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ON GENERALIZED KINEMATICS WITH THE POLAR MOMENT OF INERTIA FOR THE HOMOTHETIC MOTIONS IN $\mathbb{C}_p$

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Abstract. In this study, the polar moments of inertia for the trajectories drawn by the non linear points are obtained during the one parameter planar homothetic motions in the generalized complex plane $\mathbb{C}_p$. Then, we express the polar moment of inertia with respect to the Cauchy length formula. Moreover, for non linear three points we give new version of Holditch type theorem during the homothetic motions $\mathbb{C}_p$. Consequently, we obtain some conclusions.

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1. Introduction

Kinematics, one of the subbranch of mechanics, is the science that studies the way the geometric properties of material systems change over time. The formation of kinematics belongs to Ampère (1775-1836), who founded this science and named it [1]. Kinematics tries to determine how the object motions, what trajectory it goes on, what its location, velocity and acceleration are at any moment, without taking into account the forces acting on the object. In different spaces and dimensions, kinematics was studied by many scientists. In Euclidean and complex planes the planar motions with one parameter were expressed by Müller [2]. Then, in Lorentzian plane the same motions were given by Ergin [3] and Görmez [4]. After that, for hyperbolic planes the planar motions were expressed by Yüce and Kuruoğlu [5]. In

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addition to that, these motions for Galilean planes were obtained Akar and Yüce [6]. The Holditch theorem given by Holditch [7] is one of the remarkable expressions of kinematics. The most important part of this classic Holditch theorem given by Holditch is that the area of the trajectory during the motion is independent of the drawn curve. Therefore, thanks to this feature, many scientists started to study this theorem. On the other hand, Steiner expressed the area of the trajectory in terms of Steiner points for one-parameter planar motion [8,9]. Later, many scientists generalized Holditch theorem and the area formula in different ways and perspectives [2,10–27].

2. Preliminaries

Any generalized complex number is expressed $Z = (z, w)$ $(Z = z + iw)$ and the number system consisting of these numbers includes ordinary (when $p + q^2/4$ is negative), dual (zero) and double (positive) numbers where $i^2 = (q, p)$, $(i^2 = iq + p)$ and $z, w, q, p \in \mathbb{R}$ [28,30,32]. In this study, we assume that $q = 0$ and $i^2 = p \in \mathbb{R}$ $(-\infty < p < \infty)$. Therefore, we study in generalized complex number system

$$\mathbb{C}_p = \left\{ z + iw \, : \, z, w \in \mathbb{R}, \; i^2 = p \in \mathbb{R} \right\}.$$

Some operations defined on this system are as follows. If we take two numbers $Z_1 = z_1 + iw_1, Z_2 = z_2 + iw_2 \in \mathbb{C}_p$ therefore, we define that the addition of this numbers as

$$Z_1 \pm Z_2 = (z_1 + iw_1) \pm (z_2 + iw_2) = (z_1 \pm z_2) + i(w_1 \pm w_2).$$

In addition to that, the product in system $\mathbb{C}_p$ is

$$M^p(Z_1, Z_2) = (z_1 z_2 + p w_1 w_2) + i(z_1 w_2 + z_2 w_1)$$

[28–30]. In addition to that, we suppose that $\boldsymbol{z}_1 = z_1 + iw_1, \boldsymbol{z}_2 = z_2 + iw_2 \in \mathbb{C}_p$ are position vectors of Z_1, Z_2. Therefore, the scalar product can be expressed as

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_p = \mathrm{Re}\left(M^p \left(\mathbf{z}_1, \bar{\mathbf{z}}_2 \right) \right) = \mathrm{Re}\left(M^p \left(\bar{\mathbf{z}}_1, \mathbf{z}_2 \right) \right) = z_1 w_1 - p z_2 w_2$$

[28]. In addition, the $p-$magnitude of $Z = z + iw \in \mathbb{C}_p$ is

$$|Z|_p = \sqrt{\left| M^p(Z, \overline{Z}) \right|} = \sqrt{|z^2 - p w^2|}$$

where " $-$ " is the ordinary complex conjugation, [28]. On the other hand, the unit circle is defined as $|Z|_p = 1$ in $\mathbb{C}_p$. Therefore, the unit circles in this plane are given for the special cases of p in Figure 1.

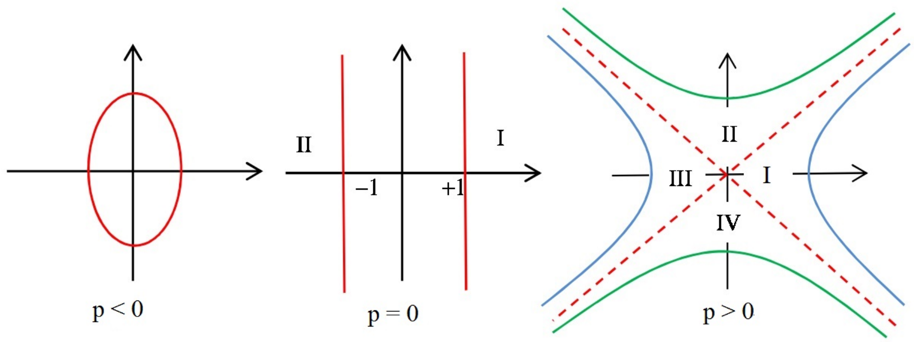

FIGURE 1. The Unit Circle in $\mathbb{C}_p$

Moreover, in $\mathbb{C}_p$ any circle can be characterized by the equation

$$\left|(z-a)^2 - p(w-b)^2\right| = r^2$$

where the center and radius of this circle $M(a,b)$ and r, respectively [28].

Now, we symbolize the number $Z = z + iw$ in $\mathbb{C}_p$ with $\overrightarrow{OT}$. Therefore, the angles γ_p are expressed by the inverse of tangent function as

$$\gamma_p = \begin{cases} \frac{1}{\sqrt{|p|}}\tan^{-1}\left(\alpha\sqrt{|p|}\right), & p<0 \\ \alpha, & p=0 \\ \frac{1}{\sqrt{p}}\tan^{-1}\left(\alpha\sqrt{p}\right), & p>0 \ (branch\ I, III) \end{cases}$$

where $\alpha \equiv w/z$ (see Figure 2).

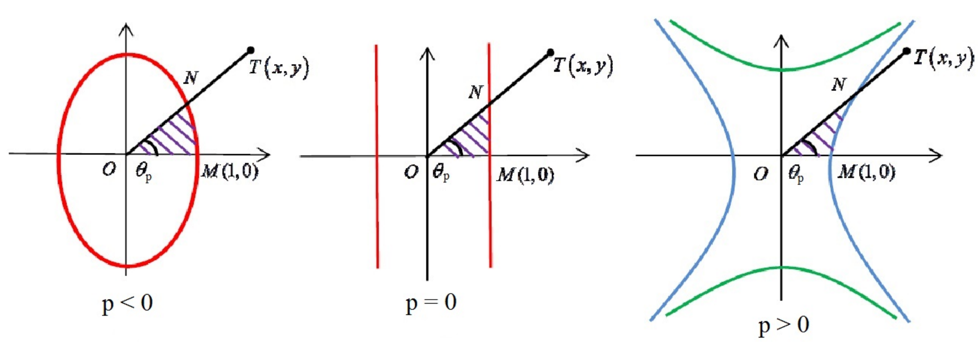

FIGURE 2. Elliptic, Parabolic and Hyperbolic Angles

We suppose that the intersection of unit circle with OT is any point N, L is the orthogonal projection of N on OM, and the tangent of the unit circle at M is QM (see Figure 3). Therefore, the $p-$trigonometric functions ($sinp$, $cosp$ and $tanp$) can be given as

$$\sin p\gamma_p = \begin{cases} \frac{1}{\sqrt{|\mathrm{p}|}} \sin\left(\gamma_p \sqrt{|p|}\right), & p < 0 \\ \gamma_p, & p = 0 \\ \frac{1}{\sqrt{\mathrm{p}}} \sinh\left(\gamma_p \sqrt{p}\right), & p > 0 \end{cases}$$

$$\cos p\gamma_p = \begin{cases} \cos\left(\gamma_p \sqrt{|p|}\right), & p < 0 \\ 1, & p = 0 \\ \cosh\left(\gamma_p \sqrt{p}\right), & p > 0 \end{cases}$$

where these functions are given for branch I when $p \geq 0$ and the ratio $\frac{QM}{OM} = \frac{NL}{OL}$ gives

$$\tan p\gamma_p = \frac{\sin p\gamma_p}{\cos \mathrm{p}\gamma_\mathrm{p}}.$$

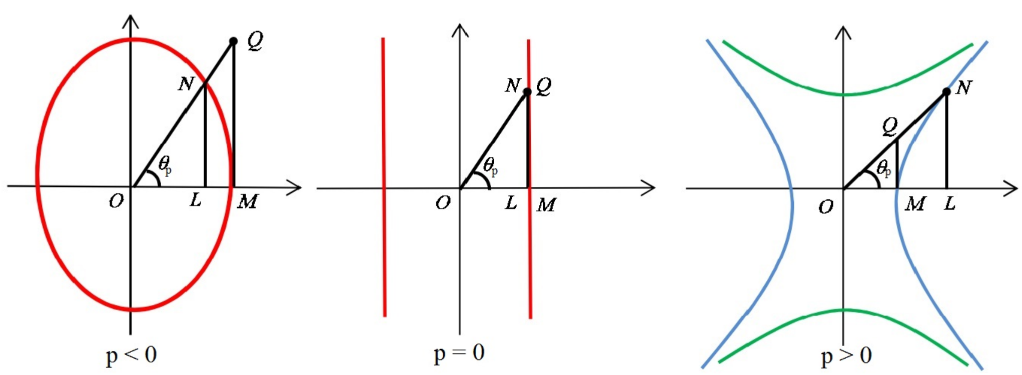

FIGURE 3. γ_p for the special cases of p

On the other hand, the Maclaurin expansions of these function for branch I can be expressed $\cos p\gamma_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n)!}\gamma_p^{2n}$ and $\sin p\gamma_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n+1)!}\gamma_p^{2n+1}$. The Euler Formula in $\mathbb{C}_p$ is expressed as $e^{i\gamma_p} = \cos p\gamma_p + i \sin p\gamma_p$ where $i^2 = p$. Moreover, the exponential forms of Z in $\mathbb{C}_p$ is

$$Z = r_p(cosp\gamma_p + i\sin p\gamma_p) = r_p e^{i\gamma_p}$$

where $r_p = |Z|_p$ [28]. Then, we can write that the $p-$rotation matrix is

$$A(\gamma_p) = \begin{bmatrix} \cos p\gamma_p & p\sin p\gamma_p \\ \sin p\gamma_p & \cos p\gamma_p \end{bmatrix}.$$

[28].

Now, we express the homothetic planar motions with one parameter in $\mathbb{C}_p$. The homothetic motions in p-complex plane, the subset of $\mathbb{C}_p$, was expressed as

$$\mathbb{C}_J = \left\{ z + Jw : x, y \in R, J^2 = p, p \in \{-1, 0, 1\} \right\}$$

by Gürses et. al [31]. Analogously, with the help of that study the homothetic planar motions with one parameter in $\mathbb{C}_p$ can be obtained as follows.

We suppose that $\mathbb{K}_p, \mathbb{K}_p'$ are the moving and fixed planes in $\mathbb{C}_p$, respectively and the vectors $\mathbf{x} = x_1 + ix_2$ and $\mathbf{x}' = x_1' + ix_2'$ are the position vectors of any point X according to the these planes, respectively. Therefore, the homothetic motions in $\mathbb{C}_p$ is characterized by the equation

$$\mathbf{x}' = (h\mathbf{x} - \mathbf{u})\, e^{i\gamma_p}$$

where $\overrightarrow{\mathrm{OO}'} = \mathbf{u}$ $(\mathbf{u}' = -\mathbf{u}e^{i\gamma_p})$, γ_p is the $p-$rotation angle of this homothetic motion, and h is the homothetic scale in C_p. In that case, the relative and absolute velocity vectors of X can be calculated as

$$\mathbf{V}_r{}' = \mathbf{V}_r e^{i\gamma_p} = h\dot{\mathbf{x}}e^{i\gamma_p} \tag{1}$$

and

$$\mathbf{V}_a{}' = \mathbf{V}_a e^{i\gamma_p} = \left(\dot{h} + i\dot{\gamma}_p h\right)\mathbf{x}e^{i\gamma_p} - (\dot{\mathbf{u}} + i\dot{\gamma}_p\mathbf{u})\, e^{i\gamma_p} + h\dot{\mathbf{x}}e^{i\gamma_p}, \tag{2}$$

respectively. Therefore, considering the equations (1) and (2) the guide velocity vector can be given as

$$\mathbf{V}_f{}' = \mathbf{V}_f e^{i\gamma_p} = \left(\dot{h} + i\dot{\gamma}_p h\right)\mathbf{x}e^{i\gamma_p} + \dot{\mathbf{u}}'.$$

Theorem 1. *The relationship between velocity vectors of the homothetic motions $\mathbb{K}_p/\mathbb{K}_p'$ in $\mathbb{C}_p$ is*

$$\boldsymbol{V}_a = \boldsymbol{V}_f + \boldsymbol{V}_r.$$

On the other hand, there are points (defined as pole points) during this homothetic motions that seem to be fixed both in $\mathbb{K}_p$ and $\mathbb{K}_p'$ in $\mathbb{C}_p$. Therefore, we consider that the pole points of the homothetic motions $\mathbb{K}_p/\mathbb{K}_p'$ be $Q = (q_1, q_2) \in \mathbb{C}_p$ and the components of these pole points can be expressed as

$$q_1 = \frac{dh(du_1 + \mathrm{p}u_2 d\gamma_\mathrm{p}) - \mathrm{p}h(du_2 + u_1 d\gamma_\mathrm{p})d\gamma_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\gamma_\mathrm{p}{}^2},$$
$$q_2 = \frac{dh(du_2 + u_1 d\gamma_\mathrm{p}) - h(du_1 + \mathrm{p}u_2 d\gamma_\mathrm{p})d\gamma_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\gamma_\mathrm{p}{}^2}.$$

where $\mathbf{V}_f = 0$. Let the position vector of the pole point Q be $\mathbf{q}$. Moreover, the guide velocity vector of the fixed point X in $\mathbb{K}_p$ can be expressed with the help of the pole points as

$$\mathbf{d}_x{}' = (dh + ihd\gamma_p)(\mathbf{x} - \mathbf{q})\, e^{i\gamma_p}.$$

In addition to that, for the homothetic motions in $\mathbb{C}_p$ the following proposition can be expressed.

Proposition 2. *We consider that two any generalized complex vectors be $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ in $\mathbb{C}_p$. Therefore, the equations*

$$\begin{aligned} &i)\ \ [\boldsymbol{u}e^{i\gamma_p}, \boldsymbol{v}e^{i\gamma_p}] = [\boldsymbol{u}, \boldsymbol{v}] \\ &ii)\ [\boldsymbol{u}, (dh + ihd\gamma_p)\,\boldsymbol{v}] = [\boldsymbol{u}, \boldsymbol{v}]dh + \tfrac{1}{2}\left[\boldsymbol{u}\bar{\boldsymbol{v}} + \bar{\boldsymbol{u}}\boldsymbol{v}\right] hd\gamma_p, \end{aligned}$$

are hold where h is homothetic scale and

$$[\boldsymbol{u}, \boldsymbol{v}] = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1.$$

In this study, we assume the open motions in branch I restricted to time interval $[t_1, t_2]$ in $\mathbb{C}_p$.

Now, we assume that $\mathbb{C}_p$ is the generalized complex plane and $\mathbb{K}_p/\mathbb{K}'_p$ is the homothetic motion in $\mathbb{C}_p$. Moreover, we consider that any point in $\mathbb{C}_p$ is $X = (x_1, x_2)$ and g is any straight line through point X in the branch I in $\mathbb{C}_p$. In this case, the Hesse form of the line g with regard to the moving generalized complex plane $\mathbb{K}_p$ can be written as $h = x_1 \cos p\psi_p - px_2 \sin p\psi_p$ where the Hesse coordinates are (h, ψ_{p}), the distance between the origin point O and the straight line g is $h = h(\psi_{\mathrm{p}})$ and the contact point of the straight line g and the envelope curve (g) is the point X [18]. In addition to that, the angle ψ_{p} in $\mathbb{C}_p$ is the angle in the positive direction made by the perpendicular descending from O to the straight line g and the principal axis of the moving plane $\mathbb{K}_p$. Similarly, the straight line g can be expressed with regard to $\mathbb{K}'_p$ as

$$k' = x'_1 \cos p\psi'_p - px'_2 \sin p\psi'_p \tag{3}$$

where the point $X' = (x'_1, x'_2)$ is the representation of X according to the fixed plane $\mathbb{K}'_p$ and the angle ψ'_p in $\mathbb{C}_p$ is the angle in the positive direction made by the perpendicular descending from the origin point O to the straight line g and the principal axis of the moving plane $\mathbb{K}'_p$ [18]. In addition to that, the relationship between the angles ψ_{p} and ψ'_p is $\psi'_p = \gamma_p + \psi_p$ where the angle γ_p is the rotation angle of the homothetic motion in $\mathbb{C}_p$ [18].

Theorem 3. *The Cauchy length formula under the homothetic motions in $\mathbb{C}_p$ is obtained*

$$L' = \frac{1}{\sqrt{|p|}} \left| phk\delta_p - A\cos p\psi_p + pB\sin p\psi_p \right| \tag{4}$$

where $\delta_p = \int_{t_1}^{t_2} d\gamma_p$, $A = \int_{t_1}^{t_2} (pu_1 - \ddot{u}_1)d\gamma_p$ and $B = \int_{t_1}^{t_2} (pu_2 - \ddot{u}_2)d\gamma_p$, h is homothetic scale and ψ_p is the angle in the positive direction made by the perpendicular descending from O to the straight line g and the principal axis of $\mathbb{K}_p$ [18].

Conclusion 1. *We suppose that the fixed straight lines which have Hesse coordinates (h, ψ_p) and the enveloping trajectories of these lines have the same Cauchy length $L' = d$ during the homothetic motions in $\mathbb{C}_p$. Therefore, all of these lines are tangent to the cycles with radius $\frac{d}{\sqrt{|p|}h\delta_p}$ and center $S_G = \left(\frac{A}{ph\delta_p}, \frac{B}{ph\delta_p}\right)$ in $\mathbb{K}_p$ where $A = \int_{t_1}^{t_2} (pu_1 - \ddot{u}_1)d\gamma_p$ and $B = \int_{t_1}^{t_2} (pu_2 - \ddot{u}_2)d\gamma_p$ [18].*

3. Main Theorems and Results

In this section, we obtain the polar moment of inertia formula of the non linear points for the homothetic motions in $\mathbb{C}_p$ and we express this moment formula with respect to the Cauchy length formula. Moreover, for non linear three points we give new version of Holditch type theorem during the homothetic motions. Consequently, we obtain some conclusions.

Now, we give a new version of the Holditch type theorem given by [13,15,17] considering non linear points for homothetic planar motion in $\mathbb{C}_p$. First of all, we assume that the non linear three fixed points X, Y and Z in $\mathbb{K}_p$ for the homothetic planar motions draw the trajectories k_X, k_Y and k_Z with polar moments of inertia T_X, T_Y and T_Z, respectively. With a special choosing, we consider $X = (0,0)$, $Y = (z+w, 0)$ and $Z = (z,u)$ $(z > u)$ in $\mathbb{K}_p$. We know that the polar moment of inertia T_X for the homothetic planar motion is calculated

$$T_X = T_O + h^2(t_0)\,\delta_p\left(\mathbf{x}\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{s}} - \bar{x}\mathbf{s}\right) + \mu_1 x_1 + \mu_2 x_2$$

where $\mu_1 = -2\int_{t_1}^{t_2} hq_2 dh$, $\mu_2 = 2\int_{t_1}^{t_2} hq_1 dh$ considering [17]. In that case, for $X = (0,0)$, $Y = (z+w, 0)$ and $Z = (z,u)$ the areas for homothetic planar motions are written by

$$T_X = T_0$$

$$T_Y = T_X + h^2\delta_p\left((z+w)^2 - 2(z+w)s_1\right) + \mu_1(z+w) \tag{5}$$

and

$$T_Z = T_X + h^2\delta_p\left(z^2 - pu^2 - 2zs_1 + 2pus_2\right) + \mu_1 z + \mu_2 u. \tag{6}$$

Therefore, using the equation (5) we get the first component of the Steiner point as

$$s_1 = \frac{T_X - T_Y}{2h^2\delta_p\left(z+w\right)} + \frac{\mu_1}{2h^2\delta_p} + \frac{z+w}{2}. \tag{7}$$

Moreover, from the equations (6) and (7)

$$T_Z = \frac{zT_Y + wT_X}{z+w} - h^2\delta_p\left(zw + pu^2\right) + 2ph^2\delta_p u s_2 + \mu_2 u$$

where $\mu_2 = 2\int\limits_{t_1}^{t_2} hq_1 dh$.

Theorem 4. *We assume that the one parameter homothetic motion in $\mathbb{C}_p$ with $S = S_G$ and the non linear points $X = (0,0)$, $Y = (z+w,0) \in \mathbb{K}_p$ move along the trajectories with the polar moments of inertia T_X and T_Y, respectively, then the point $Z = (z,u) \in \mathbb{K}_p$ draws the trajectory with the polar moment of inertia*

$$T_Z = \frac{zT_Y + wT_X}{z+w} - h^2\delta_p\left(zw + pu^2\right) - 2\sqrt{|p|}huL_{XY} + \mu_2 u \tag{8}$$

where L_{XY} is the length of the enveloping curve of (XY).

Proof. Now, we suppose that the one-parameter homothetic planar motion $S = S_G$ in $\mathbb{C}_p$. Therefore, we get $s_2 = \frac{B}{ph\delta_p}$. If we consider that $X = (0,0)$, $Y = (z+w,0)$ and $Z = (z,u)$ consequently, we have

$$T_Z = \frac{zT_Y + wT_X}{z+w} - h^2\delta_p\left(zw + pu^2\right) - 2\sqrt{|p|}huL_{XY} + \mu_2 u.$$

□

Theorem 5. Main Theorem: *We suppose that the one parameter homothetic motion in $\mathbb{C}_p$ with $S = S_G$ and the non linear points $X = (0,0)$, $Y = (z+w,0)$ and $Z = (z,u) \in \mathbb{K}_p$ move along the trajectories with the polar moments of inertia T_X, T_Y and T_Z, respectively. Therefore, the polar moment of inertia of the section between k_X, k_Y and k_Z varies depending on the distance of Z to the line XY and the points X, Y to the projection point of Z, the rotation angle of the homothetic motion, the homothetic scale and the length of the envelope curve of (XY), while it is independent of the choosing of the curves.*

Corollary 1. *We consider that the homothetic scale is $h = 1$ in equation (8). Therefore, the polar moment of inertia can be obtained as*

$$T_Z = \frac{zT_Y + wT_X}{z+w} - \delta_p\left(zw + pu^2\right) - 2\sqrt{|p|}uL_{XY}.$$

This formula is the polar moment of inertia formula given in $\mathbb{C}_p$ in [15]

Corollary 2. *We assume that X, Y and Z are linear points for the homothetic planar motions in $\mathbb{C}_p$. Therefore, $u = 0$ and using the equation (8) we get*

$$T_Z = \frac{zT_Y + wT_X}{z+w} - h^2\delta_p zw$$

In this case, this formula is a special case of the polar moment of inertia formula in [17].

Corollary 3. *Now, we assume that both $u = 0$ and $h = 1$. Therefore, the equation (8) is obtained*

$$T_Z = \frac{zT_Y + wT_X}{z+w} - \delta_p zw.$$

This equation is the polar moment of inertia formula in [13]

In this study, we generalized the studies of [13, 15, 17], which gives the Holditch type theorem regarding the polar moments of inertia for planar motions. Therefore, Holditch type theorem in this study is the most general theorem including all the studies for planar motions so far.

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