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ORTHONORMAL SYSTEM FOR A MATRIX BALL OF THE SECOND TYPE $\mathbb{B}_{m,n}^{(2)}$ AND ITS SKELETON (SHILOV'S BOUNDARY) $\mathbb{X}_{m,n}^{(2)}$

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Abstract. In this paper, an orthonormal system was found for a matrix ball of the second type and unsolved open problems of several complex analysis in the matrix bal $\mathbb{B}_{m,n}^{(3)}$ associated with the classical domain of the third type and its skeleton $\mathbb{X}_{m,n}^{(3)}$ are formulated.

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1. Introduction, preliminaries and problem statement

The theory of functions of many complex variables, or multidimensional complex analysis, currently has a fairly strictly constructed theory [1], [2], [3]. At the same time, many questions of classical complex analysis still do not have unambiguous multidimensional analogues. In the works of E. Cartan, K. Siegel, Hua Luogeng and I.I. Pyatetsky-Shapiro the matrix approach to the presentation of the theory of multidimensional complex analysis is widely used (see [4–8]).

In 1935, E. Cartan proved that there are only six possible types of irreducible, homogeneous, bounded, symmetric domains. These domains denoted by, $\Re_I, \Re_{II}, \Re_{III}$ and $\Re_{IV}$ are called classical domains:

$$\Re_I = \left\{ Z \in \mathbb{C}\left[m \times k\right] : I^{(m)} - Z\overline{Z}' > 0 \right\},$$

$$\Re_{II} = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} - Z\overline{Z} > 0, \quad \forall Z' = Z \right\},$$

$$\Re_{III} = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} + Z\overline{Z} > 0, \quad \forall Z' = -Z \right\},$$

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$$\Re_{IV} = \left\{ z \in \mathbb{C}^n : \left| zz' \right|^2 + 1 - 2\overline{z}z' > 0, \quad \left| zz' \right| < 1 \right\}.$$

The dimensions of these domains are equal, respectively to:

$$mk, m(m+1)/2, m(m-1)/2, n.$$

Consider the space of complex m^2 variables, denoted by $\mathbb{C}^{m^2}$. In some questions, it is convenient to represent the points Z of this space in the form of square $[m \times m]$-matrices, i.e. in the form of $Z = (z_{ij})_{i,j=1}^m$. With this representation of points, the space $\mathbb{C}^{m^2}$ will denote $\mathbb{C}[m \times m]$. The direct product $\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$ of n instances of $[m \times m]$-matrices space we denote by $\mathbb{C}^n[m \times m]$.

Let $Z = (Z_1, Z_2, ..., Z_n)$ be a vector composed of square matrices Z_j of order m considered over the field of complex numbers $\mathbb{C}$. We can assume that Z is an element of the space $\mathbb{C}^n[m \times m] \cong \mathbb{C}^{nm^2}$.

We define a matrix "scalar" product as below

$$\langle Z, Z \rangle = Z_1 W_1^* + ... + Z_n W_n^*$$

where W_j^* is a conjugate and transposed matrix for the matrix W_j.

Define matrix balls(see [9,10]) $\mathbb{B}_{m,n}^{(1)}$, $\mathbb{B}_{m,n}^{(2)}$ and $\mathbb{B}_{m,n}^{(3)}$ of the first, second and third types, respectively:

$$\mathbb{B}_{m,n}^{(1)} = \left\{ (Z_1, ..., Z_n) = Z \in \mathbb{C}^n \left[m \times m \right] : I - \langle Z, Z \rangle > 0 \right\},$$

$$\mathbb{B}_{m,n}^{(2)} = \left\{ Z \in \mathbb{C}^n \left[m \times m \right] : I - \langle Z, Z \rangle > 0, \quad \forall Z'_\nu = Z_\nu, \ \nu = 1, ..., n \right\},$$

$$\mathbb{B}_{m,n}^{(3)} = \left\{ (Z \in \mathbb{C}^n \left[m \times m \right] : I - \langle Z, Z \rangle > 0, \ \forall \ Z_\nu^{'} = -Z_\nu, \quad \nu = 1, ..., n \right\}.$$

The skeletons (Shilov boundaries) of matrix balls $\mathbb{B}_{m,n}^{(k)}$, denote by $\mathbb{X}_{m,n}^{(k)}, \quad k = 1, 2, 3$, i.e.,

$$\mathbb{X}_{m,n}^{(1)} = \left\{ Z \in \mathbb{C}^n \left[m \times m \right] : \langle Z, Z \rangle = I \right\},$$

$$\mathbb{X}_{m,n}^{(2)} = \left\{ Z \in \mathbb{C}^n \left[m \times m \right] : \langle Z, Z \rangle = I, \quad Z'_v = Z_\nu, \ \nu = 1, 2, ..., n \right\},$$

$$\mathbb{X}_{m,n}^{(3)} = \left\{ Z \in \mathbb{C}^n \left[m \times m \right] : I - \langle Z, Z \rangle = 0, \ \ Z_\nu^{'} = -Z_\nu, \quad \nu = 1, 2, ..., n \right\}.$$

Note that $\mathbb{B}_{1,1}^{(1)}$, $\mathbb{B}_{1,1}^{(2)}$ and $\mathbb{B}_{2,1}^{(3)}$- are unit discs, $\mathbb{X}_{1,1}^{(1)}$, $\mathbb{X}_{1,1}^{(2)}$, and $\mathbb{X}_{2,1}^{(3)}$- unit circles in the complex plane $\mathbb{C}$.

If $n = 1, m > 1$, then $\mathbb{B}_{m,1}^{(k)}, \quad k = 1, 2, 3$ - are classical domains of the first, second and third type (according to the classification of E.Cartan [4]), and the skeletons of $\mathbb{X}_{m,1}^{(1)}$, $\mathbb{X}_{m,1}^{(2)}$, and $\mathbb{X}_{m,1}^{(3)}$ - are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

Recently, scientists have achieved many significant results in classical domains and at the same time, a number of open problems have been formulated. For example, [11] studies the regularity and algebraicity of maps in classical domains, and [12] studies the harmonic Bergman functions in classical domains from a new point of view and proves the results of characterization of harmonic Bergman functions in domains of the first type. In [13] holomorphic and pluriharmonic functions for classical domains of the first type are defined, Laplace and Hua Luogeng operators are studied and a connection between these operators is found. In [14–18] studies present the properties of Bergman and

Cauchy–Szegő kernels for classical domains. For this, we use the statements of the Sommer-Mehring theorem (see [19]) on the extension of the Bergman kernel and some properties of the Bergman kernel and these domains were used as bounded full circular symmetric convexity.

In [20], the problem of holomorphic continuability of a function into a matrix ball given in pieces of its skeleton is considered. For this purpose complete orthonormal systems in a matrix ball are used. And also applied in this orthonormal systems, in the work [21,22] obtained analogues of the Laurent series with respect to the matrix ball from the space $\mathbb{C}^n[m \times m]$. To do this, we first introduce the concept of a matrix ball layer from $\mathbb{C}^n[m \times m]$, then in this matrix ball layer the properties of Bochner–Hua Luogeng integrals are used to obtain analogues of the Laurent series.

In [20] some unsolved problems are formulated relating to the matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$ from the space $\mathbb{C}^n[m \times m]$. One of them is to write out an orthonormal basis for $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$ (the existence of such bases follows from A.Cartan's theorem on complete circular domains (see [23]). In this paper $\mathbb{C}^n[m \times m]$, the second type of matrix spheres in the space $\mathbb{B}^{(2)}_{m,n}$ and its skeleton (Shilov boundary) are calculated $\mathbb{X}^{(2)}_{m,n}$ (see [24], [25]). The results obtained in this paper are analogs of the results obtained by Hua Luogeng in classical domains of the second type and G. Khudayberganov in matrix balls of the first type $\mathbb{X}^{(2)}_{m,n}$.

Note that, the matrix ball $\mathbb{B}^{(2)}_{m,n}$ is a complete circular convex bounded domain. In addition, the domain $\mathbb{B}^{(2)}_{m,n}$ and its backbone $\mathbb{X}^{(2)}_{m,n}$ are invariant with respect to unitary transformations (see [27]). We are interested in the orthonormal system spaces $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{X}^{(2)}_{m,n}$, i.e. to write out an orthonormal basis for the domains $\mathbb{B}^{(2)}_{m,n}$ and its skeleton $\mathbb{X}^{(2)}_{m,n}$. The existence of such bases follows from the theorem of A.Cartan on complete circular domains [23].

Lemma 1 [27]. *The matrix ball has the following properties:*

1) $\mathbb{B}^{(2)}_{m,n}$ *is bounded domain;*

2) $\mathbb{B}^{(2)}_{m,n}$ *is full circular domain;*

3) *Domain* $\mathbb{B}^{(2)}_{m,n}$ *and its skeleton* $\mathbb{X}^{(2)}_{m,n}$ *are invariant with respect to unitary transformations;*

4) $\mathbb{B}^{(2)}_{m,n}$ *is convex domain.*

The volume of the matrix ball of the second type is calculated using the following theorem.

Theorem 1 [24]. *Let* $m \geq 2$ *and* Z_ν *is* $m \times m$ *the symmetric matrix. Let's put*

$$J(\lambda) = \int\limits_{I - \langle Z, Z \rangle > 0} [det(I - \langle Z, Z \rangle)]^\lambda \dot{Z},$$

where $\dot{Z} = \prod\limits_{i=1}^{m} \prod\limits_{j=1}^{mn} dx_{ij} dy_{ij}, x_{ij} + iy_{ij} = z_{ij}$. *Then*

$$J(\lambda) = \frac{\pi^{\frac{m(m+1)}{2}n}}{(\lambda+1) \cdot \ldots \cdot (\lambda + mn)} \cdot \frac{\Gamma(2\lambda+3)\Gamma(2\lambda+5)\ldots\Gamma(2\lambda+2mn-1)}{\Gamma(2\lambda+mn+2)\Gamma(2\lambda+mn+3)\ldots\Gamma(2\lambda+2mn)}.$$

In particular, when $\lambda = 0$, the volume of the matrix ball of the second type is obtained

$$V(\mathbb{B}^{(2)}_{m,n}) = \frac{\pi^{\frac{m(m+1)}{2}n}}{m!} \cdot \frac{2!4! \cdot ... \cdot (2mn-3)!}{(mn+1)!(mn+2)! \cdot ... \cdot (2mn-1)!}.$$

Let $k_1, k_2, ..., k_m$ are integers satisfying the condition $k_1 \geq k_2 \geq ... \geq k_m \geq 0$. Each element of P from $GL(m)$ (i.e. the group of all non-degenerate matrices of order m) corresponds in the representation of $GL(m)$ with the signature $(k_1, k_2, ..., k_m)$ matrix

$$A_{k_1,k_2,...,k_m}(P). \tag{1}$$

Suppose that the representation (1) is unitary for unitary matrices P. It is known that (1) is a matrix having (see. [6], [29]) $\mathbb{N}_k$ rows and columns where

$$\mathbb{N}_k = N(k_1, k_2, ..., k_m) = \frac{D(k_1 + m - 1, k_2 + m - 2, ..., k_{m-1} + 1, k_m)}{D(m-1, m-2, ..., 1, 0)}, \tag{2}$$

$$D(k_1, k_2, ..., k_m) = \prod_{1 \leq i < j \leq m} (k_i - k_j), \ \ m \geq 2.$$

The trace of this matrix we will denote by

$$\chi_{k_1,k_2,...,k_m}(P) = SpA_{k_1,k_2,...,k_m}(P).$$

This value is called the character of the representation (1) (see [29]).

If P is a diagonal matrix, $P = \Lambda = [\lambda_1, \lambda_2, ..., \lambda_m]$, then (see [6], [28])

$$\chi_{k_1,k_2,...,k_m}(\Lambda) = \frac{M_{k_1,k_2,...,k_m}(\lambda_1, \lambda_2, ..., \lambda_m)}{D(\lambda_1, \lambda_2, ..., \lambda_m)},$$

where

$$M_{k_1,k_2,...,k_m}(\lambda_1, \lambda_2, ..., \lambda_m) = \det \left| \lambda_j^{k_i+m-i} \right|_{i,j=1}^{m}.$$

Denote by $u = (u_1, u_2, ..., u_m)$ a vector in m-dimensional complex space. By $u^{[\alpha]}$ we will denote a vector with components

$$\sqrt{\frac{\alpha!}{\alpha_1!\alpha_2!...\alpha_m!}} u_1^{\alpha_1} u_2^{\alpha_2} ... u_m^{\alpha_m}, \ \ \alpha = \sum_{i=1}^{m} \alpha_i, \ \ \alpha \geq 0. \tag{3}$$

The dimension of the vector $u^{[\alpha]}$ is equal to

$$\frac{(m+\alpha-1)!}{\alpha!(m-1)!}.$$

For complete circular domains, it can be assumed that the group of movements that leave the origin stationary consists of linear transformations of the form

$$v = uP, \tag{4}$$

where P is a unitary matrix. Then transformation (4) induces transformation

$$v^{[\alpha]} = u^{[\alpha]} P^{[\alpha]}, \tag{5}$$

where $P^{[\alpha]}$ denotes the α-th symmetrized Kronecker power of the matrix P (see [29]).

Expression (3) contains all monomials of degree α, i.e. any homogeneous form from $u_1, u_2, ..., u_m$ of degree α is a linear combination of expressions of the form (3). Any polynomial of $u_1, u_2, ..., u_m$ is a linear combination of expressions of the form (3) if α takes the values 0,1,2,....

1.1. **Examples to α-th symmetric Kronecker power.** For questions that arise, let's look at a special case. Let $m = 2, \;\; \alpha$ =2. Then

$$z^{[2]} = \left\{z_1^2, \;\; \sqrt{2}z_1z_2, \;\; z_2^2\right\}, \;\; w^{[2]} = \left\{w_1^2, \;\; \sqrt{2}w_1w_2, \;\; w_2^2\right\},$$

but $P^{[2]}$ has the following form:

$$P^{[2]} = \begin{pmatrix} p_{11}^2 & \sqrt{2}p_{11}p_{12} & p_{12}^2 \\ \sqrt{2}p_{11}p_{21} & p_{11}p_{22} + p_{12}p_{21} & \sqrt{2}p_{12}p_{22} \\ p_{21}^2 & \sqrt{2}p_{21}p_{22} & p_{22}^2 \end{pmatrix}.$$

Let now $m = 2, \quad \alpha$ =3. Then

$$z^{[3]} = \left\{z_1^3, \;\; \sqrt{3}z_1^2z_2, \;\; \sqrt{3}z_1z_2^2, \;\; z_2^3\right\}, w^{[3]} = \left\{w_1^3, \;\; \sqrt{3}w_1^2w_2, \;\; \sqrt{3}w_1w_2^2, \;\; w_2^3\right\},$$

but $P^{[3]}$ it has the following form:

$$P^{[3]} = \begin{pmatrix} p_{11}^3 & \sqrt{3}p_{11}^2p_{12} & \sqrt{3}p_{11}p_{12}^2 & p_{12}^3 \\ \sqrt{3}p_{12}^2p_{21} & p_{11}^2p_{22} + 2p_{11}p_{12}p_{21} & 2p_{11}p_{12}p_{21} & \sqrt{3}p_{12}^2p_{22} \\ \sqrt{3}p_{11}p_{21}^2 & 2p_{11}p_{21} + p_{12}p_{21}^2 & p_{11}p_{22}^2 + 2p_{12}p_{21}p_{22} & \sqrt{3}p_{12}p_{22}^2 \\ p_{21}^3 & \sqrt{3}p_{21}^2p_{22} & \sqrt{3}p_{21}p_{22}^2 & p_{22}^3 \end{pmatrix}.$$

2. Main results

Let's describe the vector element $Z = (Z_1, \; Z_2, ..., Z_n) \in \mathbb{B}_{m,n}^{(2)}$ in the form of a point in space $\mathbb{C}^{n\cdot\frac{m(m+1)}{2}}$:

$$z = \left\{z_{11}^{(1)}, \; z_{12}^{(1)}, ..., z_{1m}^{(1)}, \; z_{22}^{(1)}, \; z_{23}^{(1)}, ..., z_{2m}^{(1)}, ..., z_{m-1,m-1}^{(1)}, z_{m-1,m}^{(1)}, \; z_{mm}^{(1)}\right.,$$

$$z_{11}^{(2)}, \; z_{12}^{(2)}, ..., z_{1m}^{(2)}, \; z_{22}^{(2)}, \; z_{23}^{(2)}, ..., z_{2m}^{(2)}, ..., z_{m-1,m-1}^{(2)}, z_{m-1,m}^{(2)}, \; z_{mm}^{(2)},$$

$$\cdots$$

$$\left. z_{11}^{(n)}, \; z_{12}^{(n)}, ..., z_{1m}^{(n)}, \; z_{22}^{(n)}, \; z_{23}^{(n)}, ..., z_{2m}^{(n)}, ..., z_{m-1,m-1}^{(n)}, z_{m-1,m}^{(n)}, \; z_{mm}^{(n)}\right\} \in \mathbb{C}^{n\cdot\frac{m(m+1)}{2}}. \tag{6}$$

By $z^{[\alpha]}$ we will denote a vector with components

$$\sqrt{\frac{\alpha!}{\alpha_1!\alpha_2!...\alpha_{\frac{nm(m+1)}{2}}!}} \cdot \left(z_{11}^{(1)}\right)^{\alpha_1} \cdot \left(z_{12}^{(1)}\right)^{\alpha_2} \cdot ... \cdot \left(z_{1m}^{(1)}\right)^{\alpha_m} \cdot \left(z_{22}^{(1)}\right)^{\alpha_{m+1}} \cdot \left(z_{23}^{(1)}\right)^{\alpha_{m+2}} \cdot ... \cdot \left(z_{2m}^{(1)}\right)^{\alpha_{2m-1}} \times$$

$$\times \left(z_{m-1,m-1}^{(1)}\right)^{\alpha_{\frac{m(m+1)}{2}-2}} \cdot \left(z_{m-1,m}^{(1)}\right)^{\alpha_{\frac{m(m+1)}{2}-1}} \cdot \left(z_{mm}^{(1)}\right)^{\alpha_{\frac{m(m+1)}{2}}} \times$$

$$\times \left(z_{11}^{(2)}\right)^{\alpha_{\frac{m(m+1)}{2}+1}} \cdot \left(z_{12}^{(2)}\right)^{\alpha_{\frac{m(m+1)}{2}+2}} \cdot ... \cdot \left(z_{mm}^{(2)}\right)^{\alpha_{m(m+1)}} \cdot ... \cdot \left(z_{mm}^{(n)}\right)^{\alpha_{\frac{nm(m+1)}{2}}}. \tag{7}$$

$z^{[\alpha]}$ is a monomial of $z_{kj}^{(\nu)}, \nu = \overline{1,n}, k, j = \overline{1,m}$ of degree α. These components are linearly independent, and any homogeneous polynomial of degree α can be written as a linear combination of these components.

The dimension of the subspace generated by the vector $z^{[\alpha]}$ is equal to the dimension of the direct sum of subspaces with dimensions (see [6], [28], [29]):

$$\frac{\left(\frac{nm(m+1)}{2} + \alpha - 1\right)!}{\alpha!(\frac{nm(m+1)}{2} - 1)!}.$$

From the theorems 1.3.2 and 1.4.2 in the book [6] we know that the space of homogeneous polynomials is power α from $z_{kj}^{(s)}$ can be decomposed into a direct sum of subspaces invariant with respect to transformation (5). These subspaces have dimensions

$$\mathbb{N}_\alpha = N(2\alpha_1,\, 2\alpha_2,\, ...,\, 2\alpha_{\frac{m(m+1)}{2}}),$$

where $\alpha_1 +\ \alpha_2 +\ ..., +\alpha_{\frac{m(m+1)}{2}} = \alpha$ (the value of $\mathbb{N}_\alpha$ is calculated as formulas (2)).

Let $U = (U_1, U_2, ..., U_n) \in \mathbb{X}_{m,n}^{(2)}$. The following polynomials,

$$\varphi_\alpha(U) = \varphi^{(s)}_{\alpha_1, \alpha_2, ..., \alpha_{\frac{m(m+1)}{2}}}(U),\;\; s = 1, 2, ..., \mathbb{N}_\alpha. \tag{8}$$

forming the basis of a subspace of dimension $\mathbb{N}_\alpha$, where $\varphi^{(s)}_{\alpha_1, \alpha_2, ..., \alpha_{\frac{m(m+1)}{2}}}(U)$ and components of the vector $u^{[\alpha]}$ (these components are calculated as formulas (7)).

If we put

$$\psi_\alpha^{(s)}\left(U\right) = \rho_\alpha^{-\frac{1}{2}} \cdot \varphi_\alpha^{(\alpha)}\left(U\right),$$

then the following lemma is valid.

Lemma 2. *For the skeleton $\mathbb{X}_{m,n}^{(2)}$ function system*

$$(\rho_\alpha)^{-\frac{1}{2}}\varphi_\alpha^{(j)}(U),\;\; j = 1, 2, ..., \mathbb{N}_\alpha,\;\; \alpha = 0, 1, 2, ...$$

is an orthonormal system, where

$$\rho_\alpha = \int\limits_{\mathbb{X}_{m,n}^{(2)}} \left|\varphi_\alpha^{(j)}(U)\right|^2 \dot{U}.$$

Proof. By virtue of our designation, for $U \in \mathbb{X}_{m,n}^{(2)}$ we have

$$u^{[\alpha]} = \{\varphi_\alpha^{(1)}(U), \varphi_\alpha^{(2)}(U), ..., \varphi_\alpha^{(\mathbb{N}_\alpha)}(U)\}.$$

Then

$$\int\limits_{\mathbb{X}_{m,n}^{(2)}} \left(u^{[\alpha]}\right)' \overline{\left(u^{[\beta]}\right)}\dot{U} = R^{(\mathbb{N}_\alpha, \mathbb{N}_\beta)}, \tag{9}$$

where $\dot{U} = \prod_{p=1}^{m} \prod_{q=1}^{mn} dx_{pq} dy_{pq}, \;\; x_{pq} + iy_{pq} = u_{pq}$, and

$$R^{(\mathbb{N}_\alpha, N_\beta)} = \begin{cases} \text{O}, & \alpha \neq \beta, \\ \rho_\alpha \cdot I^{\mathbb{N}_\alpha}, & \alpha = \beta, \end{cases}$$

where O is a null matrix, and ρ_α does not depend on s. Therefore,

$$\int\limits_{\mathbb{X}^{(2)}_{m,n}} \varphi^{(s)}_\alpha(U) \cdot \overline{\varphi^{(l)}_\beta(U)}\, \dot{U} = \delta_{sl} \cdot \delta_{\alpha\beta} \cdot \rho_\alpha, \tag{10}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta:

$$\delta_{\alpha\beta} = \begin{cases} 0, \alpha \neq \beta, \\ 1, \alpha = \beta. \end{cases}$$

Lemma 1 is proved

Now we can calculate the constant ρ_α.

It is known that (9) is a square matrix having $\mathbb{N}_\alpha$ rows and columns (see [23]). In integral (10) we multiply both parts by $\mathbb{N}_\alpha$:

$$\mathbb{N}_\alpha \rho_\alpha = \int\limits_{\mathbb{X}^{(2)}_{m,n}} \sum_{i=1}^{\mathbb{N}_\alpha} \left| \varphi^{(i)}_\alpha(U) \right|^2 \dot{U} = \int\limits_{\mathbb{X}^{(2)}_{m,n}} Sp[A_{\alpha_1,\alpha_2,\ldots,\alpha_m}(\langle U, U \rangle)] \dot{U} =$$

$$= \int\limits_{\mathbb{X}^{(2)}_{m,n}} \chi_{\alpha_1,\alpha_2,\ldots,\alpha_m}(\langle U, U \rangle) \dot{U} = V(\mathbb{X}^{(2)}_{m,n})$$

and

$$\rho_\alpha = \frac{V(\mathbb{X}^{(2)}_{m,n})}{\mathbb{N}_\alpha},$$

where $V(\mathbb{X}^{(2)}_{m,n})$ is the volume of the matrix ball skeleton (see [25]).

The group of movements of the matrix ball $\mathbb{B}^{(2)}_{m,n}$, leaving the beginning stationary, consists of transformations of the form

$$W = U'ZV \tag{11}$$

where U and V are unitary matrices of orders m and mn, respectively. Let's arrange the elements of the matrix Z and W in the form (6).

The transformation (11) of the matrix Z into the matrix W induces some transformation of the vector z into the vector w. This transformation has the form:

$$w = z(U \otimes V),$$

here the sign $\otimes$ means the Kronecker product. Then the transformation of the vector $z^{[\alpha]}$ into the vector $w^{[\alpha]}$ consists of transformations of the form

$$w^{[\alpha]} = z^{[\alpha]}(U \otimes V)^{[\alpha]}.$$

The $z^{[\alpha]}$ subspace invariant under this transformation is split into a direct sum of subspaces with dimensions

$$q_\alpha = N(2\alpha_1, 2\alpha_2, ..., 2\alpha_{\frac{m(m+1)}{2}}) \cdot N(2\alpha_1, 2\alpha_2, ..., 2\alpha_{\frac{m(m+1)}{2}}, 0, ..., 0).$$

The components $z^{[\alpha]}$ denote by $\varphi^{(s)}_{\alpha_1,\alpha_2,...,\alpha_m}(Z), \ \ s = 1, 2, ..., q_\alpha$.

Theorem 2. *The system of functions*

$$(\rho_\alpha)^{-\frac{1}{2}} \varphi^{(s)}_\alpha(Z), \ \ s = 1, 2, ..., q_\alpha, \ \ \alpha = 0, 1, 2, ...$$

for a matrix ball $\mathbb{B}^{(2)}_{m,n}$ *is an orthonormal system, where*

$$\rho_\alpha = \int\limits_{\mathbb{B}^{(2)}_{m,n}} \left|\varphi^{(s)}_\alpha(Z)\right|^2 \dot{Z}. \tag{12}$$

Proof. If Z is converted to W by transformation (11), then $\varphi^{(s)}_{\alpha_1,\alpha_2,...,\alpha_m}(Z)$ are converted to linear combinations from $\varphi^{(s)}_{\alpha_1,\alpha_2,...,\alpha_m}(W)$ by matrix

$$A_{\alpha_1,\alpha_2,...,\alpha_{\frac{m(m+1)}{2}}}(U) \otimes A_{\alpha_1,\alpha_2,...,\alpha_{\frac{m(m+1)}{2}},0,...,0}(V), \tag{13}$$

where

$$A_{\alpha_1,\alpha_2,...,\alpha_{\frac{m(m+1)}{2}}}(U) = U^{[\alpha]}, \ \ A_{\alpha_1,\alpha_2,...,\alpha_{\frac{m(m+1)}{2}},0,...,0}(V) = V^{[\alpha]}.$$

For various $(\alpha_1, \alpha_2, ..., \alpha_m)$ representations (13) are not equivalent. Therefore,

$$\int\limits_{\mathbb{B}^{(2)}_{m,n}} \varphi^{(s)}_\alpha(Z) \cdot \overline{\varphi^{(j)}_\beta(Z)}\, \dot{Z} = \delta_{\alpha\beta} \cdot \delta_{sj} \cdot \rho_\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m), \ \ \beta = (\beta_1, \beta_2, ..., \beta_m)$, $\dot{Z} = \prod\limits_{p=1}^{m} \prod\limits_{q=1}^{mn} dx_{pq}dy_{pq}, \ \ x_{pq} + iy_{pq} = z_{pq}$, and ρ_α do not depend on i. Thus, a lot of functions

$$\{\varphi^s_\alpha(Z)\}_{s,\alpha}$$

forms an orthogonal system in the matrix ball $\mathbb{B}^{(2)}_{m,n}$. *The theorem 2 is proved.*

Now we calculate the important, for practical matters, constant ρ_α using (12). For this purpose, first of all, we will clarify the process of obtaining the functions $\varphi^s_\alpha(Z)$. The vector obtained from the matrix Z is transformed by means of the matrix (13) when Z undergoes transformation (11). For a diagonal matrix $\Lambda = [\lambda_1, \lambda_2, ..., \lambda_m]$ the matrix $A_{\alpha_1,\alpha_2,...,\alpha_m}(\Lambda)$ is also diagonal. Therefore,

$$\sum_{i=1}^{q_\alpha} \left|\varphi^{(s)}_\alpha(Z)\right|^2 = Sp[A_{\alpha_1,\alpha_2,...,\alpha_m}(\langle Z, Z\rangle)]$$

and we have

$$q_\alpha \rho_\alpha = \int\limits_{\mathbb{B}^{(2)}_{m,n}} Sp[A_{\alpha_1,\alpha_2,...,\alpha_m}(\langle Z, Z\rangle)]\dot{Z} = \int\limits_{\mathbb{B}^{(2)}_{m,n}} \chi_{\alpha_1,\alpha_2,...,\alpha_m}(\langle Z, Z\rangle)\dot{Z}.$$

it follows that

$$\rho_\alpha = \frac{V\left(\mathbb{B}^{(2)}_{m,n}\right)}{q_\alpha}$$

where $V(\mathbb{B}^{(2)}_{m,n})$ is the volume of the matrix ball skeleton (see Theorem 1).

3. Open problems

We present some unsolved problems related to matrix balls of the third type $\mathbb{B}^{(3)}_{m,n}$ and its skeletons (Shilov boundary $\mathbb{X}^{(3)}_{m,n}$):

1. Write out an orthonormal basis for $\mathbb{B}^{(3)}_{m,n}$ (the existence of such bases follows from A.Cartan's theorem on complete circular domains (see [4]).

$\mathbf{2}_{1,2.}$ Introduce the concepts of A-harmonic functions in $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$ (see [13]).

$\mathbf{3}_{1,2.}$ To obtain the criterion of holomorphic continuity into matrix balls $\mathbb{B}^{(2)}_{m,n}$ and $\mathbb{B}^{(3)}_{m,n}$ of functions defined on the skeleton part (Shilov boundaries) (see [20]).

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