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ANALYSIS OF TWO PREY AND ONE PREDATOR INTERACTION MODEL WITH DISCRETE TIME DELAY

G. SANTHOSH KUMAR[1], C. GUNASUNDARI[2,*]

[1]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, 603203, India

[2]Department of Mathematics, College of Engineering, Anna University, Chennai, 600025, India

[*]Corresponding author: gunasundari@annauniv.edu

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Abstract. Mathematical modeling is essential in both human and natural systems. In this work, we explore two prey and one predator models having Holling type I functional behaviors. We used a discrete-time delay to illustrate the permanence and boundedness of the system. We also characterized and displayed the stability and Hopf - Bifurcation for the competition model. Furthermore, we carried out the numerical simulation and experimental results to realize the impact of our model.

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1. Introduction

Mathematics plays a major role in biology with the help of a biological models. Throughout this work, we focus at the predator-prey interaction model. Relationships between two species and their consequence on one another make up the predator-prey competition. The prey species is the animal that is being consumed, and the predator is the animal that is consumed. The predator-prey correlation continues when successive decades of each species interact. By doing so, they have an influence over each other's species success and survival. As organisms evolve out of their resource masses, these types of models are potentially the fundamental components of bio and environments. Populations strive, adapt, and disperse purely to gain nourishment so that they can proceed their journey for subsistence. Depending on their individual application conditions, they can take the characteristics of resource-consumer, plant-herbivore, parasite-host, tumour cells (virus) - immune system, susceptible-infectious linkages, and so on. Because they deal with universal loss-win interactions, so that they might be

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helpful for the ecosystems. Numerous researchers work for free to research the effects of contests, including how populations evolve and disappear due to those competitions [16], [17], [18].

There are different kinds of interactions between the species during the prey-predator competition. We took two prey and one predator into the conflict with a help of Holling type I functional response [3,4]. And, also competition between prey and the predator is formulated along with discrete time delay [1]. When there is competition between prey and predator, there will never seem to be less competition between them. While engaging, the predator or prey may be affected by factors like sickness, maturation, age, and so on. At that point, the competition will get delayed to finalize the competition's outcome. There are different types of time delays, including single constant delays, discrete delays, distributed delays, state-dependent delays, and time-dependent delays among others. In our model, we choose discrete single-time delay for the analysis [7–9]. We established the model's well-posedness and demonstrated the model's local and global stability. In addition, the stability analysis has been described with and without delay.

A Hopf bifurcation is a crucial phase in a mathematical theory of bifurcations where a system's stability alters and a periodic solution develops [7]. When a pair of complex conjugate eigenvalues of the normalization more around the fixed point exceeds the complex plane imaginary axis, the fixed origin of a dynamical system seems to become unstable. Under sufficiently broad assumptions about the dynamical system, a small-amplitude limit cycle branches from the fixed point. Monitoring a branch of steady state implementations of an ODE system, which is frequently done using homotopy continuation methods, while observing the eigenvalues of the Jacobian matrix at particular points is an indirect strategy for locating Hopf bifurcation sites [13]. Griewank and Reddien (1983) published the first effective method for calculating Hopf bifurcation sites in ODE systems directly. This concept was then used in DAE systems by Reich (1995). Hopf bifurcation (also known as Poincare-Andronov-Hopf bifurcation) is the local birth or death of a periodic solution (self-excited oscillation) from the equilibrium as a parameter passes a critical point. Many researchers used the following criteria to show the principles of Hopf - bifurcation in a prey-predator competition: I non-hyperbolicity condition, (ii) transversality condition, and (iii) genericity requirement. The transversality criterion is used to prove the characteristics of Hopf bifurcation.

2. Proposed Model

Consider the following model,

$$\begin{aligned}
&(a) \quad \frac{dp_1}{dt} = \zeta_1 p_1 \left(1 - \frac{p_1}{k_1}\right) - \xi_{12} p_1 p_2 - \xi_{13} p_1 p_3, \\
&(b) \quad \frac{dp_2}{dt} = \zeta_2 p_2 \left(1 - \frac{p_2}{k_2}\right) - \xi_{21} p_1 p_2 - \xi_{23} p_2 p_3, \qquad (1) \\
&(c) \quad \frac{dp_3}{dt} = -\kappa p_3 + \xi_{31} p_1(t-\tau) p_3(t-\tau) + \xi_{32} p_2(t-\tau) p_3(t-\tau).
\end{aligned}$$

In this study, we used Holling type I functional response [15] for the competition in the above system, which included two prey and one predator contact as well as a discrete-time delay. Also, only the predator is subjected to the delay. The population stability was investigated [2]. The dynamical behavior of the three species in an ecosystem was visualized using some numerical simulations [10].

Table:1. Detailed descriptions of the system's parameters

Parameter	**Description**
p_1	Prey 1
p_2	Prey 2
p_3	Predator
ζ_1	Intrinsic growth rate of prey 1
ζ_2	Intrinsic growth rate of prey 2
k_1	Carrying capacity of prey 1
k_2	Carrying capacity of prey 2
ξ_{12}	Competition coefficient of prey 2 on 1
ξ_{21}	Competition coefficient of prey 1 on 2
ξ_{13}	Predation behavior rate on prey 1
ξ_{23}	Predation behavior rate on prey 2
ξ_{31}	Gain rate of the predator due to the predation of prey 1
ξ_{32}	Gain rate of the predator due to the predation of prey 1
τ	Time delay
κ	Natural death rate of the predator

2.1. **Positive Invariance.**

Theorem 1. Every solution of a system (1) with initial conditions $p_1(t) > 0$, $p_2(t) > 0$, $p_3(t) > 0 \; \forall$ t ≥ 0 will exist in $[0, \infty)$.

Proof. We claim that for all t $\in [0, \infty)$, $p_1(t) > 0$, $p_2(t) > 0$ and $p_3(t) > 0$. Let's prove this by contradiction. If it is not true, then one of $p_1(t) > 0$, $p_2(t) > 0$, $p_3(t) > 0$ must vanish. From the system (1) (a) with $p_1(t) > 0$,

$$p_1(t) = p_1(0)e^{\left[\int_0^t (\zeta_1(1-\frac{p_1\eta}{k_1})-\xi_{12}p_2\eta-\xi_{13}p_3(\eta))\right]} > 0,$$

Then, from (1) (b) with $p_2(t) > 0$ we get

$$p_2(t) = p_2(0)e^{\left[\int_0^t (\zeta_2(1-\frac{p_2\eta}{k_2})-\xi_{21}p_1\eta-\xi_{23}p_3(\eta))\right]} > 0,$$

Similarly, from (1) (c) with $p_3(t) > 0$

$$p_3(t) = p_3(0)e^{\left[\int_0^t(-\kappa+\frac{\xi_{31}p_1(\eta-\tau)p_3(\eta-\tau)}{p_3(\eta)}+\frac{\xi_{32}p_2(\eta-\tau)p_3(\eta-\tau)}{p_3(\eta)})\right]} > 0.$$

Here, is where the contradiction occurs. So, that $p_1(t) \geq 0$, $p_1(t) \geq 0$, $p_3(t) \geq 0 \; \forall$ t $\in [0, \infty)$. Hence, the proof. □ □

2.2. **Boundedness.**

Theorem 2. The solutions of an entire system (1) in R^3 for t≥0 are constrained.

Proof. Let

$$W'(t) = p_1'(t) + p_2'(t) + p_3'(t), \tag{2}$$

and $\Lambda > 0$ be a constant. Then,

$$W'(t) + \Lambda W = p_1'(t) + p_2'(t) + p_3'(t) + \Lambda W,$$

$$\begin{aligned} W'(t) + \Lambda W = &\zeta_1 p_1 - \frac{\zeta_1}{k_1}{p_1}^2 - \xi_{12}p_1p_2 - \xi_{13}p_1p_3 + \zeta_2 p_2 - \frac{\zeta_2}{k_2}{p_2}^2 - \xi_{21}p_1p_2 \\ &- \xi_{23}p_2p_3 - \kappa p_3 + \xi_{31}p_1(t-\tau)p_3(t-\tau) + \xi_{32}p_2(t-\tau)p_3(t-\tau) \\ &+ \Lambda(p_1+p_2+p_3), \end{aligned}$$

$$\begin{aligned} =&(\zeta_1+\Lambda)p_1 - \frac{\zeta_1}{k1}{p_1}^2 - \xi_{13}p_1p_3 + (\zeta_2+\Lambda)p_2 - \frac{\zeta_2}{k2}{p_2}^2 - \xi_{23}p_2p_3 \\ &+ (\Lambda-\kappa)p_3 + \xi_{31}p_1(t-\tau)p_3(t-\tau) + \xi_{32}p_2(t-\tau)p_3(t-\tau) \\ &- \xi_{12}p_1p_2 - \xi_{21}p_1p_2. \end{aligned}$$

If $\xi_{13} \geq \xi_{31}$, $\xi_{23} \geq \xi_{32}$,

$$\begin{aligned} \leq&(\zeta_1+\Lambda)p_1 - \frac{\zeta_1}{k1}{p_1}^2 - \xi_{13}p_1p_3 + (\zeta_2+\Lambda)p_2 - \frac{\zeta_2}{k2}{p_2}^2 - \xi_{23}p_2p_3 \\ &+ (\Lambda-\kappa)p_3 - \xi_{12}p_1p_2 - \xi_{21}p_1p_2, \end{aligned}$$

$$\begin{aligned} \leq&\frac{-\zeta_1}{k_1}\left(p_1 - k_1\frac{(\zeta_1+\Lambda)}{2\zeta_1}\right)^2 + \frac{\zeta_2}{k_2}\left(p_2 - k_2\frac{(\zeta_2+\Lambda)}{2\zeta_2}\right)^2 + (\Lambda-\kappa)p_3 \\ &- \xi_{12}p_1p_2 - \xi_{21}p_1p_2 + \frac{k_1(\zeta_1+\Lambda)^2}{6\zeta_1} + \frac{k_2(\zeta_2+\Lambda)^2}{6\zeta_2}. \end{aligned}$$

Let q=min(ξ_{12}, ξ_{21}),

$$\begin{aligned} \leq&\frac{-\zeta_1}{k_1}\left(p_1 - k_1\frac{(\zeta_1+\Lambda)}{2\zeta_1}\right)^2 + \frac{\zeta_2}{k_2}\left(p_2 - k_2\frac{(\zeta_2+\Lambda)}{2\zeta_2}\right)^2 + (\Lambda-\kappa)p_3 \\ &- 2qp_1p_2 + k_1\frac{(\zeta_1+\Lambda)^2}{6\zeta_1} + k_2\frac{(\zeta_1+\Lambda)^2}{6\zeta_2}, \\ \leq&\frac{(\zeta_1+\Lambda)^2}{6\zeta_1} + k_2\frac{(\zeta_1+\Lambda)^2}{6\zeta_2} = \mu. \end{aligned}$$

We use the inequality to conclude W′(t)+Λ W=μ and hence W=$\frac{\mu}{\kappa}$ + $ce^{-\kappa}$. By t=0, we get $W(p_1(0),p_2(0))$=$\frac{\mu}{\kappa}$ + c and then c=$W(p_1(0),p_2(0))$-$\frac{\mu}{\kappa}$. Hence, $W(p_1(t),p_2(t))$=$\frac{\mu}{\kappa}(1-e^{-\kappa t})$+$(W(p_1(0),p_2(0)))e^{-\kappa t}$ and where $0 < W(p_1(t),p_2(t)) \leq \frac{\mu}{\kappa}((1-e^{-\kappa t}))$+$(W(p_1(0),p_2(0)))e^{-\kappa t}$. By taking the limit $t \to \infty$ we have $0 < W(t) \leq \frac{\mu}{\kappa}$. Hence the theorem.

□ □

3. Analysis of the Model Without Delay

3.1. **Local stability.** At the interior equilibrium point, the nonlinear matrix of the model (1)(a)-(1)(c) with $\tau = 0$ can be written in the form of a Jacobian matrix, $J(p_1,p_2,p_3)$=

$$\begin{pmatrix} \zeta_1 - \frac{2\zeta_1 p_1}{k_1} - \xi_{12}p_2 - \xi_{13}p_3 & \xi_{12}p_1 & \xi_{13}p_1 \\ \xi_{21}p_2 & \zeta_2 - \frac{2\zeta_2 p_2}{k_2} - \xi_{21}p_1 - \xi_{23}p_3 & -\xi_{23}p_2 \\ \xi_{31}p_3 & \xi_{32}p_3 & -\kappa + \xi_{31}p_1 + \xi_{32}p_2 \end{pmatrix}.$$

The characteristic equation can be obtained as,

$$\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0, \tag{3}$$

where, $\sigma_1 = (\frac{2\zeta_1}{k_1} + \xi_{21} - \xi_{31})p_1 + (\frac{2\zeta_2}{k_2} + \xi_{12} - \xi_{32})p_2 + \kappa + (\xi_{13} + \xi_{23})p_3 - \zeta_1 - \zeta_2$,

$$\sigma_2 = {R_i}^*{R_j}^* + {R_j}^*{R_h}^* + {R_i}^*{R_h}^* - \xi_{12}\xi_{21}p_1p_2 + \xi_{23}\xi_{32}p_2p_3 + \xi_{13}\xi_{31}p_1p_3.$$

Also that,

$${R_i}^* = \zeta_1 - \frac{2\zeta_1 p_1}{k_1} - \xi_{12}p_2 - \xi_{13}p_3,$$

$${R_j}^* = \zeta_2 - \frac{2\zeta_2 p_2}{k_2} - \xi_{21}p_1 - \xi_{23}p_3,$$

$${R_h}^* = -\kappa + \xi_{31}p_1 + \xi_{32}p_2,$$

$$\sigma_3 = ({R_j}^*{R_h}^* + \xi_{23}\xi_{32}p_2p_3){R_i}^* + (\xi_{12}\xi_{31}\xi_{23} + \xi_{13}\xi_{21}\xi_{32})p_1p_2p_3 + (\xi_{13}\xi_{31}{R_j}^*p_3 - \xi_{12}\xi_{21}p_2{R_h}^*)p_1.$$

By Routh Hurwitz Criterion, the system is locally asymptotically stable, if $\sigma_1 > 0$, $\sigma_3 > 0$ and $\sigma_1\sigma_2 - \sigma_3 > 0$ are satisfied.

3.2. **Global stability:** Let us consider the following Lyapunov function [11] for proving global asymptotic stable behavior.

$$X(p_1,p_2) = p_1 - {p_1}^* - {p_1}^* log\left(\frac{p_1}{{p_1}^*}\right) + l_1\left[p_2 - {p_2}^* - {p_2}^* log\left(\frac{p_2}{{p_2}^*}\right)\right] + l_2\left[p_3 - {p_3}^* - {p_3}^* log\left(\frac{p_3}{{p_3}^*}\right)\right],$$
$$= p_1 - {p_1}^* - {p_1}^* log p_1 + {p_1}^* log {p_1}^* + l_1[p_2 - {p_2}^* - {p_2}^* log p_2 + {p_2}^* log {p_2}^*]$$
$$+ l_2[p_3 - {p_3}^* - {p_3}^* log p_3 + {p_3}^* log {p_3}^*],$$

$$\frac{dX}{dt} = \frac{\partial X}{\partial p_1}\frac{dp_1}{dt} + \frac{\partial X}{\partial p_2}\frac{dp_2}{dt} + \frac{\partial X}{\partial p_3}\frac{dp_3}{dt}, \tag{4}$$

$$\begin{aligned}\frac{dX}{dt} =& \frac{p_1 - {p_1}^*}{p_1}\left[\zeta_1 p_1\left(1 - \frac{p_1}{k_1}\right) - \xi_{12}p_1p_2 - \xi_{13}p_1p_3\right] + l_1\frac{(p_2 - {p_2}^*)}{p_2} \\ & [\zeta_2 p_2\left(1 - \frac{p_2}{k_2}\right) - \xi_{21}p_1p_2\xi_{23}p_2p_3] + l_2\frac{(p_3 - {p_3}^*)}{p_3}[-\kappa p_3 + \xi_{31}p_1p_3] \\ & + l_2\frac{(p_3 - {p_3}^*)}{p_3}[\xi_{32}p_2p_3],\end{aligned} \tag{5}$$

$$\begin{aligned}=&(p_1 - {p_1}^*)\left[\zeta_1 - \frac{\zeta_1}{k_1}p_1 - \xi_{12}p_2 - \xi_{13}p_3\right] + l_1(p_2 - {p_2}^*)\left[\zeta_2 - \frac{\zeta_2}{k_2}p_2\right] \\ & - l_1(p_2 - {p_2}^*)[\xi_{21}p_1 - \xi_{23}p_3] + l_2(p_3 - {p_3}^*)[-\kappa + \xi_{31}p_1 + \xi_{32}p_2], \\ =& \frac{-\zeta_1}{k_1}(p_1 - {p_1}^*)^2 - \frac{\zeta_2}{k_2}l_1(p_2 - p_2^*) - l_2(p_3 - {p_3}^*)[\xi_{31}(p_1 - {p_1}^*) + \xi_{32}({p_2}^* - p_2)].\end{aligned}$$

If $l_1 = \frac{\xi_{12}\xi_{32}}{\xi_{21}\xi_{23}}$ and $l_2 = \frac{\xi_{13}\xi_{23}}{\xi_{31}\xi_{32}}$ then,

$$\begin{aligned}\frac{dX}{dt} =& \frac{-\zeta_1}{k_1}(p_1 - {p_1}^*)^2 - \frac{\zeta_2}{k_2}\frac{\xi_{12}\xi_{32}}{\xi_{21}\xi_{23}}(p_2 - {p_2}^*) - \frac{\xi_{13}\xi_{23}}{\xi_{31}\xi_{32}}(p_3 - {p_3}^*)[\xi_{31}({p_1}^* - p_1) \\ & + \xi_{32}({p_2}^* - p_2)] < 0.\end{aligned} \tag{6}$$

Therefore, the system is globally asymptotically stable near $E^*({p_1}^*, {p_2}^*, {p_3}^*)$.

4. Analysis of the Model With Delay

At the interior equilibrium point E^*, the delayed model's characteristic equation (1)(a)-(1)(c) generally means,

$$\lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 + e^{-\lambda\tau}(\Upsilon_1\lambda^2 + \Upsilon_2\lambda + \Upsilon_3), \tag{7}$$

$$f(\lambda) + e^{-\lambda\tau}\Upsilon(\lambda) = 0, \tag{8}$$

where, $f(\lambda) = \lambda^3 + f_1\lambda^2 + f_2\lambda + f_3$ and $\Upsilon(\lambda) = \Upsilon_1\lambda^2 + \Upsilon_2\lambda + \Upsilon_3$,

$f_1 = \frac{2\zeta_1 p_1}{k_1} + \xi_{12}p_2 + \xi_{13}p_3 - \zeta_1 - \zeta_2 + \frac{2\zeta_2 p_2}{k_2} + \xi_{21}p_1 + \xi_{23}p_3 + \kappa$,

$$f_2 = \zeta_1\zeta_2 - 2\zeta_1\tfrac{\zeta_2}{k_2}p_2 - \zeta_1 p_1\xi_{21} - \zeta_1 p_3\xi_{23} - \zeta_1\kappa - 2\tfrac{\zeta_1}{k_1}\zeta_2 p_1 + 4\tfrac{\zeta_1}{k_1}\tfrac{\zeta_2}{k_2}p_1p_2 \\ + 2\tfrac{\zeta_1}{k_1}{p_1}^2\xi_{21} + 2\tfrac{\zeta_1}{k_1}p_1p_3\xi_{23} + 2\tfrac{\zeta_1}{k_1}p_1\kappa - \xi_{12}p_2\zeta_2 + 2\xi_{12}{p_2}^2\tfrac{\zeta_2}{k_2} + \xi_{12}\xi_{21}p_1p_2 \\ + \xi_{12}\xi_{23}p_2p_3 + \xi_{12}p_2\kappa - \xi_{13}\zeta_2p_3 + 2\xi_{13}p_2p_3\tfrac{\zeta_2}{k_2} + \xi_{13}\xi_{21}p_1p_3 + \xi_{13}\xi_{23}{p_3}^2 \\ + \xi_{13}p_3\kappa - \zeta_2\kappa + 2\tfrac{\zeta_2}{k_2}\kappa p_2 + \xi_{21}p_1\kappa + \xi_{23}p_3\kappa - \xi_{12}\xi_{21}p_1p_2,$$

$$f_3 = \zeta_1\zeta_2\kappa - 2\zeta_1\tfrac{\zeta_2}{k_2}p_2\kappa - \zeta_1p_1\xi_{21}\kappa - p_1p_3\xi_{23}\kappa - 2\tfrac{\zeta_1}{k_1}\zeta_2p_1\kappa + 4p_1p_2\tfrac{\zeta_1}{k_1}\tfrac{\zeta_2}{k_2} \\ + 2\tfrac{\zeta_1}{k_1}{p_1}^2\xi_{21}\kappa + 2\tfrac{\zeta_1}{k_1}p_1p_3\xi_{23}\kappa - \xi_{12}p_2\zeta_2\kappa + 2\xi_{12}{p_2}^2\tfrac{\zeta_2}{k_2}\kappa + \xi_{12}\xi_{21}p_2p_1\kappa \\ + \xi_{12}\xi_{23}p_2p_3\kappa - \xi_{13}\zeta_2p_3\kappa + 2\xi_{13}p_2p_3\tfrac{\zeta_2}{k_2}\kappa + \xi_{13}\xi_{21}p_1p_3\kappa + \xi_{13}\xi_{23}{p_3}^2\kappa \\ - \xi_{12}\xi_{21}p_1p_2\kappa,$$

$$\Upsilon_1 = -(\xi_{31} + \xi_{32}),$$

$$\Upsilon_2 = \zeta_1\xi_{31} + \zeta_1\xi_{32} - 2\tfrac{\zeta_1}{k_1}p_1\xi_{31} - 2\tfrac{zeta_1}{k_1}p_1\xi_{32} - \xi_{12}\xi_{31} - p_2 - \xi_{12}\xi_{32}p_2 - \xi_{13}\xi_{31}p_3 \\ - \xi_{13}\xi_{32}p_3 + \zeta_2\xi_{31} + \zeta_2\xi_{32} - 2\tfrac{\zeta_2}{k_2}p_2\xi_{31} - 2\tfrac{\zeta_2}{k_2}p_2\xi_{32} - \xi_{23}\xi_{32}p_3 + \xi_{23}\xi_{32}p_2 \\ + \xi_{13}\xi_{31}p_1,$$

$$\Upsilon_3 = 2\zeta_1\tfrac{\zeta_2}{k_2}p_2\xi_{31} + 2\zeta_1\tfrac{\zeta_2}{k_2}p_2\xi_{32} - \zeta_1\zeta_2\xi_{31} - \zeta_1\zeta_2\xi_{32} + \zeta_1p_1\xi_{21}\xi_{31} + \zeta_1p_1\xi_{21}\xi_{32} \\ + \zeta_1p_3\xi_{23}\xi_{31} + \zeta_1p_3\xi_{23}\xi_{32} - \zeta_1p_2\xi_{23}\xi_{32} + 2\tfrac{\zeta_1}{k_1}\zeta_2p_1\xi_{31} + 2\tfrac{\zeta_1}{k_1}\zeta_2p_1\xi_{32} \\ - 4\tfrac{\zeta_1}{k_1}\tfrac{\zeta_2}{k_2}p_1p_2\xi_{31} - 4\tfrac{\zeta_1}{k_1}\tfrac{\zeta_2}{k_2}p_1p_2\xi_{32} - 2\tfrac{\zeta_1}{k_1}{p_1}^2\xi_{21}\xi_{31} - 2\tfrac{\zeta_1}{k_1}{p_1}^2\xi_{21}\xi_{32} - 2\tfrac{\zeta_1}{k_1}p_1p_3\xi_{23}\xi_{31} \\ - 2\tfrac{\zeta_1}{k_1}p_1p_3\xi_{23}\xi_{32} + 2\tfrac{\zeta_1}{k_1}p_1p_2\xi_{23}\xi_{32} + p_2\zeta_2\xi_{12}\xi_{31} + p_2\zeta_2\xi_{12}\xi_{32} - 2{p_2}^2\tfrac{\zeta_2}{k_2}\xi_{12}\xi_{31} \\ - 2{p_2}^2\tfrac{\zeta_2}{k_2}\xi_{12}\xi_{32} - p_1p_2\xi_{12}\xi_{21}\xi_{31} - \xi_{12}\xi_{21}\xi_{32}p_1p_2 - \xi_{12}\xi_{23}\xi_{31}p_2p_3 - \xi_{12}\xi_{23}\xi_{32}p_2p_3 \\ + \xi_{12}\xi_{23}\xi_{32}{p_2}^2 + \xi_{13}\xi_{31}\zeta_2p_3 + \xi_{13}\xi_{32}\zeta_2p_3 - 2p_2p_3\tfrac{\zeta_2}{k_2}\xi_{13}\xi_{31} - 2p_2p_3\tfrac{\zeta_2}{k_2}\xi_{13}\xi_{32} \\ - \xi_{13}\xi_{21}\xi_{31}p_1p_3 - \xi_{13}\xi_{21}\xi_{32}p_1p_3 - \xi_{13}\xi_{23}\xi_{31}{p_3}^2 - \xi_{13}\xi_{23}\xi_{32}{p_3}^2 + \xi_{13}\xi_{23}\xi_{32}p_2 \\ + \xi_{12}\xi_{21}\xi_{31}p_1p_2 + \xi_{12}\xi_{21}\xi_{32}p_1p_2 - \xi_{12}\xi_{31}\xi_{23}p_1p_2 - \xi_{13}\xi_{21}\xi_{32}p_1p_2 - \xi_{13}\xi_{31}p_1\zeta_2 \\ + 2p_1p_2\tfrac{\zeta_2}{k_2}\xi_{12}\xi_{31} + \xi_{13}\xi_{31}\xi_{21}{p_1}^2 + \xi_{13}\xi_{31}\xi_{23}p_1p_3.$$

Let us consider $\lambda = i\omega$ be a root of the system (8) where ω is a real number. Substitute λ in (8) to separate the real and imaginary portions, and we get,

$$f_3 - \omega^2 f_1 = (\omega^2\Upsilon_1 - \Upsilon_3)cos\omega\tau - \omega\Upsilon_2 sin\omega\tau, \tag{9}$$

$$f_2\omega - \omega^3 = (\Upsilon_3 - \Upsilon_1\omega^2)sin\omega\tau - \omega\Upsilon_2 cos\omega\tau. \tag{10}$$

Squaring and adding (9) and (10), we obtain

$$\omega^6 + \omega^4 N_1 + \omega^2 N_2 + N_3 = 0, \tag{11}$$

where, $N_1 = {f_1}^2 - 2f_2 - {\omega_1}^2 > 0$,

$N_2 = {f_2}^2 - 2f_3f_1 + 2\Upsilon_1\Upsilon_3 - {\Upsilon_2}^2$,

$N_3 = {f_3}^2 - {\Upsilon_3}^2$.

By Descartes's rule, if $N_3 < 0$, then there exists a unique positive root ${\omega_0}^2$ and then (7) has a pair of imaginary roots $\pm i\omega_0$.
From (9) and (10), we obtain

$$cos\omega\tau = \frac{\omega\Upsilon_2(\omega^3 - \omega f_2) - (f_3 - \omega^2 f_1)(\Upsilon_3 - \Upsilon_1\omega^2)}{(\Upsilon_3 - \omega^2\Upsilon_1)^2 + (\omega\Upsilon_2)^2}. \tag{12}$$

Then τ_k, corresponding to $\omega = \omega_0$ is given by,

$$\tau_k = \frac{1}{\omega_0}cos^{-1}\left[\frac{\omega\Upsilon_2(\omega^3 - \omega f_2) - (f_3 - \omega^2 f_1)(\Upsilon_3 - \Upsilon_1\omega^2)}{(\Upsilon_3 - \omega^2\Upsilon_1)^2 + (\omega\Upsilon_2)^2}\right] + \frac{2k\pi}{\omega_0}, k = 0, 1, 2, ... \tag{13}$$

The model (1)(a) - (2)(c) is stable around E* for $\tau < \tau_0$ according to Buttler's lemma. Now, we differentiate (8) for τ,

$$f'(\lambda)\frac{d\lambda}{dt} + e^{-\lambda\tau}\Upsilon'(\lambda)\frac{d\lambda}{dt} + \Upsilon(\lambda)e^{-\lambda\tau}(-\lambda - \tau\frac{d\lambda}{d\tau}) = 0. \tag{14}$$

Hence,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{f'(\lambda)}{-\lambda f(\lambda)} + \frac{\Upsilon'(\lambda)}{\lambda\Upsilon(\lambda)} - \frac{\tau}{\lambda},$$

$$\begin{aligned}\left[\left(\frac{d\lambda}{dt}\right)^{-1}\right]_{\lambda=i\omega_0} =& \frac{{\omega_0}^2(3{\omega_0}^4 + (2{f_1}^2 - 4f_2){\omega_0}^2 + {f_2}^2 - 2f_1f_3)}{({\omega_0}^4 - f_2{\omega_0}^2)^2 + (f_3\omega_0 - f_1{\omega_0}^3)^2} \\ &- \frac{2\Upsilon_1\Upsilon_3{\omega_0}^2 - 2{\Upsilon_1}^2{\omega_0}^4 - {\Upsilon_2}^2{\omega_0}^2}{(\Upsilon_2{\omega_0}^2)^2 + (\Upsilon_3\omega_0 - \Upsilon_1{\omega_0}^3)^2} - \frac{\tau}{i\omega_0}.\end{aligned} \tag{15}$$

Using (11) we obtain,

$$\begin{aligned}Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} =& \frac{3{\omega_0}^6 - 2{\Upsilon_1}^2{\omega_0}^4 + (2{f_1}^2 + 2\Upsilon_1\Upsilon_3 - {\Upsilon_2}^2 - 4f_2){\omega_0}^2}{\psi^2} \\ &+ \frac{{f_2}^2 - 2f_1f_3}{\psi^2},\end{aligned} \tag{16}$$

where, $\psi^2 = ({\omega_0}^4 - f_2{\omega_0}^2)^2 + (f_3\omega_0 - f_1{\omega_0}^3)^2 = (\Upsilon_2{\omega_0}^2)^2 + (\Upsilon_3\omega_0 - \Upsilon_1{\omega_0}^3)^2$ If $2{\Upsilon_1}^2{\omega_0}^4 < 0$ then $Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\lambda=i\omega_0} > 0$. Hence, $\frac{d}{d\tau}(Re(\lambda)) > 0$. As a result, the Hopf-bifurcation condition is satisfied, and the system exhibits periodic oscillations at $t > 0$.

5. Numerical Simulation

By selecting suitable sets of parameter values, we proved the results numerically. We present the simulations in terms of stability using Mathematica software as follows.

Figure 1 (A) depicts the coexistence of all populations with $\tau = 0$, $\zeta_1 = 2$, $\xi_{12} = 0.3$, $\zeta_2 = 0.5$, $\xi_{23} = 0.3$, $\xi_{31} = 3$ and $\xi_{32} = 1.6$. When the time delay increases then a periodic oscillation occurs between the population of the species as shown in Figures 1 (B), (C), and (D).

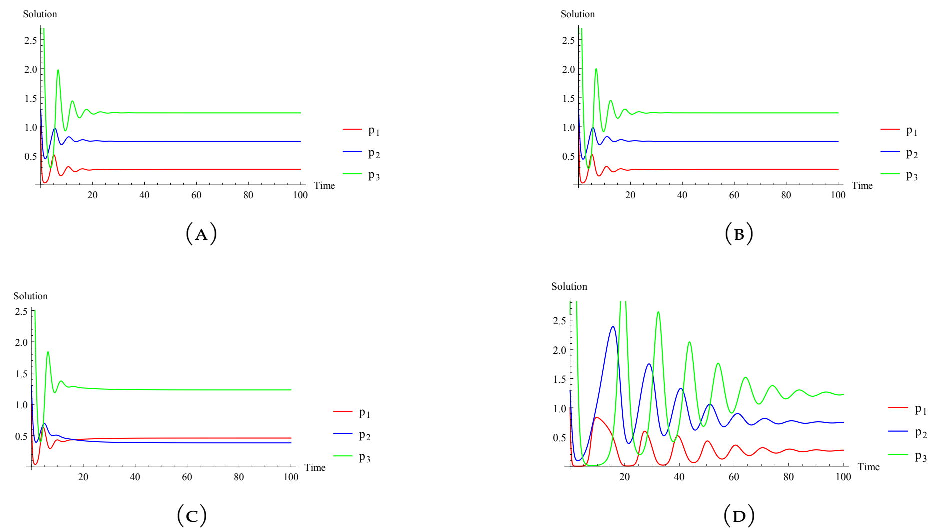

Figure 1. The coexistence and time series assessment of the deterministic system of all three populations at (A) $\zeta_1 = 2, \zeta_2 = 0.5$, $\tau = 0$ (B) $\zeta_1 = 2, \zeta_2 = 0.5$, $\tau = 0.01$ (C) $\zeta_1 = 2, \zeta_2 = 0.5$, $\tau = 0.05$ and (D) $\zeta_1 = 2, \zeta_2 = 0.5$, $\tau = 1$.

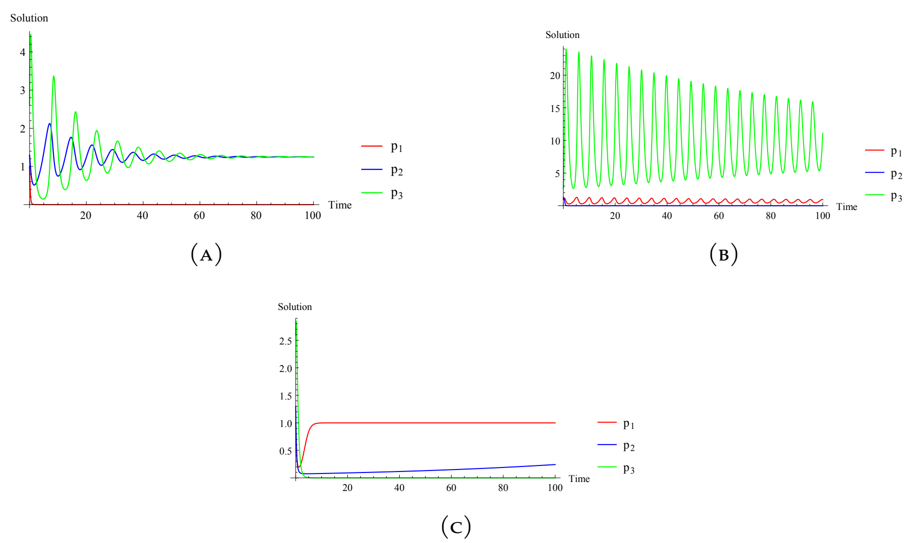

Figure 2. The time series evaluation of two populations when $\tau = 0.01$ at (A) $\xi_{23} = 1, \xi_{31} = 1, \xi_{32} = 1.6$ (B) $\xi_{23} = 0.3, \xi_{31} = 0.003, \xi_{32} = 1.6$ and (C) $\xi_{23} = 0.3, \xi_{31} = 3, \xi_{32} = 1.6$

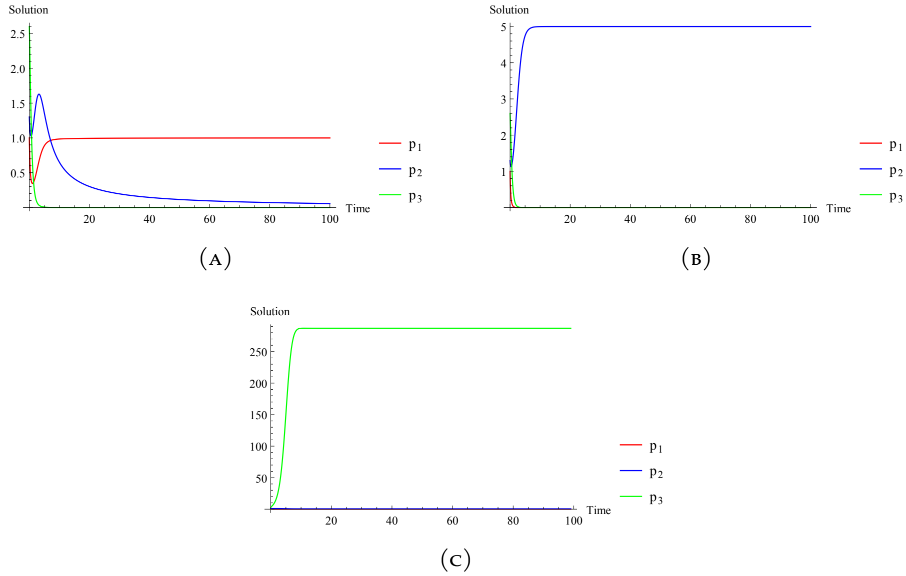

FIGURE 3. Persistence of single species when $\tau = 0.01$ at (A) $\xi_{12} = 0.3, \xi_{21} = 0.2, \xi_{32} = 1.6, \kappa = 0.001$ (B) and (C) $\xi_{12} = 2.9, \xi_{21} = 1.09, \xi_{32} = 2.05, \kappa = 1$

Figure 2 (A) shows that only the second prey and predator populations are alive, whereas the first prey population has gone extinct. When the value of the competition coefficient decreases, a wide oscillation occurs for the predator population which is shown in Figure 2 (B). Here the first prey population shows no fluctuation whereas the second prey population got extinct. Similarly, the two prey populations exist in Figure 2 (C), but the predator population has vanished.

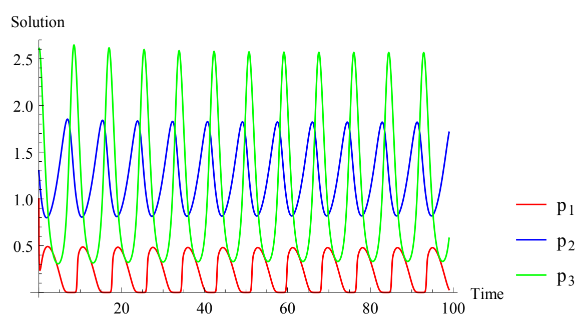

FIGURE 4. Oscillatory behavior of three populations with $\tau = 0.01$ at $\xi_{21} = 0.2, \xi_{23} = 0.3, \xi_{31} = 0.003, \xi_{32} = 1.6$

Only the second prey species persist, in Figure 3 (A) whereas the other two population increase cease to exist when the competition coefficient gets increases. The wave pattern emerged purely for the predator population in Figure 3 (B), whereas the other two prey species became extinct [12]. Similarly, only the first prey lives in Figure 3 (C), while the others become extinct. In particular, the second prey population persists for a short period $(t > 0)$ before dissipating.

The periodic solution has been obtained over all three populations in Figure 4. The wave amplitudes of the predator population are greater than those of the other two prey populations.

6. Conclusion

The Lotka-Volterra predator-prey model is a key model of the ecosystem. It assumes that the predator has only one prey and vice versa [5,6]. It also indicates that no other variables, such as illness, changing weather, pollution, and so on, are present. However, many research articles have been published recently that have produced a variety of prey-predator models with a variety of functional responses. Many researchers formulated the predator-prey model with disturbances. In this work, we analyzed the interaction of two prey and one predator in an ecosystem with a discrete-time delay and a Holling type I functional response. We analyzed the system's well-posedness, such as positive invariance and boundedness. The stability analysis examined both has been locally and globally, with and without time delay. Descartes' rule and Buttler's lemma are also used to describe and prove the characteristics of Hopf - bifurcation [14]. Finally, numerical simulations were carried out to see how the population of the species that participated in the competition changed dramatically.

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