2m+2 & 2m+3 & 2m+4 & \cdots & 3m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-1)m+1 & (m-1)m+2 & (m-1)m+3 & \cdots & m(m-1)+m & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} m^2+m \\ m^2+2m \\ m^2+3m \\ \vdots \\ 2m^2 \end{bmatrix}$$

The discrepancy matrix of the system $P \odot x = q$ is obtained as follows,

$$D_{Pq} = \begin{bmatrix} m^2+m-1 & m^2+m-2 & m^2+m-3 & \cdots & m^2 \\ m^2+m-1 & m^2+m-2 & m^2+m-3 & \cdots & m^2 \\ m^2+m-1 & m^2+m-2 & m^2+m-3 & \cdots & m^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ m^2+m-1 & m^2+m-2 & m^2+m-3 & \cdots & m^2 \end{bmatrix}$$

and

$$RD_{Pq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & .. & 1 & 1 & 1 \end{bmatrix}$$

where in RD_{Pq} matrix every row contains a element with value 1 more than one time. We can conclude the system has many solutions.

$$x^* = \begin{bmatrix} m^2+m-1 \\ m^2+m-2 \\ m^2+m-3 \\ \vdots \\ m^2 \end{bmatrix}$$

(2) if $q = P_j$ for some $1 \leq j \leq m$ then we have to show that system has many solution. Suppose that $q = P_k$, then k^{th} column of D_{Pq} matrix is a zero vector, $(k-1)^{th}$ column vector is a 1 vector, (Since 1 vector denote the vector which has all entries 1), $(k-2)^{th}$ column vector is a 2 vector,

$(k-r)^{th}$ column vector is a r vector, $(k+1)^{th}$ column is a (-1) vector, Similarly $(k+h)^{th}$ column is a $(-h)$ vector.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & \cdots & m \\ m+1 & m+2 & m+3 & \cdots & \cdots & 2m \\ 2m+1 & 2m+2 & 2m+3 & \cdots & \cdots & 3m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-1)m+1 & (m-1)m+2 & (m-1)m+3 & \cdots & \cdots & m(m-1)+m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = P_j,$$

The discrepancy matrix is obtained as follows,

$$D_{Pq} = \begin{bmatrix} r & \cdots & 1 & 0 & -1 & \cdots & -h \\ r & \cdots & 1 & 0 & -1 & \cdots & -h \\ r & \cdots & 1 & 0 & -1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r & \cdots & 1 & 0 & -1 & \cdots & -h \end{bmatrix}$$

The reducible discrepancy matrix is obtained as follows,

$$RD_{Pq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & .. & 1 & 1 & 1 \end{bmatrix}$$

Since RD_{P_q} we can say that system has many solution and the maximal solution is obtained as follows,

$$x^* = \begin{bmatrix} r \\ r-1 \\ \vdots \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ \vdots \\ -h \end{bmatrix}$$

□

Theorem 4.3. *Let $P \in M_{m\times m}(\mathbf{V})$ be a J-matrix over the tropical semiring $\mathbf{V}$ and $P \odot x = q$ be a linear system with the $m \times 1$ normal vector q of the form*

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_m \end{bmatrix}$$

then,

(1) *The system has many solution if and only if $q_i = q_j$ for all $1 \le i, j \le m$.*
(2) *The system has no solution if and only if $q_i \ne q_j$, $1 \le i, j \le m$.*

Proof. Assume that J-matrix over the tropical semiring $\mathbf{V}$

$$\begin{bmatrix} j & j & j & \cdots & j \\ j & j & j & \cdots & j \\ j & j & j & \cdots & j \\ \vdots & \vdots & \vdots & \ddots & j \\ j & j & j & \cdots & j \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_m \end{bmatrix}$$

The discrepancy matrix of the system is obtained as follows,

$$D_{Pq} = \begin{bmatrix} q_1 - j & q_1 - j & q_1 - j & \cdots & q_1 - j \\ q_2 - j & q_2 - j & q_2 - j & \cdots & q_2 - j \\ q_3 - j & q_3 - j & q_3 - j & \cdots & q_3 - j \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_m - j & q_m - j & q_m - j & \cdots & q_m - j \end{bmatrix}$$

(1) Assume $q_i = q_j = k$, for all $1 \le i, j \le m$ that implies $d_{ij}{}^s$ are equal for all $1 \le i, j \le m$ that implies every elements of D_{Pq} are minimum. We obtain the reducible discrepancy matriix RD_{Pq} as follows,

$$RD_{Pq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & .. & 1 & 1 & 1 \end{bmatrix}$$

Since RD_{P_q} we conclude system has many solutions.

Conversely assume that system has a many solutions. We have to show that $q_i = q_j$ for all

$1 \leq i, j \leq m$. Already we have

$$\tilde{p_{1j}} = k, \forall 1 \leq j \leq m$$

$$\tilde{p_{2j}} = k, \forall 1 \leq j \leq m$$

$$\vdots$$

$$\tilde{p_{mj}} = k, \forall 1 \leq j \leq m$$

suppose $q_i \neq q_j$ for some $1 \leq i, j \leq m$ then we can split into two cases.

(a) If $q_i = r < q_j = k$ then i-th row of the D_{Pq} matrix contains column minimum element of every column and i-th row of RD_{Pq} matrix contains all element as 1 and all other rows does not contains any element with a value one. We conclude that the system has no solution. Which is a contradiction to our assumption.

(b) Similarly for the case $q_j = k < q_i = r$ also we get contradiction. Let $q_i - j = w$ then maximal solution

$$x^* = \begin{bmatrix} w \\ w \\ w \\ \vdots \\ w \end{bmatrix}$$

(2) Assume $q_i \neq q_j$ for some $1 \leq i, j \leq m$, If $q_i = r < q_j = k$ then i-th row of the D_{Pq} matrix contains column minimum element of every column. i-th row of RD_{Pq} matrix contains all element as 1 and all other rows does not contains element with a value one. We conclude that the system has no solution. Conversely assume that system has no solution. Suppose $q_i = q_j$ then by the first part of the Theorem 4.3 we know that system has a many solutions. So by the contradiction we can prove $q_i \neq q_j$ for some $1 \leq i, j \leq m$.

□

Corollary 4.4. *Let $P \in M_{m \times m}(\mathbf{V})$ be a J-matrix and $P \odot x = q$ be a linear system over the tropical semiring $\mathbf{V}$ with normal vector*

$$q = \begin{bmatrix} j \\ j \\ j \\ \vdots \\ j \end{bmatrix}$$

then the system has a solution.

Proof. By the first part of the Theorem 4.3 directly we can prove that system has many solutions. □

Theorem 4.5. *Let $P \in M_{m\times m}(\mathbf{V})$ be a γ-diagonal matrix and if $P \odot x = q$ be a linear system over the tropical semiring $\mathbf{V}$ with normal vector*

$$q = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \\ \vdots \\ \gamma \end{bmatrix}$$

then system has a solution.

Proof. Assume that the system with $\gamma-$ diagonal matrix,

$$\begin{bmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & \gamma & 0 & \cdots & 0 \\ 0 & 0 & \gamma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \\ \vdots \\ \gamma \end{bmatrix}$$

The discrepancy matrix is obtained as follows,

$$D_{Pq} = \begin{bmatrix} 0 & \gamma & \gamma & \cdots & \gamma \\ \gamma & 0 & \gamma & \cdots & \gamma \\ \gamma & \gamma & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \gamma & \cdots & 0 \end{bmatrix}$$

The reducible discrepancy matrix is obtained by splitting two cases as follows, if $\gamma < 0$ then

$$RD_{Pq} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & 0 \end{bmatrix}$$

$$x^* = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \\ \vdots \\ \gamma \end{bmatrix}$$

all other maximal solutions are

$$x^* = \begin{bmatrix} \gamma \\ v_1^1 \\ v_1^2 \\ \vdots \\ v_1^{m-1} \end{bmatrix}; v_1^1 \leq \gamma, v_1^2 \leq \gamma, ..., v_1^{m-1} \leq \gamma$$

$$x^* = \begin{bmatrix} v_2^1 \\ \gamma \\ v_2^2 \\ \vdots \\ v_2^{m-1} \end{bmatrix}; v_2^1 \leq \gamma, v_2^2 \leq \gamma, ..., v_2^{m-1} \leq \gamma$$

similarly we get as follows,

$$x^* = \begin{bmatrix} v_m^1 \\ v_m^2 \\ v_m^3 \\ \vdots \\ v_m^{m-1} \\ \gamma \end{bmatrix}; v_m^1 \leq \gamma, v_m^2 \leq \gamma, ..., v_m^{m-1} \leq \gamma$$

if $0 < \gamma$ then

$$R_{Pq} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & .. & 0 & 0 & 1 \end{bmatrix}$$

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

It is a unique solution. □

Theorem 4.6. *Let $P \odot x = q$ be a tropical linear system with the circulant matrix $P \in M_{m\times m}$ over the tropical semiring $\mathbf{V}$, where q is a $m \times 1$ normal vector, if $q = C_j$, for some $1 \leq j \leq m$, where C_j is j-th column of the circulant matrix then the system has a solution.*

Proof. Assume that $P \odot x = q$ be a tropical linear system with the circulant matrix.

$$\begin{bmatrix} c_0 & c_{m-1} & c_{m-2} & \cdots & c_1 \\ c_1 & c_0 & c_{m-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \vdots & c_{m-2} \\ \vdots & \vdots & \vdots & \vdots & c_{m-1} \\ c_{m-1} & c_{m-2} & c_2 & \cdots & c_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = C_j$$

where $q = C_j$, for some $1 \leq j \leq m$. Suppose that $q = C_k$, $1 \leq k \leq m$, where C_k denotes the k-th column of the circulant matrix then k-th column of D_{Pq} matrix is zero and corresponding k-th column of RD_{Pq} matrix contains all entries of value one. Which implies that the system has a solution. □

5. Relationship between Normalization and Discrepancy Methods

We finished the scenarios in which the particular matrices over the tropical semirings have a solution or do not have a solution in the preceding sections. We used discrepancy method to determine the general form of the maximal solution of several specific matrices over tropical semirings. We also compared the normalization and discrepancy approaches in this section by comparing the data obtained from both methods. To understand the relationship between the two methodologies, we present the following theorems.

Theorem 5.1. *A tropical linear system $P \odot x = q$ where $P \in M_{m\times m}(S)$ every row of associated normalized matrix has atleast one column minimum element if and only if every row of RD_{Pq} has atleast one element with value of 1.*

Proof. Assume that every row of the associated normalized matrix has atleast one column minimum element. Which gives that system has a solution. If the system has a solution then by the Theorem 3.2 we can say that every row of RD_{Pq} has atleast one element with the value 1. Conversely assume that every row of RD_{Pq} has atleast one element of value 1. By the Theorem 3.2 we can say that system has a solution. By the Theorem 3.1 we can prove that every row of associated normalized matrix has atleast one column minimum element. □

Theorem 5.2. *Let the system $P \odot x = q$ with $P \in M_{m\times n}(S)$ if every row of associated normalized matrix has atleast one column minimum element if and only if every row of D_{Pq} has atleast one column minimum element.*

Proof. Assume that every row of the associated normalized matrix has atleast one column minimum element. Suppose any one of the rows of discrepancy matrix has no column minimum element then by the Theorem 5.1 that implies some row of RD_{Pq} has no element with value 1. which is the contradiction. Conversely assume that every row of the D_{Pq} has atleast one column minimum element. We know that every row of the RD_{Pq} has atleast one element with a value 1 which implies that system has a solution. By Theorem 3.1 every row of associated normalized matrix has atleast one column minimum element. Hence the proof. □

Theorem 5.3. *Let the system $P \odot x = q$ with $P \in M_{m\times n}(S)$, RD_{Pq} matrix has exactly one element with a value 1 if and only if the associated normalized matrix has exactly one column minimum element.*

Conclusion

In this article to determine the solutions of tropical linear systems, we employed with discrepancy method. We talked about the conditions in tropical systems which came up with a unique solution, many solutions and no solution. We used the discrepancy approach to find all the solutions once the system had numerous. Also we spoken about how to use the normalization and discrepancy methods to determine the maximal solution of the linear equations over the tropical semirings. We employed certain matrices row natural arithmetic matrices, column natural matrices, γ - diagonal matrices, J-matrices, and circulant matrices and studied the general form of the maximal solutions. We have also written several theorems about the generalized maximal solutions of specific linear systems over the tropical semirings. We presented some results on relationship between two methods. In future we may try to concentrate on the attack of cryptographical algorithms with obtained solutions.

References

[1] I. Simon, Limited subsets of a free monoid, in: 19th Annual Symposium on Foundations of Computer Science (Sfcs 1978), IEEE, Ann Arbor, MI, USA, 1978: pp. 143-150. https://doi.org/10.1109/SFCS.1978.21.

[2] I. Simon, Recognizable sets with multiplicities in the tropical semiring, in: M.P. Chytil, V. Koubek, L. Janiga (Eds.), Mathematical Foundations of Computer Science 1988, Springer-Verlag, Berlin/Heidelberg, 1988: pp. 107-120. https://doi.org/10.1007/BFb0017135.

[3] J.E. Pin, Tropical Semirings, J. Gunawardena. Idempotency (Bristol, 1994), Cambridge University Press, Cambridge, pp. 50-69, 1998. https://hal.science/hal-00113779.

[4] W. Burnside, On an unsettled question in the theory of discontinuous groups, Quart. J. Pure Appl. Math. 33 (1902), 230-238.

[5] L. Aceto, Z. Ésik, A. Ingólfsdóttir, Equational theories of tropical semirings, Theor. Computer Sci. 298 (2003), 417-469. https://doi.org/10.1016/S0304-3975(02)00864-2.

[6] R. Coninghame-Green, Minimax algebra, Lecture Notes in Economics and Mathematical Systems 166, Springer Science & Business Media, New York, 2012.

[7] D. Maclagan, B. Sturmfels, Introduction to tropical geometry, Graduate Studies in Mathematics, Volume 161, American Mathematical Society, Providence, 2021.

[8] D. Speyer, B. Sturmfels, Tropical mathematics, Math. Mag. 82 (2009), 163-173. https://doi.org/10.1080/0025570X.2009.11953615.

[9] D. Krob, Some automata-theoretic aspects of min-max-plus semirings, In: J. Gunawardena (ed.) Idempotency, Cambridge University Press, Cambridge, pp. 70-79, (1998).

[10] M. Akian, S. Gaubert, A. Guterman, Contemporary mathematics, American Mathematical Society, Providence, 2009.

[11] D. Joó, K. Mincheva, On the dimension of polynomial semirings, J. Algebra. 507 (2018), 103-119. https://doi.org/10.1016/j.jalgebra.2018.04.007.

[12] [1] A. Omanović, P. Oblak, T. Curk, Application of tropical semiring for matrix factorization, Uporabna Inform. 28 (2020), 205-208. https://doi.org/10.31449/upinf.99.

[13] S. Gaubert, M. Plus, Methods and applications of (max,+) linear algebra, in: R. Reischuk, M. Morvan (Eds.), STACS 97, Springer, Berlin, Heidelberg, 1997: pp. 261-282. https://doi.org/10.1007/BFb0023465.

[14] D. Grigoriev, Complexity of solving tropical linear systems. Comput. Complex, 22 (2013), 71-88.

[15] Z. Izhakian, Tropical and Idempotent Mathematics, American Mathematical Society, Providence, 2009.

[16] F. Olia, S. Ghalandarzadeh, A. Amiraslani, S. Jamshidvand, Solving linear systems over tropical semirings through normalization method and its applications, J. Algebra Appl. 20 (2021), 2150159. https://doi.org/10.1142/S0219498821501590.

[17] S. Jamshidvand, S. Ghalandarzadeh, A. Amiraslani, F. Olia, On the maximal solution of a linear system over tropical semirings, Math. Sci. 14 (2020), 147-157. https://doi.org/10.1007/s40096-020-00325-w.

[18] B. Amutha, R. Perumal, Public key exchange protocols based on tropical lower circulant and anti circulant matrices, AIMS Math. 8 (2023), 17307-17334. https://doi.org/10.3934/math.2023885.

[19] J.S. Golan, Semirings and their applications, Springer, New York, 2013.

[20] P. Butkovič, Max-linear systems: theory and algorithms, Springer, London, 2010. https://doi.org/10.1007/978-1-84996-299-5.

[21] B. Heidergott, G.J. Olsder, J. van der Woude, Max plus at work modelling and analysis of synchronized system: a course on max-plus algebra and its application, Princeton University Press, Princeton, 2006.

[22] A. Davydow, An algorithm for solving an overdetermined tropical linear system using the analysis of stable solutions of subsystems, J. Math. Sci. 232 (2018), 25-35. https://doi.org/10.1007/s10958-018-3856-3.