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ADVANCED HAAR WAVELET METHOD WITH ITS CONVERGENCE TO THE BRATU'S TYPE NUMERICAL BOUNDARY VALUE PROBLEMS

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Abstract. The purpose of this study is to present a robust and efficient method based on the Haar wavelet. Without any linearization, perturbation, or discretization, a numerical solution to Bratu's-type boundary value problems has been presented. Convergence and error analysis are addressed to ensure the completeness of the Haar wavelet method. Some Bratu-type problem are addressed in order to test the validity, reliability, and generality of the present method. Analysis and results of Haar solutions are contrasted with exact solution and results from other existing methods.

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1. Introduction

In this research, we describe an efficient and uncomplicated method for numerically solving Bratu's type boundary value problems (BVPs). In 1914, G. Bratu introduced and made a point of highlighting the Bratu's dilemma. [1]. This problem is also called "Liouville-Bratu" problem since it is highly regarded by mathematicians Liouville. [2,3]. consider the Bratu's problem as

$$y''(\mu) + \lambda e^{y(\mu)} = 0,\ 0 < \mu < 1, \tag{1}$$

subject to the conditions

$$y(0) = 0,\ y(1) = 0. \tag{2}$$

For parameter $\lambda > 0$, analytic solution of Bratu's problem is

$$y(\mu) = -2\,ln\left[\frac{cosh(\frac{\theta}{2})(\mu-\frac{1}{2})}{cosh(\frac{\theta}{4})}\right], \tag{3}$$

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θ satisfied the condition

$$\theta = \sqrt{2\lambda}\, cosh\left(\frac{\theta}{\sqrt{16}}\right). \tag{4}$$

Equation (1) has zero solution if $\lambda \geqslant \lambda_c$, one solution $\lambda = \lambda_c$ and two solutions if $\lambda \leqslant \lambda_c$, value λ_c satisfies ,

$$1 = \frac{1}{\sqrt{16}}\sqrt{2\lambda_c}\, sinh\left(\frac{\theta}{\sqrt{16}}\right), \tag{5}$$

and $\lambda_c = 3.513830719$.

In real-world applications, Bratu's problem appears in many models as medical experiments, combustion theory, biotechnology, model of expansion of the universe [4–6].

Numerous scholars have utilised Bratu's issue (1) to assess the precision of their numerical techniques as follows: G. Hariharan et al. [7] applied Chebyshev wavelets method, Author [8] used the wavelet method of Legendre type. Decomposition method (DM) [9–11,13], Optimal homotopy asymptotic method [29], Runge-Kutta method [14], Hybrid method [30], Perturbation method [27,28], Operational matrix method [26] have used to evaluate Bratu's problem (1).

Here, we establish a wavelet technique which is mainly based on Haar wavelet scheme, to obtain the Haar solution of BVPs of Bratu's kind.

In order to describe physical issues that occur in the actual world, differential equations must be solved analytically and numerically. In Haar wavelet method, solution y and nonlinear term N(y) write in infinite series as: If

$$y = \sum_{n=0}^{\infty} y_i, \tag{6}$$

term N(y) expressed in the following series

$$N(y) = N\left(\sum_{i=0}^{\infty} y_i\right) = \sum_{i=0}^{\infty} a_i, \tag{7}$$

where $a_i = a_i(y_0, y_1, \cdots, y_i)$ are Haar coefficients and written as

$$a_i = \frac{1}{i!}\frac{d^i}{d\mu^i}\left[N\left(\sum_{k=0}^{\infty} \mu^k i_k\right)\right]_{\mu=0}, \; n > 0 \tag{8}$$

μ is a real constant.

It is important to note that these expression requires a very time-consuming and laborious technique to handle the Haar functions. Here the computational work has been carried out by using the programmes like Matlab, Python.

Algorithm:

For $n > 1$,

$$c_i^1 = y_i, \tag{9}$$

For $2 > j > i$,

$$c_n^j = \frac{1}{i}\sum_{k=0}^{i-j}(k+1)y_{k+1}c_{i-1-k}^{j-1}. \tag{10}$$

Then a_i are written as

$$a_0 = N(y_0),\ and\ a_i = \sum_{j=1}^{i} N^k(y_0)\ c_i^j,\ for\ n > 1. \tag{11}$$

This paper's remainder is structured as follows: Multi Resolution analysis is given in Section 2. Haar wavelet and approximation of Haar function is given in Section 3. The Haar wavelet method for resolving BVPs of the Bratu type is covered in Section 3 and Section 4. The suggested technique's convergence analysis is shown in Section 5. Results and discussion for Bratu's type BVPs subjected to the wavelet approach for various values of λ. is given in Section 6. In the end, conclusions of the study are summarized in section 7.

2. Multi Resolution Analysis (MRA)

The multi-resolution analysis (MRA) is described to understand the Haar wavelet as an orthonormal wavelet. MRA is the best technique to learn deeply about wavelets. Given a real-valued square-integrable function $y(x) \in L^2(\mathbb{R})$, then multi-resolution analysis of y(x) contributes to the cause of generating a sequence of subspace W_k, W_{k+1}, W_{k+2}..... so that the projections of square integrable function towards $L^2(\mathbb{R})$ provide improved approximations of function y(x) as $k \to \infty$. The function y(x) can be approximated at various resolution levels, which correspond to different translations for the subspace ....,W_{k-1}, W_k, W_{k+1}......

To understand multi-resolution analysis (MRA) mathematically, an MRA consists of a family of increasing closed subspaces $W_k \subset L^2(\mathbb{R}), k \in \mathbb{Z}$ together with the following group of properties:

(I) ....$W_{-2} \subset W_{-1} \subset W_0 \subset W_1 \subset W_2 \subset$ ....

(II) $\bigcup_{k=\infty}^{\infty} W_k$ is dence in $L^2(\mathbb{R})$ i.e. closure $|\bigcup_{k=-\infty}^{\infty} W_k| = L^2(\mathbb{R})$ and $\bigcap_{i=\infty}^{\infty} W_k = \{0\}$.

(III) $f(x) \in W_k \iff f(2x) \in W_{k+1},\ k \in \mathbb{Z}$.

(IV) $f(x) \in W_k \iff f(x-k) \in W_k,\ k \in \mathbb{Z}$.

(V) there is a function $\zeta_{j,k} \in W_k$ such that $\zeta_{j,k} = 2^{\frac{j}{2}}\zeta(2^j\ x - k),\ j,k \in \mathbb{Z}$

(VI) $\exists\, \zeta$ in W_0 such that $\zeta(x-k), k \in \mathbb{Z}$ form an orthogonal set of basis for W_0.

where k is the resolution level, and this integer index set $k \in \mathbb{Z}$ is connected with resolution levels, and W_k denotes approximation spaces of level k. For each W_k, there exists a complement in W_{k+1}.

In the formation of a wavelet function, $\zeta(x)$, a collection of nested sequences $< W_k >$ has been derived that satisfies the properties mentioned above. That is, for the existence of the wavelet function $\zeta(x)$ which yields a multi-resolution analysis (MRA), then these properties are essential. Here, $\zeta(x) \in L^2(\mathbb{R})$

represents the scaling function. Then:

$$W_k = clos_{L^2(\mathbb{R})}(\zeta_{j,k} : k \in \mathbb{Z}),\ j,k \in \mathbb{Z} \tag{12}$$

scaling function

$$\zeta_{j,k}(x) = 2^{\frac{j}{2}}\zeta(2^j\ x - k),\ j,k \in \mathbb{Z} \tag{13}$$

defined in subspace W_k of $L^2(\mathbb{R})$, $\zeta(x-k), k \in \mathbb{Z}$ satisfies the above properties and $\zeta(x-k), k \in \mathbb{Z}$ forms one of the orthonormal sets of basis that consists of a characteristic of linear independence. Hence, the function defined in equation (13) is the collection of functions ($\zeta_{j,k}(x) : j,k \in \mathbb{Z}$) which will forms an orthogonal basis for the space W_k or $L^2(\mathbb{R})$, which is called an orthogonal basis with mother wavelet ζ (see Goswami et al., [16] and Mallat et al., [17]). For interested readers to have a better understanding, Debnath et al., [18] consulted the notion of a monograph

MRA is a tool for developing wavelet theory using translation and scaling features. The basis which is formed by the set of functions for the space. The basis and operational matrices of integration are discussed in the following section.

3. Haar Wavelet and Haar Operational Matrices of Integration

In 1910, Hungarian mathematician Alfred Haar (Lepik at.al., [19]; Chen, Hsiao et.al., [20]) established a well-known wavelet which is based on the functions named the Haar wavelet. A Haar wavelet is one of the classes of wavelets, and they are step functions piece-wise constant functions that have only finitely many pieces. Each piece carries only three constant values $1, -1$ and zero on the real line. The Haar wavelet is an uneven rectangular pulse pair and has the simplest orthonormal series together with the property of compact support in the interval $[0, 1]$. This wavelet is very well localized in the time domain but it is not continuous. Due to mathematical simplicity Chen et. al., [21]; Lepik, [22] discussed the Haar wavelet method and how it has become an effective method for solving many differential equation as well as integral equations arising in the modelling of numerous scientific physical issues. The Haar wavelet family over the given interval $x \in [0, 1]$ consists the following functions :

$$H_i(x) = \begin{cases} 1, & \text{if } \alpha \leqslant x < \beta \\ -1, & \text{if } \beta \leqslant x < \gamma \\ 0, & \text{elsewhere} \end{cases} \tag{14}$$

where:

$$\alpha = \frac{k}{m},\ \beta = \frac{k+\frac{1}{2}}{m},\ \gamma = \frac{k+1}{m}$$

where, $m = 2^j$, $j = 0$ and $1, 2, ..., J$ and $k = 0$ and $1, 2, ..., .m - 1$, j denotes the level of wavelet, k represent the translation parameter, J be the maximum resolution level.

The index number for the Haar function H_i in equation (14) is computed by the formula $i = m + k + 1$.

In the case of a minimum index of $i = 1$ for the Haar function, select $m = 1$, $k = 0$. The index number i can reach the maximum value of $i = 2M = 2^{j+1}$ when including all levels of wavelet.
Function $H_1(x)$ is the scaling function for $i = 1$ for the family of wavelet

$$H_1(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < 1 \\ 0, & \text{elsewhere} \end{cases} \tag{15}$$

for $i = 2$, the function $H_2(x)$ is the mother wavelet and represented as:

$$H_2(x) = \begin{cases} 1, & \text{if } 0 \leqslant x < \frac{1}{2} \\ -1, & \text{if } \frac{1}{2} \leqslant x < 1 \\ 0, & \text{elsewhere} \end{cases} \tag{16}$$

In addition, $H_2(x)$ is the fundamental square wave or the mother Haar wavelet which spans the whole interval $(0, 1)$. Akmal et.al., [23]; raza et.al., [24] discussed all other subsequent curves that are generated from Haar wavelet function $H_i(x)$ with two operations translation and dilation. In particular, Haar wavelets are orthogonal functions on $[0, 1]$.
In addition, the Haar operational matrix integration formula is presented for determining the solution of Bratu's problem numerically.
The operational matrix $P_{i,\xi}$ of order $m \times m$ is derived from integration of Haar wavelet family with the help of following integral as:

$$P_{i,\xi}(x) = \int_A^x \int_A^x ...... \int_A^x H_i(t)dt^{\xi} = \frac{1}{\xi - 1} \int_A^x (x-t)^{\xi-1} H_i(t)\, dt. \tag{17}$$

where,

$\xi = 1, 2, ..., $ and $i = 1, 2, ..., 2m$.

The explicit form of integrals in equation (17) can be written as:

$$P_{i,\xi}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^{\xi}}{\xi!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{[(x-\alpha)^{\xi} - 2(x-\beta)^{\xi}]}{\xi!}, & \text{if } \beta \leqslant x < \gamma \\ \frac{[(x-\alpha)^{\xi} - 2(x-\beta)^{\xi} + (x-\gamma)^{\xi}]}{\xi!} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{18}$$

The following integrals are used to derive the solution of Bratu's problem using the Haar wavelet method:

$$P_{i,1}(x) = \int_0^x H_i(t)dt \tag{19}$$

$$P_{i,1}(x) = \begin{cases} (x-\alpha) & \text{if } \alpha \leqslant x < \beta \\ (\gamma - x) & \text{if } \beta \leqslant x < \gamma \\ 0 & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{20}$$

$$P_{i,\xi+1}(x) = \int_0^x P_{i,\xi}(t)dt \tag{21}$$

and

$$C_{i,\xi}(x) = \int_0^1 P_{i,\xi}(t)dt,\ \xi = 1, 2, ... \tag{22}$$

for $\xi = 2, 3$ $P_{i,\xi+1}(x)$ consists the following functions as:

$$P_{i,2}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^2}{2!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{1}{4m^2} - \frac{(\gamma-x)^2}{2!} & \text{if } \beta \leqslant x < \gamma \\ \frac{1}{4m^2} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{23}$$

$$P_{i,3}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^3}{3!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{(x-\beta)}{4m^2} - \frac{(\gamma-x)^3}{3!} & \text{if } \beta \leqslant x < \gamma \\ \frac{(x-\beta)}{4m^2} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{24}$$

$$P_{i,4}(x) = \begin{cases} 0, & \text{if } 0 \leqslant x < \alpha \\ \frac{(x-\alpha)^4}{4!}, & \text{if } \alpha \leqslant x < \beta \\ \frac{(x-\beta)}{8m^2} - \frac{(\gamma-x)^4}{4!} + \frac{1}{192m^4} & \text{if } \beta \leqslant x < \gamma \\ \frac{(x-\beta)}{8m^2} + \frac{1}{192m^4} & \text{if } \gamma \leqslant x < 1 \end{cases} \tag{25}$$

Next, for $\xi = 4, 5, ...$ values of equation (18) can be evaluated. And also keep that,

$P_{1,\xi}(x) = \frac{x^\xi}{\xi!}$, $C_{1,\xi}(x) = \frac{1}{(\xi+1)!}$, $\xi = 1, 2, ...$

Here, the expression for the Haar operational matrix of integration of finite order has been calculated for the approximation of the solution to the SPBVPs by developing the numerical algorithm.

3.1. **Function approximation by Haar Wavelet.** For a better understanding of the Haar wavelet as a type of orthonormal wavelet, this section describes the function approximation by the Haar wavelet.

3.2. **Function with One Variable :** Pandit et al., [25] introduced that any real- valued twice integrable function $y_c(x) \in L^2[0,1]$ can be expressed as a linear combination of Haar wavelet function as below:

$$y_c(x) = \sum_{i=1}^{\infty} a_i H_i(x) \ = \ a_1 H_1(x) + a_2 H_2(x) + ... \tag{26}$$

where the Haar coefficient $a_i, i = 1, 2, 3...$ can be calculated by

$$a_i =< y_c, H_i >= 2^j \int_0^1 \ y_c(x) \bar{H}_i(x) dx. \tag{27}$$

This series ends with the countable terms if $y_c(x)$ is approximated by piece-wise terms and piece-wise itself during each sub-interval (x_{i-1}, x_i). If the function $y_c(x)$ is piece-wise constant then the countable terms of series $y_c(x)$ can be resolved as fixed terms as:

$$y_c(x) = \sum_{i=1}^{2M} a_i H_i(x) = a_{2M}^T H_{2M}(x) \tag{28}$$

equation (28) can be written in matrix form as $y_c = a^T H$. The scalar function a_{2M}^T and Haar wavelet function vector $H_{2M}(x)$ are defined as below:

$$a_{2M}^T = [a_1, a_2, a_3, ..., a_{2M}] and \ H_{2M}(x) = [H_1(x), H_2(x), H_3(x), ...H_{2M}(x)]^T \tag{29}$$

Here T is the transposition operator, and M is a power of two.

Since the vector a is in discrete form and the Haar matrix H is of order $m = 2^{j+1}$, where $j = 0, 1, 2, ...J$, that is

$$H = \begin{bmatrix} h_1(x_1) & h_1(x_2) & ...h_1(x_m) \\ h_2(x_1) & h_2(x_2) & ...h_2(x_m) \\ . & . & . \\ . & . & . \\ . & . & . \\ h_m(x_1) & h_m(x_2) & ...h_m(x_m) \end{bmatrix} \tag{30}$$

In order to find the Haar coeffient, the points $x_l = \frac{l-\frac{1}{2}}{2m}, l = 1, 2, ...2m$ are calculate by discretizing Haar function $H_i(x)$ by breaking the interval $[0, 1]$ into $2m$ parts with length $\Delta t = \frac{1}{2m}$ to find the matrix H.

For different value of m, For $m = 8$ the H square Haar matrix would be as below:

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (31)$$

Moreover, by the definition of Haar wavelet H_8, it is orthogonal.
By the definition of Haar wavelet H_8, it is orthogonal. equation (30) has been used to construct the Haar matrix of any order which developed the matrix $p_{i,1}, p_{i,2}$ i.e. first and second order integration of Haar matrix. Further, these developed matrices are required for approximation of the problem.

4. Wavelet Scheme for the Solution of Bratu's Type BVPs

We investigate the generic two point BVPs of the form for the solution of the Bratu's type problems (1) and (2).

$$y''(\mu) + \lambda\, N(y(\mu)) = 0, \quad (32)$$

with conditions

$$y(0) = \delta,\ Y(1) = \sigma. \quad (33)$$

In this method, solution $y(\mu)$ writes in infinite series form as

$$y(\mu) = \sum_{n=0}^{\infty} c_n \mu^i, \quad (34)$$

the term $N(y(\mu))$ write in series of wavelet coefficients a_n as,

$$N(y(\mu)) = N\left(\sum_{n=0}^{\infty} c_n \mu^n\right) = \sum_{n=0}^{\infty} a_n \mu^i. \quad (35)$$

Now calculate the Haar coefficients c_n
Using the condition $y(0) = \delta$ on (34), we have

$$c_0 = \delta. \quad (36)$$

When the equations (34)-(36) are substitute in (32), we obtain

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)\mu^n + \lambda \sum_{n=0}^{\infty} a_i \mu^n = 0, \quad (37)$$

then we have the equation as

$$c_{n+2} = \frac{-\lambda\, a_n(c_0, c_1, \cdots, c_n)}{(1+n)(2+n)},\ n > 0 \tag{38}$$

a_n ate in terms of Haar solution coefficients $c_n's$ are calculated by using numerical algorithm.
For the order of n^{th} Haar solution of problem, we can write

$$y_n(\mu) = \sum_{j=0}^{n} c_j \mu^j. \tag{39}$$

From equation (36) and (37), (39) becomes

$$y_n(\mu,\ c_1) = \delta + c_1\,\mu - \lambda \sum_{j=0}^{n-2} \frac{a_j}{(1+j)(2+j)} \mu^{j+2}. \tag{40}$$

Now using the condition $y(1) = 0$ on n^{th} order Haar approximation, we have

$$y_n(1,\ c_1) = 0, \tag{41}$$

after calculating the value of c_1, equation (40) gives the n^{th} order Haar approximate solution.

5. Convergence Analysis of Haar Wavelet Method for Bratu's Problem

Numerous writers examine the convergence of the Haar wavelet method (HWM) and its variants for the initial, boundary value issues. [34,36,38].
Here, we discuss about the wavelet technique's convergence for BVPs of type Bratu problem. Equation (40) may be expressed as follows

$$y = \delta + M(y) \tag{42}$$

where

$$M(Y) = M\left(\sum_{j=0}^{\infty} c_j \mu^j\right) = \sigma\mu - \lambda \sum_{j=0}^{\infty} \frac{a_j}{(2+j)(1+j)} \mu^{j+2}. \tag{43}$$

Equation (40) can be rewritten as

$$y_n(\mu) = \delta + \sigma\mu - \lambda \sum_{j=1}^{n-1} \frac{a_{j-1}}{j(j+1)} \mu^{j+1}. \tag{44}$$

Using (43) and (44), equation (40) is written as

$$y_n = \delta + M(y_{n-1}),\ n > 1. \tag{45}$$

Convergence of Haar solution y_n defined by (25) is given by the theorem.

Theorem 1. Suppose $z = C[0,1]$ be a real space with the norm $||y|| = \max_{0<\mu<1} |y(\mu)|,\ y \in z$.
Suppose $M(y)$ be the linear operator given in (19) and satisfied Lipschitz criteria:
$||M(y) - M(x)|| < R\ ||y - x||,\ \forall\ y,\ x \in z$ with real constant R, $0 < R < 1$. If $||\delta|| < \infty$, then the

subsequence $\{y_n\}$ given in equation (45) converges to the point y.

Proof. For convergence of $\{y_n\}$, first we prove

$$||y_{n+1} - y_n|| < R^n \; ||\delta||, \tag{46}$$

we use principle of mathematical induction.

For $n = 1$, we have

$$||y_2 - y_1|| = ||M(y_1) - M(y_0)|| < R \; ||y_1 - y_0|| = R \; ||\delta||, \tag{47}$$

so (46) is true for $n = 1$.

For $n = j$, by induction principle (45) is true

$$||y_{j+1} - y_j|| = ||M(y_j) - M(y_{j-1})|| < R^k \; ||y_j - y_{j-1}|| = R^j \; ||\delta||. \tag{48}$$

Then, for $n = 1 + j$,

$$||y_{j+2} - y_{j+1}|| = ||M(y_{j+1}) - M(y_j)|| < R^{j+1} \; ||y_{j+1} - y_j|| = R^{j+1} \; ||\delta||. \tag{49}$$

From the relations (47)-(49), we observe that (46) is true ∀ n.

Next, we have to show $\{y_n\}$ is convergent, so, $\{y_n\}$ is Cauchy in real space z.

For each $m, n \in N, \;\; m > n$, we have

$$\begin{aligned} ||y_m - y_n|| &= ||(y_m - y_{m-1}) + (y_{m-1} - y_{m-2}) + \cdots + (y_{n+1} - y_n)|| \\ &< \;\; ||(y_m - y_{m-1})|| + ||(y_{m-1} - y_{m-2})|| + \cdots + ||(y_{m+1} - y_n)|| \\ &< R^{m-1} \; ||\delta|| + R^{m-2} \; ||\delta|| + \cdots + R^{n+1} \; ||\delta|| + R^n \; ||\delta|| \\ &\le R^n(1 + R + R^2 + \cdots + R^{m-n-1}) \; ||\delta|| \\ &< R^n \left(\frac{1 - R^{m-n}}{1 - R} \right) ||\delta|| \end{aligned} \tag{50}$$

and since $0 < R < 1$, so $1 - R^{m-n} < 1$ and $||\delta|| < \infty$, then (50) written as

$$||y_m - y_n|| < \frac{R^n}{1 - R} ||\delta|| \tag{51}$$

as $n \to \infty$ in (51), we have $||y_m - y_n|| \to 0$.

So, $\{y_m\}$ is convergent in real space z. So there exist a y such that $\lim_{m \to \infty} y_m = y$. Therefore y_m converges to point y.

6. Numerical Results and Discussion

The hybrid scheme given in Section 3 is implemented Bratu's type BVPs for different values of parameter λ and the outcomes are compared with existing techniques given in [8,29]. We introduce the error functions as follows to evaluate the robustness and efficacy of the present method:

If $y(\mu)$ and $y_n(\mu)$ are exact and nth-order Haar solution of Bratu's problem, then the error $e_n(\mu)$ is as follows

$$e_n(\mu) = |y(\mu) - y_n(\mu)|, \tag{52}$$

and maximum absolute error (MAE)me_n is

$$me_n = \max_{\mu\in[0,1]} |y(\mu) - y_n(\mu)|. \tag{53}$$

Numerical computations have been done on MATLAB R2022b.

Problem:- Consider

$$y''(\mu) + \lambda e^{y(\mu)} = 0,\ 0 < \mu < 1, \tag{54}$$

with conditions

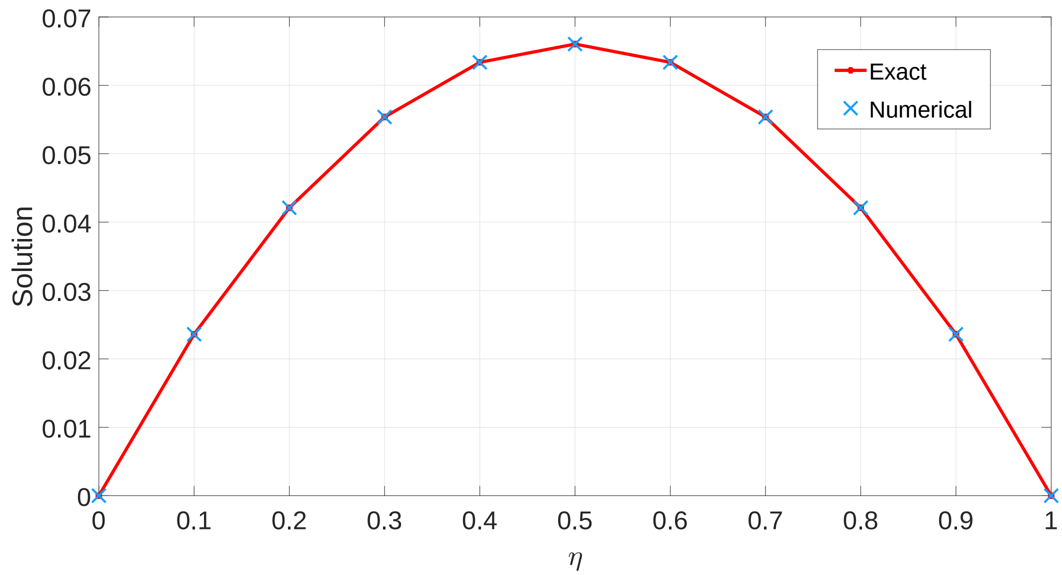
$$y(0) = 0,\ y(1) = 0. \tag{55}$$

Haar coefficients for the term $\lambda\, e^{y(\mu)}$ of (54) are calculated as

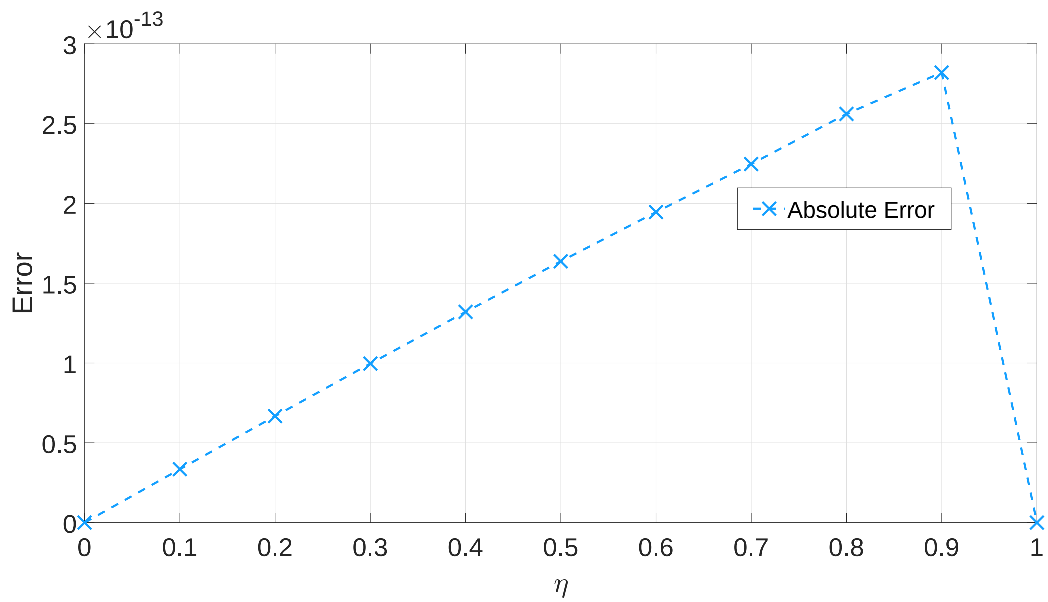
$$\begin{aligned}
a_0 &= \lambda\, e^{c_0}\\
a_1 &= \lambda\, c_1\, e^{c_0}\\
a_2 &= \lambda\, \frac{c_1^2}{2}\, e^{c_0} + \lambda\, c_2\, e^{c_0}\\
c_3 &= \lambda\, \frac{c_1^3}{3!}\, e^{c_0} + \lambda\, c_2\, c_1\, e^{c_0} + \lambda\, c_3\, e^{c_0}\\
&\vdots
\end{aligned}$$

For the parameter λ, we use the wavelet method described in Section 4 as follows
When $\lambda = 0.5$

$$\begin{aligned}
y_{16}(\mu) = &-(3.276e-09)\,\mu^{16} - (8.013e-09)\,\mu^{15} + (2.694e-08)\,\mu^{14} + (1.167e-07)\,\mu^{13}\\
&-(1.717e-07)\,\mu^{12} - (1.496e-06)\,\mu^{11} + (3.095e-07)\,\mu^{10} + (1.770e-05)\,\eta^{9}\\
&+(1.625e-05)\,\mu^{8} - (1.974e-04)\,\mu^{7} - (4.369e-04)\,\mu^{6} + (2.102e-03)\,\mu^{5}\\
&+(8.994e-03)\,\mu^{4} - (2.177e-02)\,\mu^{3} - (0.25)\,\mu^{2} + (0.26127)\,\mu.
\end{aligned}$$



(A)



(B)

Fig. 1. (a) Haar solution $y_{20}(\theta)$ and exact solution $y(\theta)$ of Problem 1.
(b) The absolute error $e_{20}(\theta)$ in Haar solution $y_{20}(\theta)$ of Problem 1.

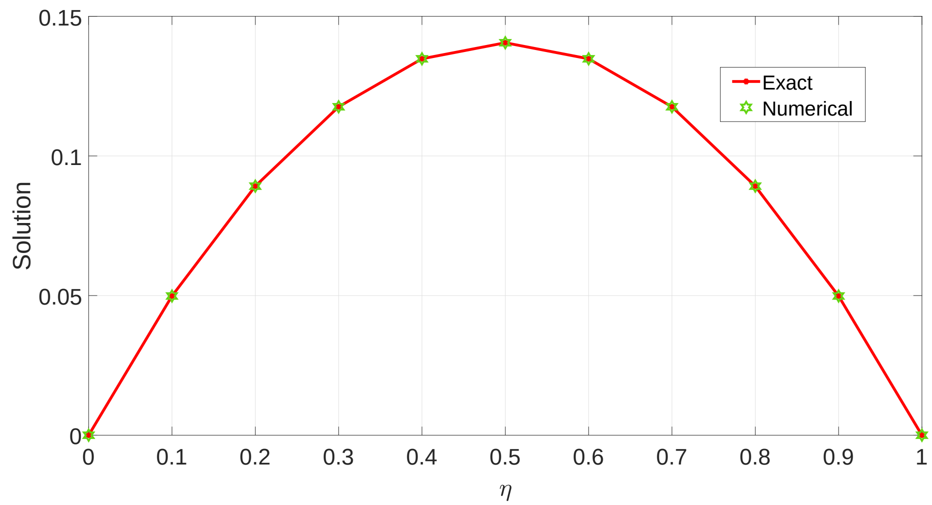
TABLE 1. Analysis of absolute errors for Problem 1.

			Haar wavelet method		
μ	BCM[18]	HM [13]	$e_{16}(\mu)$	$e_{18}(\mu)$	$e_{20}(\mu)$
0	0	0	0	0	0
0.1	-	-	7.98$e-11$	4.15$e-12$	3.34$e-14$
0.2	5.80$e-10$	2.20$e-09$	1.59$e-10$	8.29$e-12$	6.67$e-14$
0.3	-	-	2.37$e-10$	1.23$e-11$	9.96$e-14$
0.4	5.97$e-10$	6.01$e-09$	3.15$e-10$	1.64$e-11$	1.32$e-13$
0.5	-	-	3.90$e-10$	2.03$e-11$	1.63$e-13$
0.6	5.97$e-10$	6.01$e-09$	4.64$e-10$	2.41$e-11$	1.94$e-13$
0.7	-	-	5.34$e-10$	2.78$e-11$	2.24$e-13$
0.8	5.80$e-10$	2.20$e-09$	5.88$e-10$	3.09$e-11$	2.56$e-13$
0.9	-	-	5.51$e-10$	3.01$e-11$	2.82$e-13$
1.0	5.74$e-20$	3.14$e-20$	2.05$e-42$	2.49$e-42$	2.50$e-42$

For parameter $\lambda = 1$, we have $y_n(\mu)$ for other values of n as

$$\begin{aligned} y_{14}(\mu) = {} & (7.1006e-05)\,\mu^{14} + (1.0170e-06)\,\mu^{13} - (3.6489e-05)\,\mu^{12} - (4.5261e-05)\,\mu^{11} \\ & + (1.4890e-04)\,\mu^{10} + (4.1650e-04)\,\mu^{9} - (3.7594e-04)\,\mu^{8} - (2.86062e-03)\,\mu^{7} \\ & - (1.0713e-03)\,\mu^{6} + (1.6930e-02)\,\mu^{5} + (2.9092e-02)\,\eta^{4} - (9.1558e-02)\,\mu^{3} \\ & - (0.5)\,\mu^{2} + (0.5493)\,\mu. \end{aligned}$$

Now that we have a graphical representation, we can check how closely the approximate answer and the precise solution match in terms of accuracy.



(A)

FIG. 2. Analysis of Haar $y_{10}(\theta)$ and exact solution $\mu(\theta)$ of Problem 1.

TABLE 2. Analysis of induced absolute error for Problem 1.

				Present technique	
μ	DM[18]	VIMpd[13]	VIM [13]	$e_{14}(\mu)$	$e_{20}(\mu)$
0	-	-	0	0	0
0.1	2.68e−03	4.46e−05	-	1.252e−08	2.887e−09
0.2	2.02e−03	1.28e−04	2.44e−05	2.492e−08	5.745e−09
0.3	1.52e−04	1.94e−04	-	3.704e−08	8.539e−09
0.4	2.20e−03	2.64e−04	4.21e−05	4.875e−08	1.123e−08
0.5	3.01e−03	3.51e−04	-	5.991e−08	1.380e−08
0.6	2.20e−03	4.76e−04	4.21e−05	7.058e−08	1.622e−08
0.7	1.52e−04	6.77e−04	-	8.155e−08	1.843e−08
0.8	2.02e−03	1.08e−03	2.44e−05	9.505e−08	2.024e−08
0.9	2.68e−03	1.59e−03	-	1.038e−07	1.961e−08
1.0	-	-	1.79e−17	5.919e−42	2.946e−42

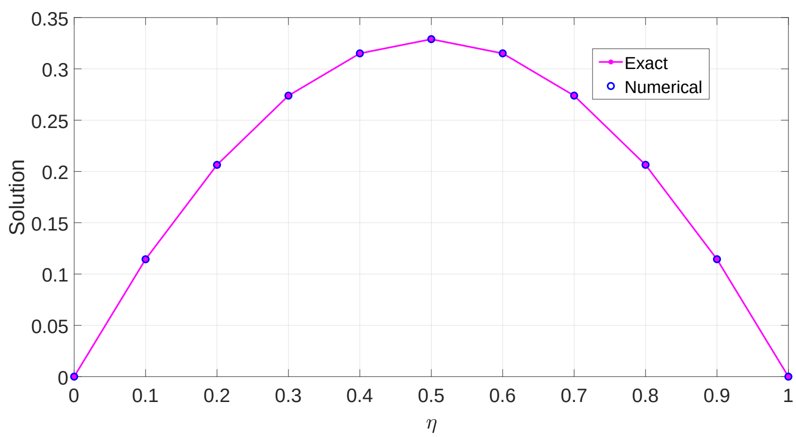
Fig. 1(a), Fig. 2, Fig. 3(a) depict the numerical result of Haar solution $\mu_n(\theta)$ for $n = 10, 12, 18$ and exact $\mu(\theta)$. Here, we observe that the approximate solution and exact solution coincide nearly. And error between exact $\mu(\theta)$ and Haar solution $\mu_n(\theta)$ for different $n = 10, 12, 18$ of Problem 1 has given in Fig. 1(b), Fig. 2(b), and in Fig. 3(b). It is significant to observe from these Figures that numerical solution errors decrease as order is increased.

The Table 1 shows the analysis of errors $e_n(\theta)$ calculated by the present Haar wavelet method for $n = 10, 12, 18$ and the method given in literature as Legendre method (LM) [8], homotopy method (HM) [29] and Taylor method (TM) [39]. To solve problem 1, the present method yields results with a greater accuracy and lesser error. [8,29].

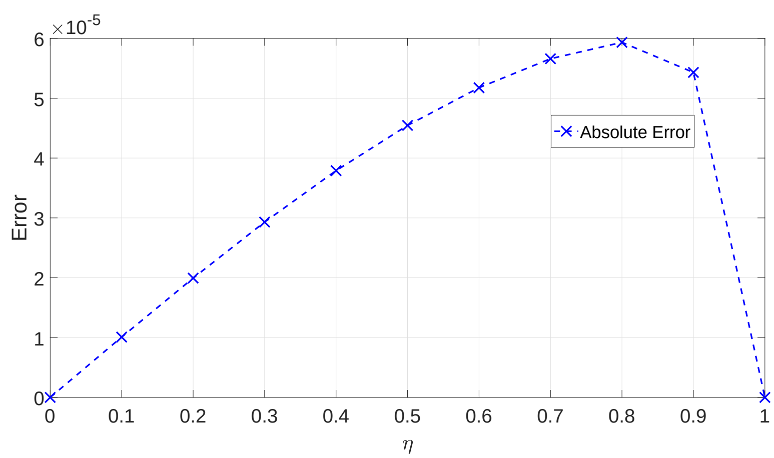
If parameter $\lambda = 2$.

Using the Haar wavelet method given in section 3, we obtain the Haar solution as,

$$y_{15}(\mu) = (1.047e-03)\,\mu^{15} - (6.324e-04)\,\mu^{14} - (3.128e-03)\,\mu^{13} - (1.895e-03)\,\eta^{12} + (5.419e-03)\,\mu^{11} + (1.056e-02)\,\mu^{10} - (1.668e-03)\,\mu^{9} - (2.863e-02)\,\mu^{8} - (2.841e-02)\,\mu^{7} + (4.409e-02)\,\mu^{6} + (0.13404)\,\mu^{5} + (0.036724)\,\mu^{4} - (0.41624)\,\mu^{3} - \mu^{2} + (1.24872)\,\mu.$$



(A)



(B)

FIG. 3. (a) Comparison between approximate solution $\mu_{12}(\theta)$ and exact solution $\mu(\theta)$ of Problem 2.

(b) Absolute error $e_{12}(\theta)$ in the approximate solution $\mu_{12}(\theta)$ of Problem 1.

TABLE 3. Analysis of absolute errors of Problem 1.

				Haar wavelet method	
μ	DM [8][18]	VIMpd [39][13]	LTM [29][20]	$e_{15}(\mu)$	$e_{20}(\mu)$
0	-	-	-	0	0
0.1	1.52$e-$02	3.65$e-$03	2.13$e-$03	5.00$e-$05	1.00$e-$05
0.2	1.47$e-$02	7.22$e-$03	4.21$e-$03	9.90$e-$05	1.99$e-$05
0.3	5.89$e-$03	1.39$e-$03	6.19$e-$03	1.45$e-$04	2.93$e-$05
0.4	3.25$e-$03	1.78$e-$02	8.00$e-$03	1.88$e-$04	3.78$e-$05
0.5	6.98$e-$03	2.10$e-$02	9.60$e-$03	2.25$e-$04	4.54$e-$05
0.6	3.25$e-$03	2.31$e-$02	1.09$e-$03	2.56$e-$04	5.17$e-$05
0.7	5.89$e-$03	2.36$e-$02	1.19$e-$02	2.79$e-$04	5.66$e-$05
0.8	1.47$e-$02	2.18$e-$02	1.24$e-$02	2.86$e-$04	5.93$e-$05
0.9	1.52$e-$02	1.68$e-$02	1.09$e-$02	2.39$e-$04	5.43$e-$05
1.0	-	-	-	1.14$e-$41	2.33$e-$42

Fig. 1(a), Fig. 2(a), Fig. 3(a) shows that Haar $\mu_n(\theta)$ for $n = 20, 10, and 12$ and exact solution $\mu(\theta)$ for Problem 1 is overlapped at the same point. Absolute error in exact $\mu(\theta)$ and Haar solution $\mu_n(\theta)$ for $n = 20, 10$ and 12, figures 1(b), and 3(b) demonstrate that errors in Haar solution decrease with increasing order. Table 2 shows that the suggested method outperforms the efficient wavelet method (EWM) in terms of results. [7] and homotopy analysis scheme (HAS) [29].

7. Conclusion

The numerical solution to non-linear Bratu's type (1)–(2) BVPs was successfully obtained using a novel wavelet approach in this study, along with a fast methodology to create the Haar function. The present method can handle nonlinear problems of type (1) without requiring the linearization of nonlinear terms, discretization of the variables, or any disturbed parameter, therefore the findings are more physically plausible than those of the previous methods. We have demonstrated that the suggested method is convergent in Theorem 1.

In section 6, we analysed non-linear IVPs of Bratu's kind to evaluate the resilience and efficacy of the suggested method. In Problem 1's Figures 1(a), 2(a), and 3(a), as well as Figures 4(a), 5(a), and 6(a), we analysed that Haar solution $\mu_n(\theta)$ are overlapped to exact solution $\mu(\theta)$, This show that the suggested approach solves Bratu's problem extremely effectively. Furthermore, it should be observed that when we raise the order of numerical solution error are diminishing in Figures 1(b), 2(b), and 3(b) of Problem 1 and similarly in Figures 4(b), 5(b), and 6(b) of Problem 1. Since more terms may be added to the approximate series solution, the accuracy of our obtained solutions can be increased. Additionally, Tables 1 and 2 show that the present method is significantly more accurate than the techniques given in [7,8,29,39].

References

[1] G. Bratu, Sur les équations intégrales non linéaires, Bull. Soc. Math. France. 2 (1914), 113–142. https://doi.org/10.24033/bsmf.943.

[2] I. M. Gel'fand, Some problems in the theory of quasi-linear equations, Uspekhi Mat. Nauk, 14 (1959), 87–158.

[3] J. Jacobsen, K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, J. Differ. Equ. 184 (2002), 283–298. https://doi.org/10.1006/jdeq.2001.4151.

[4] D.A. Frank-Kamenetski, Diffusion and heat exchange in chemical kinetics, Princeton University Press, Princeton, NJ, 1955.

[5] S. Chandrasekhar, Introduction to the study of Stellar structure, Dover, New York, 1967.

[6] R. Buckmire, Application of a Mickens finite-difference scheme to the cylindrical Bratu-Gelfand problem, Numer. Methods Part. Differ. Equ. 20 (2004), 327–337. https://doi.org/10.1002/num.10093.

[7] G. Hariharan, P. Pirabaharan, An efficient wavelet method for initial value problems of Bratu-type arising in engineering, Appl. Math. Sci. 7 (2013), 2121–2130.

[8] S.G. Venkatesh, S.K. Ayyaswamy, S. Raja Balachandar, The Legendre wavelet method for solving initial value problems of Bratu-type, Computers Math. Appl. 63 (2012), 1287–1295. https://doi.org/10.1016/j.camwa.2011.12.069.

[9] A.M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu-type equations, Appl. Math. Comput. 166 (2005), 652–663. https://doi.org/10.1016/j.amc.2004.06.059.

[10] T. Öziş, A. Yıldırım, Comparison between Adomian's method and He's homotopy perturbation method, Comput. Math. Appl. 56 (2008), 1216–1224. https://doi.org/10.1016/j.camwa.2008.02.023.

[11] A.R. Vahidi, M. Hasanzade, Restarted Adomian's decomposition method for the Bratu-Type problem, Appl. Math. Sci. 6 (2012), 479–486.

[12] D.M. Tefera, A. Andargie, The Adomian Decomposition Method with Discretization for Second Order Initial Value Problems, Int. J. Comput. Sci. Appl. Math. 5 (2019), 56–60. https://doi.org/10.12962/j24775401.v5i2.5148.

[13] Z. Odibat, The optimized decomposition method for a reliable treatment of IVPs for second order differential equations, Phys. Scr. 96 (2021), 095206. https://doi.org/10.1088/1402-4896/ac065e.

[14] H.D. Debela, H.B. Yadeta, S.B. Kejela, Numerical solutions of second order initial value problems of Bratu-type equation using higher ordered Rungu-Kutta method, Int. J. Sci. Res. Publ. 7 (2017), 187–197.

[15] H.B. Fenta, G.A. Derese, Numerical solution of second order initial value problems of Bratu-type equations using sixth order Runge-Kutta seven stages method, Int. J. Comput. Sci. Appl. Math. 5 (2019), 10–14. https://doi.org/10.12962/j24775401.v5i1.3806.

[16] J.C. Goswami, A.K. Chan, Fundamentals of wavelets: theory, algorithms, and applications, 2nd ed., John Wiley and Sons, Inc., Hoboken, (2011).

[17] S. Mallat, A Wavelet tour of signal processing. 2nd ed., Academic Press, New York, (1999).

[18] L. Debnath, F.A. Shah, Wavelet transforms and their applications, Springer, New York, (2004).

[19] U. Lepik, H. Hein, Haar wavelets: with applications, Springer, New York, (2014).

[20] C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems, IEE Proc. - Control Theory Appl. 144 (1997), 87–94. https://doi.org/10.1049/ip-cta:19970702.

[21] M.Q. Chen, C. Hwang, Y.P. Shih, The computation of wavelet-Galerkin approximation on a bounded interval, Int. J. Numer. Meth. Eng. 39 (1996), 2921–2944. https://doi.org/10.1002/(sici)1097-0207(19960915)39:17<2921::aid-nme983>3.0.co;2-d.

[22] Ü. Lepik, Numerical solution of evolution equations by the Haar wavelet method, Appl. Math. Comput. 185 (2007), 695–704. https://doi.org/10.1016/j.amc.2006.07.077.

[23] A. Raza, A. Khan, K. Ahmad, Solution of partially singularly perturbed system of initial and boundary value problems using non-uniform Haar wavelet, TWMS J. Appl. Eng. Math. 11 (2021), 1246–1259.

[24] A. Raza, A. Khan, Haar wavelet series solution for solving neutral delay differential equations, J. King Saud Univ. - Sci. 31 (2019), 1070–1076. https://doi.org/10.1016/j.jksus.2018.09.013.

[25] S. Pandit, R. Jiwari, K. Bedi, M.E. Koksal, Haar wavelets operational matrix based algorithm for computational modelling of hyperbolic type wave equations, Eng. Comput. 34 (2017), 2793–2814. https://doi.org/10.1108/ec-10-2016-0364.

[26] W.M. Abd-Elhameed, Y.H. Youssri, E.H. Doha, A novel operational matrix method based on shifted Legendre polynomials for solving second-order boundary value problems involving singular, singularly perturbed and Bratu-type equations, Math. Sci. 9 (2015), 93–102. https://doi.org/10.1007/s40096-015-0155-8.

[27] Y. Aksoy, M. Pakdemirli, New perturbation-iteration solutions for Bratu-type equations, Comput. Math. Appl. 59 (2010), 2802–2808. https://doi.org/10.1016/j.camwa.2010.01.050.

[28] B.S.H. Kashkari, S.S.Z. Abbas, Solution of initial value problem of Bratu - type equation using modifications of homotopy perturbation method, Int. J. Comput. Appl. 162 (2017), 44–49.

[29] M.A. Darwish, B.S. Kashkari, Numerical solutions of second order initial value problems of Bratu-type via optimal homotopy asymptotic method, Amer. J. Comput. Math. 4 (2014), 47–54.

[30] M.A.Z. Raja, R. Samar, E.S. Alaidarous, et al. Bio-inspired computing platform for reliable solution of Bratu-type equations arising in the modeling of electrically conducting solids, Appl. Math. Model. 40 (2016), 5964–5977. https://doi.org/10.1016/j.apm.2016.01.034.

[31] J.S. Duan, Convenient analytic recurrence algorithms for the Adomian polynomials, Appl. Math. Comput. 217 (2011), 6337–6348. https://doi.org/10.1016/j.amc.2011.01.007.

[32] R. Rach, A convenient computational form for the Adomian polynomials, J. Math. Anal. Appl. 102 (1984), 415–419. https://doi.org/10.1016/0022-247x(84)90181-1.

[33] M. Kaliyappan, S. Hariharan, Symbolic computation of Adomian polynomials based on Rach's rule, Br. J. Math. Comput. Sci. 5 (2015), 562–570. https://doi.org/10.9734/bjmcs/2015/14067.

[34] Y. Cherruault, G. Adomian, Decomposition methods: a new proof of convergence, Math. Comput. Model. 18 (1993), 103–106. https://doi.org/10.1016/0895-7177(93)90233-o.

[35] Umesh, M. Kumar, Numerical solution of singular boundary value problems using advanced Adomian decomposition method, Eng. Comput. 37 (2020), 2853–2863. https://doi.org/10.1007/s00366-020-00972-6.

[36] S.S. Ray, New Approach for General convergence of the Adomian decomposition method, World Appl. Sci. J. 32 (2014), 2264–2268.

[37] U. Umesh, M. Kumar, Numerical solution of Lane-Emden type equations using Adomian decomposition method with unequal step-size partitions, Eng. Comput. 38 (2020), 1–18. https://doi.org/10.1108/ec-02-2020-0073.

[38] Umesh, M. Kumar, Approximate solution of singular IVPs of Lane-Emden type and error estimation via advanced Adomian decomposition method, J. Appl. Math. Comput. 66 (2020), 527–542. https://doi.org/10.1007/s12190-020-01444-2.

[39] E. Keshavarz, Y. Ordokhani, M. Razzaghi, The Taylor wavelets method for solving the initial and boundary value problems of Bratu-type equations, Appl. Numer. Math. 128 (2018), 205–216. https://doi.org/10.1016/j.apnum.2018.02.001.