Asia Pac. J. Math. 2023 **10**:30

EXISTENCE OF SOLUTIONS FOR SOME UNILATERAL PROBLEMS WITH DEGENERATE COERCIVITY

YOUSSEF AKDIM[1], SOUMIA LALAOUI RHALI[2], YOUSSEF OUMOUACHA[2,*]

[1]LAMA Laboratory, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, P.O. Box 1796 Atlas Fez, Morocco

[2]Laboratory LSI, Department of Mathematics, Physics and Informatics, Faculty Polydisciplinary of Taza, University Sidi Mohamed Ben Abdellah, P.O. Box 1223 Taza Gare, Morocco

[*]Corresponding author: youssef.oumouacha@usmba.ac.ma

Received Aug. 30, 2023

Abstract. In this paper, we investigate the existence and regularity of solutions to unilateral problem associated to the equation of the type:

$$\operatorname{div} a(x, u, \nabla u) = \operatorname{div} F \quad \text{in } \Omega,$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and a is a Carathéodory function having degenerate coercivity.

2020 Mathematics Subject Classification. 35J70, 35J60.

Key words and phrases. degenerate coercivity; degenerate elliptic equations; unilateral problem.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$. Let $|F| \in L^m(\Omega)$ for some $m \geq 1$. Consider the following nonlinear Dirichlet problem

$$Au = -\operatorname{div} F, \tag{1.1}$$

where $Au = -\operatorname{div} a(\cdot, u, \nabla u)$ is nonlinear elliptic differential operator of monotone type defined on $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ with p is a real such that $1 < p < N$.

Problem (1.1) in the coercive case has been studied in [8], [7], where the authors have proved the existence and regularityd of solutions. We refer to the references therein, for more results in different particular cases.

In a recent work [4], Benkirane and Youssfi considered the Dirichlet (1.1) in noncoercive case, more precisely when the Carathéodory function $a(x, s, \xi)$ satisfying the following degenerate coercivity

DOI: 10.28924/APJM/10-30

condition

$$a(x,s,\xi)\cdot\xi \geq h^{p-1}(|s|)|\xi|^p, \tag{1.2}$$

with h is a continuous decreasing function such that its primitive H is unbounded. And they have obtained the regularity of solutions in terms of the summability of the datum F.

The existence results for similar problems to (1.1) with datum f rather than $\operatorname{div} F$ have been obtained by several authors. In this direction, we cite [1] where the authors studied the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

They proved the existence and some regularity of solutions under various assumptions on the summability of the function f. see also [10]. Recently, in [2] Ayadi and Souilah studied the obstacle problem associated to (1.3) using the penalization method.

In [5], Boccardo and Cirmi have obtained the existence and uniqueness of solution to the obstacle problem for the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Where the data f belongs to $L^1(\Omega)$ (see also [6]).

In the present paper we deal with the existence and regularity results for the unilateral problem associated to the equation (2.1) under the condition of degenerate coercivity (1.2). Mainly, we will prove that the results obtain in [4] remain true in the case of unilateral problem. Due to the lack of coercivity of the nonlinear elliptic operator A, the results are obtained by means of approximation through suitable coercive problems.

The paper is organized as follows: In Section 2, we introduce the assumptions and state our main results. In Section 3 we obtained a priori estimates needed to prove the existence results in Section 4.

2. Basic assumptions and main results

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, p be a real number such that $1 < p < N$. Taking ψ a measurable function on Ω such that

$$\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

we define the convex set $\mathcal{K}_\psi$ by

$$\mathcal{K}_\psi = \left\{ v \in W_0^{1,p}(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega \right\}.$$

Let us consider the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x,u,\nabla u) = \operatorname{div} F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (i.e., $a(\cdot, s, \xi)$ is measurable on Ω, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and $a(x,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$, a.e. $x \in \Omega$), such that the following assumptions holds for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, for every $\xi \neq \eta \in \mathbb{R}^N$:

$$a(x,s,\xi) \cdot \xi \geq h^{p-1}(|s|)|\xi|^p, \tag{2.2}$$

$$|a(x,s,\xi)| \leq \beta \left(j(x) + |s|^{p-1} + |\xi|^{p-1} \right), \tag{2.3}$$

$$(a(x,s,\xi) - a(x,s,\eta)) \cdot (\xi - \eta) > 0, \tag{2.4}$$

where $h : \mathbb{R}^+ \to]0,\infty[$ is a decreasing continuous function such that its primitive H is unbounded

$$H(s) = \int_0^s h(t)dt, \tag{2.5}$$

$j(x)$ is a positive function lying in $L^{p'}(\Omega)$ and β is a positive constant.

As regards the datum, we suppose that $F \in (L^m(\Omega))^N$ for some $m > 1$.

The first result concerns the existence of bounded solutions with the data having high summability.

Theorem 2.1. *Assume that (2.2)-(2.5) hold. Let $|F|$ belongs to $L^m(\Omega)$, with*

$$m > \frac{N}{p-1}. \tag{2.6}$$

Then there exists a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is solution of the unilateral problem

$$\begin{cases} u \in \mathcal{K}_\psi, \\ \displaystyle\int_\Omega a(x,u,\nabla u) \cdot \nabla(u - v)dx \leq \int_\Omega F \cdot \nabla(u-v)dx, \\ \forall v \in \mathcal{K}_\psi. \end{cases} \tag{2.7}$$

Now we take the limit case $|F| \in L^m(\Omega)$ with $m = \frac{N}{p-1}$. The solutions we get do not belong in general to $W_0^{1,p}(\Omega)$. We will introduce a different formulation of unilateral problem, along with a new definition of gradient for a measurable function.

For a fixed $k \geq 0$ and $s \in \mathbb{R}$, we recall the definition of the usual truncation function $T_k(s)$

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we denote by $\mathcal{T}_0^{1,p}(\Omega)$ the space of the measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. We recall the following result (see [3], Lemma 2.1):

Lemma 2.2. *For every $u \in \mathcal{T}_0^{1,p}(\Omega)$ there exists a unique measurable map $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{a.e. in } \Omega.$$

Moreover, if $u \in W_0^{1,p}(\Omega)$, then v coincides with the usual distributional gradient ∇u.

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Theorem 2.3. *Assume that (2.2)-(2.5) hold. Let $|F| \in L^m(\Omega)$ with*

$$m = \frac{N}{p-1}. \tag{2.8}$$

Assume in addition that

$$\lim_{t\to\infty} \frac{1}{th(t)} = 0. \tag{2.9}$$

Then there exists a measurable function u which is solution of the unilateral problem

$$\begin{cases} u \geq \psi \text{ a.e. in } \Omega, \\ u \in \mathcal{T}_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \leq \int_\Omega F\cdot\nabla T_k(u-v)dx, \\ \forall v \in \mathcal{K}_\psi \cap L^\infty(\Omega). \end{cases} \tag{2.10}$$

If we weaken the summability of $|F|$ we obtain unbounded solution. In the interest of simplification, we assume that the function h in (2.2) is defined as follows

$$h(s) = \frac{1}{(1+|s|)^\theta}, \qquad \text{with } 0 \leq \theta < 1. \tag{2.11}$$

Let us set

$$\tilde{m} = \frac{Np'}{N-\theta(N-p)}.$$

Theorem 2.4. *Assume that (2.3), (2.4) and (2.2) (with (2.11)) hold. Let $|F|$ belongs to $L^m(\Omega)$, with*

$$\tilde{m} \leq m < \frac{N}{p-1}. \tag{2.12}$$

Then there exists a function $u \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, with

$$r = (1-\theta)(m(p-1))^*, \tag{2.13}$$

which is a solution of the unilateral problem (2.7).

As before we define

$$\bar{m} = \max\left(p', \frac{Np'}{p((1-\theta)N+\theta)}\right).$$

Theorem 2.5. *Assume that (2.3), (2.4) and (2.2) (with (2.11)) hold. Let $F \in L^m(\Omega)$ with*

$$\bar{m} \leq m < \tilde{m}. \tag{2.14}$$

Then there exists a measurable function u wich is solution of the unilateral problem (2.10). Moreover, u belongs to $W_0^{1,q}(\Omega) \cap L^r(\Omega)$, with

$$r = (1-\theta)(m(p-1))^*, \tag{2.15}$$

and

$$q = \frac{(1-\theta)Nm(p-1)}{N-\theta m(p-1)}. \tag{2.16}$$

3. A priori estimates

To prove existence results stated in the precedent section we introduce the following approximating problems. Let $n \in \mathbb{N}$

$$\begin{cases} u_n \in \mathcal{K}_\psi, \\ \langle A_n u_n, u_n - v\rangle \leq \displaystyle\int_\Omega F \cdot \nabla(u_n - v)dx, \\ \forall v \in \mathcal{K}_\psi, \end{cases} \tag{3.1}$$

where $A_n u_n = -\operatorname{div} a(x, T_n(u_n), \nabla u_n)$. For a fixed n, thanks to the hypotheses (2.2)-(2.4) A_n is a nonlinear operator of Leray-Lions type, and since $\operatorname{div} F$ belongs to $W^{-1,p'}(\Omega)$, by well-known results (see [11]) there exists at least a solution u_n so the approximate problem (3.1).

Lemma 3.1. *Let $|F|$ be in $L^m(\Omega)$ with $m > \frac{N}{p-1}$ and let u_n be a solution of the approximate problem (3.1). Then the sequence u_n is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let us start with the estimate in $L^\infty(\Omega)$. Define, for s in $\mathbb{R}$

$$G_k(s) = s - T_k(s).$$

For $\varepsilon > 0$ and $k > \|\psi\|_\infty$, let $v = u_n - T_\varepsilon(G_k(u_n))$ v belongs to $W_0^{1,p}(\Omega)$, and for ε small enough we obtain $v \geq \psi$. So the function v belongs to $\mathcal{K}_\psi$ and can be used as test function in (3.1), giving

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_\varepsilon(G_k(u_n))dx \leq \int_\Omega F \cdot \nabla T_\varepsilon(G_k(u_n))dx. \tag{3.2}$$

using assumption (2.2), It follows that

$$h^{p-1}(k+\varepsilon) \int_{\{k<|u_n|\leq k+\varepsilon\}} |\nabla u_n|^p dx \leq \int_{\{k<|u_n|\leq k+\varepsilon\}} F \cdot \nabla u_n dx.$$

Now, we can follow the same steps as in the proof of Theorem 5 [4] to obtain

$$\|u_n\|_\infty \leq H^{-1}\left(\frac{1}{NC_N^{\frac{1}{N}}}\|F\|^{p'}_{(L^m(\Omega))^N}\left(\int_0^{|\Omega|}\frac{d\sigma}{\sigma^{\frac{N-1}{N}\frac{pm}{pm-p'}}}\right)^{1-\frac{p'}{pm}}\right) := c_\infty, \tag{3.3}$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Now we prove that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. Let us consider ψ as test function in (3.1), then using (2.2) and (2.3) we get

$$\begin{aligned} h^{p-1}(c_\infty)\int_\Omega |\nabla u_n|^p dx - \beta\int_\Omega (j(x)+|T_n(u_n)|^{p-1}+|\nabla u_n|^{p-1}).\nabla\psi \\ \leq \int_\Omega F\cdot\nabla(u_n-\psi)dx. \end{aligned} \tag{3.4}$$

The second term on the left-hand-side of (3.4), is evaluated by using Young's inequality with $\nu > 0$. We will denote by $C_i, i = 1, 2\dots$ some generic constants

$$\begin{aligned} \beta\int_\Omega (j(x)+&|T_n(u_n)|^{p-1}+|\nabla u_n|^{p-1}).\nabla\psi \\ &\leq C_1+\beta\int_\Omega |\nabla u_n|^{p-1}.\nabla\psi \\ &\leq C_1+\beta\nu\int_\Omega |\nabla u_n|^p dx + C_2(\nu)\int_\Omega |\nabla\psi|^p dx \\ &\leq C_3(\nu)+\beta\nu\int_\Omega |\nabla u_n|^p dx. \end{aligned} \tag{3.5}$$

Concerning the term on the right-hand-side of (3.4)

$$\begin{aligned} \int_\Omega F\cdot\nabla(u_n-\psi)dx &\leq \int_\Omega |F|\cdot|\nabla u_n|dx + \int_\Omega |F|\cdot|\nabla\psi|dx \\ &\leq C_4(\nu)\int_\Omega |F|^{p'}dx + \frac{1}{p}\int_\Omega |\nabla\psi|^p dx + \nu\int_\Omega |\nabla u_n|^p dx. \end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we deduce that

$$\left(h^{p-1}(c_\infty)-\nu(\beta+1)\right)\int_\Omega |\nabla u_n|^p dx \leq C_5(\nu),$$

we can now choose ν such that $h^{p-1}(c_\infty) = 2\nu(\beta+1)$, to get

$$\frac{1}{2}h^{p-1}(c_\infty)\int_\Omega |\nabla u_n|^p dx \leq C_6.$$

The last inequality tells us that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$.

□

Lemma 3.2. *Assume that m satisfies (2.12), let $|F|$ belongs to $L^m(\Omega)$, and let u_n be a solution of (3.1). Then the norms of u_n in $L^r(\Omega)$ (where r is defined by (2.13))and in $W_0^{1,p}(\Omega)$ are bounded by a constants which depend on $\theta, m, N, |\Omega|$ and the norm of F in $L^m(\Omega)$.*

Proof. Let us start with the estimate in $L^r(\Omega)$. For $k \in \mathbb{N}$ with $k > \|\psi\|_{L^\infty(\Omega)}$, let $v = u_n - T_1(G_k(u_n))$ (which is an admissible test function). Choosing v as test function in (3.1), we have

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_1(G_k(u_n))dx \leq \int_\Omega F\cdot\nabla T_1(G_k(u_n))dx, \tag{3.7}$$

which gives

$$\int_{B_k} a(x, T_n(u_n), \nabla u_n)\cdot\nabla u_n dx \leq \int_{B_k} F\cdot\nabla u_n dx, \tag{3.8}$$

where we have set

$$B_k = \{x \in \Omega : k < |u_n| \leq k+1\}.$$

Hence,using (2.2) Hölder's inequality, we get

$$\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx \leq \int_{B_k} F \cdot \nabla u_n dx$$
$$\leq \left[\int_{B_k} |F|^{p'} dx\right]^{\frac{1}{p'}} \left[\int_{B_k} |\nabla u_n|^p dx\right]^{\frac{1}{p}},$$

and since h is decreasing, we obtain

$$\int_{B_k} |\nabla u_n|^p dx \leq (2+k)^{\theta p} \int_{B_k} |F|^{p'} dx. \tag{3.9}$$

If γ is a positive number (to be fixed later). Let $p^* = \frac{pN}{N-p}$ be the Sobolev conjugate exponent of p, by Sobolev's inequality and the previous inequality we can write

$$\begin{aligned}
\left[\int_\Omega |u_n|^{\gamma p^*} dx\right]^{\frac{p}{p^*}} &\leq C \int_\Omega |\nabla(|u_n|^\gamma)|^p dx \\
&= C \int_\Omega |u_n|^{p(\gamma-1)} |\nabla u_n|^p dx \\
&= C \sum_{k=0}^{\infty} \int_{B_k} |u_n|^{p(\gamma-1)} |\nabla u_n|^p dx \\
&\leq C \sum_{k=0}^{\infty} (1+k)^{p(\gamma-1)} \int_{B_k} |\nabla u_n|^p dx \\
&\leq C \sum_{k=0}^{\infty} (1+k)^{p(\gamma-1)} (2+k)^{\theta p} \int_{B_k} |F|^{p'} dx \\
&\leq C \sum_{k=0}^{\infty} (2+k)^{p(\theta+\gamma-1)} \int_{B_k} |F|^{p'} dx.
\end{aligned} \tag{3.10}$$

Therefore, using the fact that on the set B_k we have $|u_n| > k$ and then Hölder's inequality to get

$$\begin{aligned}
\left[\int_\Omega |u_n|^{\gamma p^*} dx\right]^{\frac{p}{p^*}} &\leq C \sum_{k=0}^{\infty} \int_{B_k} |F|^{p'} (2+|u_n|)^{p(\theta+\gamma-1)} dx \\
&= C \int_\Omega |F|^{p'} (2+|u_n|)^{p(\theta+\gamma-1)} dx \\
&\leq C \left[\int_\Omega |F|^m dx\right]^{\frac{p'}{m}} \left[\int_\Omega (2+|u_n|)^{\frac{pm(\theta+\gamma-1)}{m-p'}} dx\right]^{1-\frac{p'}{m}} \\
&\leq C \|F\|^{p'}_{(L^m(\Omega))^N} \left[1 + \int_\Omega |u_n|^{\frac{pm(\theta+\gamma-1)}{m-p'}} dx\right]^{1-\frac{p'}{m}}.
\end{aligned} \tag{3.11}$$

We now choose γ such that

$$\frac{pm(\theta+\gamma-1)}{m-p'} = \gamma p^*,$$

namely

$$\gamma = \frac{(1-\theta)(m(p-1))^*}{p^*}.$$

It turns out that $\gamma p^* = r$, so that from (3.11) we have

$$\left[\int_\Omega |u_n|^r dx\right]^{\frac{p}{p^*}} \leq C\|F\|^{p'}_{(L^m(\Omega))^N}\left[1+\int_\Omega |u_n|^r\, dx\right]^{1-\frac{p'}{m}},$$

since $1-\frac{p'}{m} < \frac{p}{p^*}$ (due to the fact that $m < \frac{N}{p-1}$), one has

$$\int_\Omega |u_n|^r dx \leq C,$$

therefore, u_n is bounded in $L^r(\Omega)$.

Now we turn to the estimates in $W_0^{1,p}(\Omega)$. On one hand we write

$$\int_\Omega |\nabla(|u_n|^\gamma)|^p dx = C\int_\Omega |u_n|^{p(\gamma-1)}|\nabla u_n|^p dx.$$

It's clear that $\gamma \geq 1$ since $m \geq \frac{p'N}{N-\theta(N-p)}$, thus

$$\int_{\{|u_n|>1\}} |\nabla u_n|^p dx \leq C\int_\Omega |u_n|^{p(\gamma-1)}|\nabla u_n|^p dx.$$

Hence, from (3.10) and (3.11) we obtain

$$\int_{\{|u_n|>1\}} |\nabla u_n|^p dx \leq C.$$

On the other hand,inequality (3.9), written for $k=0$, and then using the Hölder inequality implies

$$\int_{\{|u_n|\leq 1\}} |\nabla u_n|^p dx \leq 2^{\theta p}\|F\|^{p'}_{(L^m(\Omega))^N}|\Omega|^{1-\frac{p'}{m}}.$$

We conclude that

$$\int_\Omega |\nabla u_n|^p dx \leq C,$$

which completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume that m satisfies (2.14), let $|F|$ belongs to $L^m(\Omega)$, and let u_n be a solution of (3.1). Then, there exist two positive constants c_1, c_2 depending on Ω, N, F, θ and s, such that, for any $n \in \mathbb{N}$,*

$$\|u_n\|_{L^r(\Omega)} \leq c_1, \tag{3.12}$$

and

$$\|u_n\|_{W_0^{1,q}(\Omega)} \leq c_2, \tag{3.13}$$

where r and q are defined by (2.15) and (2.16) respectively. Moreover, the sequence $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$.

Proof. Let $k \in \mathbb{N}$ and $\tau \geq 0$. If we take $u_n - T_1(G_k(u_n))$ as test function in (3.1), and use hypothesis (2.2), we obtain

$$\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx \leq \int_{B_k} F \cdot \nabla u_n dx, \tag{3.14}$$

where we have set

$$B_k = \{x \in \Omega : k < |u_n| \leq k+1\},$$

hence, using Hölder's inequality, we get

$$\begin{aligned}\int_{B_k} h^{p-1}(|u_n|)|\nabla u_n|^p dx &\leq \int_{B_k} F \cdot \nabla u_n dx \\ &\leq \left[\int_{B_k} |F|^{p'} dx\right]^{\frac{1}{p'}} \left[\int_{B_k} |\nabla u_n|^p dx\right]^{\frac{1}{p}},\end{aligned}$$

and since h is decreasing, we obtain

$$\int_{B_k} |\nabla u_n|^p dx \leq (2+k)^{\theta p} \int_{B_k} |F|^{p'} dx. \tag{3.15}$$

On the other hand, let $\gamma > 0$ be a real number which will be chosen later

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} dx &= \sum_{k=0}^{+\infty} \int_{B_k} \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} dx \\ &\leq \sum_{k=0}^{+\infty} \frac{1}{(1+k)^\gamma} \int_{B_k} |\nabla u_n|^p dx\end{aligned}$$

using (3.15)

$$\begin{aligned}\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma} dx &\leq \sum_{k=0}^{\infty} \frac{(2+k)^{\theta p}}{(1+k)^\gamma} \int_{B_k} |F|^{p'} dx \\ &\leq 2^{\theta p} \sum_{k=0}^{\infty} (1+k)^{\theta p - \gamma} \int_{B_k} |F|^{p'} dx \\ &\leq 2^{\theta p} \sum_{k=0}^{\infty} \int_{B_k} |F|^{p'} (1+|u_n|)^{\theta p - \gamma} dx \\ &= 2^{\theta p} \int_\Omega |F|^{p'} (1+|u_n|)^{\theta p - \gamma} dx \\ &\leq 2^{\theta p} \|F\|^{p'}_{(L^m(\Omega))^N} \left[\int_\Omega (1+|u_n|)^{\frac{(\theta p - \gamma)m}{m-p'}} dx\right]^{1-\frac{p'}{m}}.\end{aligned} \tag{3.16}$$

Let r and q be as in (2.15) and (2.16). Then one can check that $r = q^* = Nq/(N-q)$. Therefore, using the Sobolev inequality and (3.16), we have

$$\begin{aligned}\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} &\le c\int_\Omega |\nabla u_n|^q dx = c\int_\Omega \frac{|\nabla u_n|^q}{(1+|u_n|)^{\frac{\gamma q}{p}}}(1+|u_n|)^{\frac{\gamma q}{p}}dx \\ &\le c\left[\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}dx\right]^{\frac{q}{p}}\left[\int_\Omega (1+|u_n|)^{\frac{\gamma q}{p-q}}dx\right]^{\frac{p-q}{p}} \\ &\le c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[\int_\Omega (1+|u_n|)^{\frac{(\theta p-\gamma)m}{m-p'}}dx\right]^{(1-\frac{p'}{m})\frac{q}{p}}\left[\int_\Omega (1+|u_n|)^{\frac{\gamma q}{p-q}}dx\right]^{\frac{p-q}{p}}.\end{aligned} \tag{3.17}$$

We choose now γ such that $\frac{\lambda q}{p-q} = r$, that is

$$\gamma = \frac{pN - m(p-1)(N-\theta(N-p))}{N-m(p-1)}. \tag{3.18}$$

Thanks to the choice of γ in (3.18), we get

$$\frac{(\theta p-\gamma)m}{m-p'} = r.$$

Consequently, we get

$$\begin{aligned}\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} &\le c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[\int_\Omega (1+|u_n|)^r dx\right]^{1-\frac{p'q}{mp}} \\ &\le c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[|\Omega| + \int_\Omega |u_n|^r dx\right]^{1-\frac{p'q}{mp}}.\end{aligned} \tag{3.19}$$

Since $m < \frac{N}{p-1}$, the last exponent in (3.19) is smaller than $\frac{q}{r}$. Therefore, we obtain

$$\left[\int_\Omega |u_n|^r dx\right]^{\frac{q}{r}} \le c2^{\theta q}\|F\|^{\frac{qp'}{p}}_{(L^m(\Omega))^N}\left[|\Omega| + \int_\Omega |u_n|^r dx\right]^{\frac{q}{r}}. \tag{3.20}$$

Thus, we conclude the boundedness of u_n in $L^r(\Omega)$. Going back to (3.17), this in turn implies an estimate for the norm of u_n in $W_0^{1,q}(\Omega)$.

To prove the last part of Lemma 3.3, we take $u_n - T_k(u_n-\psi)$ as test function in (3.1) to get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx \le \int_\Omega F\cdot\nabla T_k(u_n-\psi)dx. \tag{3.21}$$

Since we are on the set $\{|u_n-\psi|\le k\}$, we set $M = k + \|\psi\|_{L^\infty(\Omega)}$. For $n > M$, we have

$$\begin{aligned}\int_\Omega a(x, T_n(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx &= \int_\Omega a(x, T_M(u_n), \nabla u_n)\cdot\nabla T_k(u_n-\psi)dx \\ &= \int_{\{|u_n-\psi|\le k\}} a(x, T_M(u_n), \nabla u_n)\cdot\nabla(u_n-\psi)dx.\end{aligned}$$

using (2.2) in (3.21), we obtain

$$h^{p-1}(M)\int_{\{|u_n-\psi|\le k\}}|\nabla u_n)|^p dx-\int_{\{|u_n-\psi|\le k\}}a(x,T_M(u_n),\nabla u_n)\cdot\nabla\psi dx \le \int_{\{|u_n-\psi|\le k\}}F\cdot\nabla(u_n-\psi)dx. \tag{3.22}$$

and therefore, using (2.3), Höder's and Young's inequalities, one can estimate the second term on the left-hand side

$$\begin{aligned}\int_{\{|u_n-\psi|\le k\}}a(x{,}T_M(u_n),\nabla u_n)\cdot\nabla\psi dx \\ &\le \int_{\{|u_n-\psi|\le k\}}\beta(j(x)+|T_M(u_n)|^{p-1}+|\nabla u_n|^{p-1})\cdot|\nabla\psi|dx \\ &\le C+\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^{p-1}\cdot|\nabla\psi|dx \\ &\le C+\frac{1}{p'}\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p.\end{aligned}$$

For the term in the right-hand side, since $|F|$ belongs to $L^{p'}(\Omega)$ and using höder's and Young's inequalities, we obtain

$$\begin{aligned}\int_{\{|u_n-\psi|\le k\}}F\cdot\nabla(u_n-\psi)dx &\le \int_{\{|u_n-\psi|\le k\}}|F||\nabla u_n|dx+\int_{\{|u_n-\psi|\le k\}}|F||\nabla\psi|dx \\ &\le C+\frac{1}{p}\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p.\end{aligned}$$

therefore

$$\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p\le C,$$

replacing k with $k+\|\psi\|_\infty$ in the last inequality, we get

$$\int_{\{|u_n|\le k\}}|\nabla u_n|^p\le\int_{\{|u_n-\psi|\le k\}}|\nabla u_n|^p\le C. \tag{3.23}$$

So that the Lemma 3.3 is completely proved. □

4. Proof of the results

4.1. **Proof of theorem 2.1.** The Lemma 3.1 guarantees the existence of a subsequence, still denoted by $\{u_n\}$, and a measurable function u such that

$$u_n\to u \quad \text{and a.e. in } \Omega, \tag{4.1}$$

and

$$u_n\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{4.2}$$

Since $u_n(x)\ge\psi(x)$ a.e. in Ω, $\forall n\in\mathbb{N}$

$$u\ge\psi \quad \text{a.e. in } \Omega. \tag{4.3}$$

Moreover, we shall prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{4.4}$$

In order to get (4.4), it is sufficient to show that

$$u_n \to u \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{4.5}$$

We write

$$\begin{aligned} &\int_\Omega (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)) \cdot (\nabla u_n - \nabla u) dx \\ &\quad = \int_\Omega a(x,u_n,\nabla u_n) \cdot (\nabla u_n - \nabla u) dx - \int_\Omega a(x,u_n,\nabla u) \cdot (\nabla u_n - \nabla u) dx. \end{aligned} \tag{4.6}$$

For $n > c_\infty$, if we take u as test function in (3.1), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla(u_n - u) dx \le \int_\Omega F \cdot \nabla(u_n - u) dx. \tag{4.7}$$

Since F belongs at least to $\left(L^{p'}(\Omega)\right)^N$, the term -div F is in $W^{-1,p'}(\Omega)$, so that by (4.2) we obtain

$$\lim_{n\to\infty} \int_\Omega F \cdot (\nabla u_n - \nabla u)\, dx = 0, \tag{4.8}$$

this and (4.7) yield

$$\lim_{n\to\infty} \int_\Omega a\left(x,u_n,\nabla u_n\right) \cdot \left(\nabla u_n - \nabla u\right) dx \le 0. \tag{4.9}$$

In view of the growth assumption (2.3) and vitali's theorem we have

$$a(x,u_n,\nabla u) \to a(x,u,\nabla u) \quad \text{strongly in } \left(L^{p'}(\Omega)\right)^N. \tag{4.10}$$

It follows by (4.2) that

$$\lim_{n\to\infty} \int_\Omega a(x,u_n,\nabla u) \cdot (\nabla u_n - \nabla u)\, dx = 0 \tag{4.11}$$

Therefore, we obtain

$$\lim_{n\to\infty} \int_\Omega \left(a\left(x,u_n,\nabla u_n\right) - a\left(x,u_n,\nabla u\right)\right) \cdot \left(\nabla u_n - \nabla u\right) dx = 0. \tag{4.12}$$

Using Lemma 5 in [9], we get (4.5). Also we have

$$\nabla u_n \to \nabla u \quad \text{strongly in } (L^p(\Omega))^N \text{ and a.e. in } \Omega. \tag{4.13}$$

We can now pass to the limit. Let $v \in \mathcal{K}_\psi$, choosing v as test function in (3.1). For every $n > c_\infty$ one has

$$\int_\Omega a(x,u_n,\nabla u_n) \cdot \nabla(u_n - v) dx \le \int_\Omega F \cdot \nabla(u_n - v) dx. \tag{4.14}$$

The right-hand side converges, as n tends to infinity, to

$$\int_\Omega F \cdot \nabla(u - v) dx. \tag{4.15}$$

In view of Fatou's Lemma, we obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla u dx \le \liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n dx. \tag{4.16}$$

Thanks to (4.13) and Vitali's theorem, we get

$$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) \quad \text{strongly in } (L^{p'}(\Omega))^N,$$

thus

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v dx \longrightarrow \int_\Omega a(x,u,\nabla u)\cdot\nabla v dx. \tag{4.17}$$

Consequently, by combining (4.15)–(4.17), it is then possible to pass to the limit as n tends to infinity in (4.14) to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla(u-v)dx \le \int_\Omega F\cdot\nabla(u-v)dx.$$

4.2. **Proof of theorem 2.3.** Under the hypotheses of Theorem 2.3, we have the following results.

Lemma 4.1. *Let u_n be a solution of (3.1). Then there exists a measurable function u such that*

$$u_n\to u \quad \textit{in measure}, \tag{4.18}$$

$$u_n\to u \quad \textit{a.e. in } \Omega, \tag{4.19}$$

and

$$T_k(u_n)\rightharpoonup T_k(u) \quad \textit{weakly in } W_0^{1,p}(\Omega) \textit{ for every } k>0. \tag{4.20}$$

Proof. Firstly, we show that u_n is a Cauchy sequence in measure. Let $\eta\ge 0$, if we use as test function $v=u_n-\eta T_k(u_n)$ in (3.1) which is an admissible test function for η small enough, we get

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u_n)dx \le \int_\Omega F\cdot\nabla T_k(u_n)dx.$$

For $n>k$, we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx \le \int_\Omega F\cdot\nabla T_k(u_n)dx,$$

using (2.2) gives

$$\int_\Omega h^{p-1}(|T_k(u_n)|)|\nabla T_k(u_n)|^p dx \le \int_\Omega F\cdot\nabla T_k(u_n)dx,$$

being h deacrising function and using Hölder's inequality, it follows

$$h^{p-1}(k)\int_\Omega |\nabla T_k(u_n)|^p dx \le \left[\int_\Omega |F|^{p'}dx\right]^{\frac{1}{p'}}\left[\int_\Omega |\nabla T_k(u_n)|^p dx\right]^{\frac{1}{p}},$$

that is

$$\int_\Omega |\nabla T_k(u_n)|^p dx \le \frac{1}{h^p(k)}\|F\|^{p'}_{L^{p'}(\Omega)}. \tag{4.21}$$

We use now Sobolev inequality with exponents p and p^* ($p^* = \frac{Np}{N-p}$), so there exist a constant C which doesn't depending on n such that

$$\|T_k(u_n)\|^p_{L^{p^*}(\Omega)} \leq C\|\nabla T_k(u_n)\|^p_{L^p(\Omega)},$$

hence, by (4.21) we deduce

$$\int_\Omega |T_k(u_n)|^{p^*} dx \leq \frac{C}{h^{p^*}(k)} \|F\|^{\frac{p^*}{p-1}}_{L^{p'}(\Omega)}.$$

Since the set $\{x \in \Omega : |u_n(x)| > k\} \subset \Omega$, we have

$$|\{|u_n| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|^{\frac{p^*}{p-1}}_{L^{p'}(\Omega)}. \tag{4.22}$$

For t and $\varepsilon > 0$ we can write

$$\{|u_n - u_m| > t\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > t\}. \tag{4.23}$$

Since $\lim_{k\to\infty} \frac{1}{kh(k)} = 0$, for sufficiently large k, by inequality (4.22)

$$|\{|u_n| > k\}| < \frac{\varepsilon}{3} \quad \text{and} \quad |\{|u_m| > k\}| < \frac{\varepsilon}{3}. \tag{4.24}$$

By (4.21), the sequence $\{|T_k(u_n)|\}$ is bounded in $W_0^{1,p}(\Omega)$, we can assume that it is a Cauchy sequence in measure. Thus the existence of some N such that for $n, m \geq N$, we have

$$|\{|T_k(u_n) - T_k(u_m)| > t\}| < \frac{\varepsilon}{3}. \tag{4.25}$$

Therefore

$$|\{|u_n - u_m| > t\}| < \varepsilon, \quad \forall n, m \geq N. \tag{4.26}$$

Then u_n is a Cauchy sequence in measure, thus there exists a subsequence still denoted u_n which converge almost everywhere to some measurable function u, and from (4.21) it follows that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ for every } k > 0.$$

□

Lemma 4.2. *Let u_n be a solution of (3.1). Then*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{a.e. in } \Omega. \tag{4.27}$$

To prove the precedent lemma, it is sufficient to show that

$$\nabla u_n \longrightarrow \nabla u \quad \text{in measure}. \tag{4.28}$$

For this purpose, we will need the following estimates:

Lemma 4.3. *There exists a constant $C > 0$ such that, for every $k > 0$, we have*

$$|\{|u| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}}, \tag{4.29}$$

$$|\{|\nabla u_n| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}} + \frac{1}{k^p h^p(k)} \|F\|_{L^{p'}(\Omega)}^{p'}, \tag{4.30}$$

and

$$|\{|\nabla u| > k\}| \leq \frac{C}{k^{p^*} h^{p^*}(k)} \|F\|_{L^{p'}(\Omega)}^{\frac{p^*}{p-1}} + \frac{1}{k^p h^p(k)} \|F\|_{L^{p'}(\Omega)}^{p'}. \tag{4.31}$$

Proof. The proof is entirely similar to the corresponding one of Lemma 3 in [4]. □

Proof of Lemma 4.2. Let $\lambda > 0$, we set for some $k > 0$ and $\varepsilon > 0$

$$E_1 = \{x \in \Omega : |u_n| > k\} \cup \{x \in \Omega : |u| > k\} \cup \{x \in \Omega : |\nabla u_n| > k\} \cup \{x \in \Omega : |\nabla u| > k\},$$

$$E_2 = \{x \in \Omega : |u_n - u| > \varepsilon\},$$

$$E_3 = \{x \in \Omega : |u_n - u| \leq \varepsilon, |u_n| \leq k, |u| \leq k, |\nabla u_n| \leq k, |\nabla u| \leq k, |\nabla u_n - \nabla u| \geq \lambda\}.$$

We have that:

$$\{x \in \Omega : |\nabla u_n - \nabla u| \geq \lambda\} \subset E_1 \cup E_2 \cup E_3. \tag{4.32}$$

Fixed $\sigma > 0$, we will prove that there exists N such that

$$|\{x \in \Omega : |\nabla u_n - \nabla u| \geq \lambda\}| \leq \sigma \quad \text{for avery } n \geq N. \tag{4.33}$$

By virtue of the inequalities (4.22), (4.29), (4.30) and (4.31) and using the fact that $\lim_{k\to\infty} \frac{1}{kh(k)} = 0$, we can conclude that there exists some k_σ such that

$$|E_1| \leq \frac{\sigma}{3} \quad \text{for all } n \text{ and } k \geq k_\sigma. \tag{4.34}$$

On the other hand, since the sequence u_n is convergent in measure to u, guarantees that there exists N_1 such that

$$|E_2| \leq \frac{\sigma}{3}, \quad \text{for all } n \geq N_1. \tag{4.35}$$

Now, we prove that there exists N_2 depending on k and σ, and verifying

$$|E_3| \leq \frac{\sigma}{3}, \quad \text{for all } n \geq N_2. \tag{4.36}$$

In order to get (4.36) we need the following standard lemma:

Lemma 4.4. *Let (X, T, m) a measurable space, such that $|X| < +\infty$. Let γ be a measurable function $\gamma : X \to [0, +\infty[$ such that $|\{x \in X : \gamma(x) = 0\}| = 0$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that:*

$$\int_A \gamma \mathrm{d}m \leq \delta \Rightarrow |A| \leq \sigma.$$

Assumption (2.4) implies that there exists a real valued function $\gamma : \Omega \to [0, +\infty[$ such that

$$|\{x \in \Omega : \gamma(x) = 0\}| = 0, \tag{4.37}$$

and

$$(a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) \geq \gamma(x) \quad \text{a.e. in } \Omega, \tag{4.38}$$

for every $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N$ such that $|s| \leq k$, $|\xi| \leq k$, $|\eta| \leq k$ and $|\xi - \eta| \geq \lambda$. Thus, we get

$$\int_{E_3} \gamma(x) dx \leq \int_{E_3} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u) dx,$$

in E_3 we have $|u| \leq k$ and $|u_n - u| \leq \varepsilon$, so that

$$\begin{aligned}
\int_{E_3} \gamma(x) dx &\leq \int_{E_3} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_k(u))) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx \\
&\leq \int_{\Omega} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla T_k(u))) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx \\
&= \int_{\Omega} a\left(x, u_n, \nabla u_n\right) \cdot \nabla T_\varepsilon\left(u_n - T_k(u)\right) dx \\
&\quad - \int_{\Omega} a\left(x, u_n, \nabla T_k(u)\right) \cdot \nabla T_\varepsilon\left(u_n - T_k(u)\right) dx,
\end{aligned}$$

We evaluate the term

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx. \tag{4.39}$$

Let $v = u_n - \tau T_\varepsilon(u_n - T_k(u))$ where $\tau \geq 0$. For τ small enough, v is an admissible test function in (3.1). It follows that

$$\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx \leq \int_{\Omega} F \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx,$$

thus, for $n > k + \varepsilon$ we have

$$\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx \leq \int_{\Omega} F \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx,$$

passing to the limit as n tends to infinity

$$\begin{aligned}
\lim_{n \to \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx &\leq \int_{\Omega} F \cdot \nabla T_\varepsilon(u - T_k(u)) dx \\
&= \int_{k < |u| \leq k + \varepsilon} F \cdot \nabla T_{k+\varepsilon}(u) dx,
\end{aligned}$$

therfore

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx \leq 0. \tag{4.40}$$

Concerning the second integral

$$\int_{\Omega} a(x, u_n, \nabla T_k(u_n)) \cdot \nabla T_\varepsilon(u_n - T_k(u)) dx. \tag{4.41}$$

Since we have the inclusion

$$\{x \in \Omega : |u_n - T_k(u)| \leq \varepsilon\} \subset \{x \in \Omega : |u_n| \leq k + \varepsilon\},$$

we can write

$$\int_\Omega a(x, u_n, \nabla T_k(u_n)) \cdot \nabla T_\varepsilon(u_n - T_k(u))dx$$
$$= \int_{\{|u_n - T_k(u)| \le \varepsilon\}} a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u_n)) \cdot \nabla(T_{k+\varepsilon}(u_n) - T_k(u))dx.$$

By (4.20) and using Vitali's theorem we can conclude that $\forall h, k > 0$

$$a(x, T_h(u_n), \nabla T_k(u)) \to a(x, T_h(u), \nabla T_k(u)) \quad \text{strongly in } (L^{p'}(\Omega))^N.$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \int_{\{|u_n - T_k(u)| \le \varepsilon\}} & a(x, T_{k+\varepsilon}(u_n), \nabla T_k(u_n)) \cdot \nabla(T_{k+\varepsilon}(u_n) - T_k(u))dx \\
&= \int_{\{|u - T_k(u)| < \varepsilon\}} a(x, T_{k+\varepsilon}(u), \nabla T_k(u)) \cdot \nabla(T_{k+\varepsilon}(u) - T_k(u))dx \\
&= \int_{\{k < |u| \le k+\varepsilon\}} a(x, u, \nabla T_k(u)) \cdot \nabla T_{k+\varepsilon}(u)dx \\
&= \int_{\{k < |u| \le k+\varepsilon\}} a(x, u, 0) \cdot \nabla T_{k+\varepsilon}(u)dx \\
&= 0.
\end{aligned}$$

Hence, there exists N_2 (dependent on k and σ) such that

$$\int_{E_3} \gamma(x)dx \le \frac{\delta}{3} \quad \text{for every } n \ge N_2.$$

By Lemma 4.3, we conclude that

$$|E_3| \le \frac{\sigma}{3}, \quad \text{for all } n \ge N_2. \tag{4.42}$$

We take $N = \max(N_1, N_2)$, then

$$|\{x \in \Omega : |\nabla u_n - \nabla u| \ge \lambda\}| \le \sigma \quad \text{for every } n \ge N.$$

this completes the proof of Lemma 4.2. □

We can now pass to the limit in (3.1) to obtain a solution of (2.10). Let $v \in \mathcal{K}_\psi \cap L^\infty(\Omega)$; the function $u_n - T_k(u_n - v)$ is an admissible test function in (3.1). This choice yields

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - v)dx \le \int_\Omega F \cdot \nabla T_k(u_n - v)dx, \tag{4.43}$$

Taking into account the fact that the integral in (4.43) is on the subset of the set

$$\{x \in \Omega, |u_n| \le k + \|v\|_{L^\infty(\Omega)}\},$$

we set $M = k + \|v\|_{L^\infty(\Omega)}$ and let $n > M$, we can rewrite (4.43) as

$$\int_{\{|u_n - v| \le k\}} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot \nabla T_k(u_n - v)dx \le \int_{\{|u_n - v| \le k\}} F \cdot \nabla T_k(u_n - v)dx. \tag{4.44}$$

The right-hand side of (4.44) converge to

$$\int_{\{|u-v|\leq k\}} F\cdot\nabla(u-v)dx. \tag{4.45}$$

In view of Fatou's Lemma and (4.18) and (4.27), we obtain

$$\int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla u dx \leq \liminf_{n\to\infty}\int_{\{|u_n-v|\leq k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla u_n dx. \tag{4.46}$$

Since $T_M(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, so as a consequence of (2.3), $|a(x,T_M(u_n),\nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega)$. Thus together with (4.19) and (4.27), $a(x,T_M(u_n),\nabla T_M(u_n))$ is weakly convergent to $a(x,T_M(u),\nabla T_M(u))$ in $(L^{p'}(\Omega))^N$. Therefore

$$\int_{\{|u_n-v|\leq k\}} a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla v dx \longrightarrow \int_{\{|u-v|\leq k\}} a(x,u,\nabla u)\cdot\nabla v dx. \tag{4.47}$$

From (4.45)–(4.47), it is then possible to pass to the limit as n tends to infinity in (4.44) to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \leq \int_\Omega F\cdot\nabla T_k(u-v)dx.$$

4.3. **Proof of theorem 2.4.** In Order to prove Theorem 2.4 we need the following convergence

Lemma 4.5. *let u_n be a solution of (3.1). Then*

$$T_k(u_n)\longrightarrow T_k(u) \quad \textit{strongly in } W_0^{1,p}(\Omega) \textit{ for all } k>0. \tag{4.48}$$

Proof. from Lemma 3.2, we have that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, so that

$$T_k(u_n)\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{4.49}$$

Observe that for $n>k$, one has

$$\begin{aligned}
&\int_\Omega \left(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\right)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad- \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\quad= \int_{\{|u_n|<k\}} a(x,T_n(u_n),\nabla u_n)\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx\\
&\qquad- \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla\left(T_k(u_n)-T_k(u)\right)dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx \\
&\quad - \int_{\{|u_n| \geq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx \\
&\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx \\
&= \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx \\
&\quad - \int_{\{|u_n| \geq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) dx \\
&\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx.
\end{aligned}
$$

We shall prove that the previous integral converges to zero. Indeed, on one hand, by choosing $v = u_n - (T_k(u_n) - T_k(u))$ as a test function in (3.1), which is an admissible test function, we obtain

$$
\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u)) dx \leq \int_\Omega F \cdot \nabla(T_k(u_n) - T_k(u)) dx. \tag{4.50}
$$

Thanks to (4.49), we have

$$
\lim_{n\to\infty} \int_\Omega F \cdot \nabla(T_k(u_n) - T_k(u)) dx = 0.
$$

Therefore,

$$
\lim_{n\to\infty} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u)) dx \leq 0. \tag{4.51}
$$

On the other hand, By the growth assumption (2.3), we get

$$
\begin{aligned}
\int_\Omega |a(x, T_n(u_n), \nabla u_n)|^{p'} dx &\leq \beta c \int_\Omega (j^{p'}(x) + |u_n|^p + |\nabla u_n|^p) dx \\
&\leq \beta c \left(\|j\|^{p'}_{L^{p'}(\Omega)} + \|u_n\|^p_{L^p(\Omega)} + \|u_n\|^p_{W_0^{1,p}(\Omega)} \right).
\end{aligned}
$$

therefore the sequence $\{a(x, T_n(u_n), \nabla u_n)\}$ is bounded in $(L^{p'}(\Omega))^N$. Then, it converges weakly to some l in $(L^{p'}(\Omega))^N$ and we obtain

$$
\lim_{n\to\infty} \int_{\{|u_n| \geq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) dx = \int_{\{|u| \geq k\}} l \cdot \nabla T_k(u) dx = 0. \tag{4.52}
$$

By virtue of Lemma 3.2 and Vitali's theorem, we obtain

$$
a(x, T_n(u_n), \nabla T_k(u)) \to a(x, u, \nabla T_k(u)) \text{ strongly in } \left(L^{p'}(\Omega)\right)^N,
$$

it follows from (4.49) that

$$
\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx = 0, \tag{4.53}
$$

which with (4.51) and (4.52) allow us to get

$$
\lim_{n\to\infty} \int_\Omega \left(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))\right) \cdot \nabla\left(T_k(u_n) - T_k(u)\right) dx = 0. \tag{4.54}
$$

Now we can apply Lemma 5 of [9] to conclude (4.48) □

From the precedent lemma, we also have

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \tag{4.55}$$

We are now in position to prove Theorem 2.4. Choosing $v \in \mathcal{K}_\psi$ as test function in (3.1), we get

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(u_n - v)dx \leq \int_\Omega F \cdot \nabla(u_n - v)dx. \tag{4.56}$$

The right-hand side of (4.56) converges as n tends to infinity to

$$\int_\Omega F \cdot \nabla(u - v)dx. \tag{4.57}$$

In view of Fatou's Lemma, we obtain

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla u dx \leq \liminf_{n \to \infty} \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx. \tag{4.58}$$

Combining (4.55) and the assumptions on the function $a(x, s, \xi)$, we have

$$a(x, T_n(u_n), \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } (L^{p'}(\Omega))^N.$$

Thus

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla v dx \longrightarrow \int_\Omega a(x, u, \nabla u) \cdot \nabla v dx. \tag{4.59}$$

Then we passing to the limit thanks to the previous results, we prove the Theorem 2.4.

4.4. **Proof of theorem 2.5.** In virtue of Lemma 3.3, there exists a subsequence, still denoted by u_n, which is weakly convergent to some function u in $W_0^{1,q}(\Omega) \cap L^r(\Omega)$. Moreover,

$$u_n \to u \quad \text{a.e. in } \Omega. \tag{4.60}$$

From (3.23), $T_k(u_n)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, which implies together with (4.60) that

$$T_k(u_n) \to T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{4.61}$$

Lemma 4.6. *Let u_n be a sequence of solutions of the problems (3.1) with the same assumptions as in the statement of Theorem 2.5. Then there exists a subsequence, denoted by u_n such that*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{a.e. in } \Omega. \tag{4.62}$$

Proof. Fix λ such that $1 < \lambda < \frac{q}{p}$, define

$$I_{n,\Omega} = \int_\Omega \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx. \tag{4.63}$$

We shall prove that the integral $I_{n,\Omega}$ converges to zero. We split it on the sets

$$C_k = \{x \in \Omega : |u(x)| > k\} \quad \text{and} \quad \bar{C}_k = \{x \in \Omega : |u(x)| \leq k\},$$

to get

$$\begin{aligned}I_{n,\Omega} =& \int_{C_k} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx \\ &+ \int_{\bar{C}_k} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla u))(\nabla u_n - \nabla u)\}^\lambda dx \\ =& I_{n,C_k} + I_{n,\bar{C}_k}.\end{aligned}$$

Let $\gamma = \frac{q}{p\lambda}$, using assumption (2.3) then Hölder's inequality with exponents γ and γ' in the first integral I_{n,C_k}, one has

$$\begin{aligned}I_{n,C_k} \leq& c\left(\int_\Omega j^{p'}(x)dx\right)^\lambda |C_k|^{1-\lambda} \\ &+ c\left(\int_{C_k} (|u_n| + |\nabla u_n| + |\nabla u|)^q dx\right)^{\frac{1}{\gamma}} |C_k|^{1-\frac{1}{\gamma}}.\end{aligned}$$

thus, by means of estimate (3.12), we get

$$I_{n,C_k} \leq c(|C_k|^{1-\lambda} + |C_k|^{1-\frac{1}{\gamma}}),$$

we thus obtain

$$\lim_{k\to\infty} \lim_{n\to\infty} I_{n,C_k} = 0. \tag{4.64}$$

Concerning the second integral $I_{n,\bar{C}_k}$, we have

$$\begin{aligned}I_{n,\bar{C}_k} &= \int_{\bar{C}_k} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &\leq \int_\Omega \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &= J_{n,\Omega}.\end{aligned}$$

Again, we split the integral $J_{n,\Omega}$ on the sets

$$D_{k,l} = \{x \in \Omega : |u_n - T_k(u)| > l\}, \quad \bar{D}_{k,l} = \{x \in \Omega : |u_n - T_k(u) \leq l\}, \quad (l \in \mathbb{N}),$$

obtaining

$$\begin{aligned}J_{n,\Omega} =& \int_{D_{k,l}} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ &+ \int_{\bar{D}_{k,l}} \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u)))(\nabla u_n - \nabla T_k(u))\}^\lambda dx \\ =& J_{n,D_{k,l}} + J_{n,\bar{D}_{k,l}}.\end{aligned}$$

the measure of the set $D_{k,l}$ tends to zero as l tends to ∞ uniformly in n and k

$$\lim_{l\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} J_{n,D_{k,l}} = 0. \tag{4.65}$$

Since on $\bar{D}_{k,l}$ we can write $\nabla(u_n - T_k(u)) = \nabla T_l(u_n - T_k(u))$, we have

$$J_{n,\bar{D}_{k,l}} = \int_\Omega \{(a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u))) \cdot \nabla T_l(u_n - T_k(u))\}^\lambda dx$$

therefore, using the Hölder inequality, we get

$$\begin{aligned} J_{n,\bar{D}_{k,l}} &\leq |\Omega|^{1-\lambda} \left[\int_\Omega (a(x, T_n(u_n), \nabla u_n) - a(x, T_n(u_n), \nabla T_k(u))) \cdot \nabla T_l(u_n - T_k(u)) dx \right]^\lambda \\ &= |\Omega|^{1-\lambda} (J_{n,\bar{D}_{k,l},1} - J_{n,\bar{D}_{k,l},2})^\lambda, \end{aligned}$$

where

$$J_{n,\bar{D}_{k,l},1} = \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_l(u_n - T_k(u)) dx,$$

and

$$J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_n(u_n), \nabla T_k(u)) \cdot \nabla T_l(u_n - T_k(u)) dx.$$

The integral in $J_{n,\bar{D}_{k,l},2}$ is on the set $\{|u_n - T_k(u)| \leq l\}$, which is a subset of the set $\{|u_n| \leq l + k\}$; hence, if we take $n \geq l + k := M$, we get

$$J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_M(u_n), \nabla T_k(u)) \cdot \nabla T_l(u_n - T_k(u)) dx.$$

Using the almost everywhere convergence (4.60) and the Vitali theorem we get

$$a(x, T_M(u_n), \nabla T_k(u)) \to a(x, T_M(u), \nabla T_k(u)) \quad \text{strongly in } (L^{p'}(\Omega))^N.$$

As consequence of (4.60) and(4.61), we have

$$\nabla T_l(u_n - T_k(u)) \rightharpoonup \nabla T_l(u - T_k(u)) \quad \text{weakly in } (L^{p'}(\Omega))^N, \tag{4.66}$$

so that

$$\lim_{n\to\infty} J_{n,\bar{D}_{k,l},2} = \int_\Omega a(x, T_M(u), \nabla T_k(u)) \cdot \nabla T_l(u - T_k(u)) dx = 0. \tag{4.67}$$

To evaluate the integral $J_{n,\bar{D}_{k,l},1}$. Let $\tau \geq 0$, we choose $u_n - \tau T_l(u_n - T_k(u))$ as test function in (3.1), to get

$$J_{n,\bar{D}_{k,l},1} \leq \int_\Omega F \cdot \nabla T_l(u_n - T_k(u)) dx.$$

using (4.66)

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega F \cdot \nabla T_l(u_n - T_k(u)) dx &= \int_\Omega F \cdot \nabla T_l(u - T_k(u)) dx \\ &= \int_{\{|u|>k\}} F \cdot \nabla T_l(u) dx. \end{aligned}$$

Therefore

$$\lim_{l\to\infty} \lim_{k\to\infty} \lim_{n\to\infty} J_{n,\bar{D}_{k,l},1} = 0. \tag{4.68}$$

Gathering results (4.64), (4.65), (4.68) and (4.67), we obtain

$$\lim_{n\to\infty} I_n = 0.$$

Since the integrand function in I_n is non-negative, we have

$$\|\{(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))(\nabla u_n-\nabla u)\}^\lambda\|_{L^1(\Omega)}\to 0,$$

thus,there exist a subsequence still denoted by u_n,

$$(a(x,T_n(u_n),\nabla u_n)-a(x,T_n(u_n),\nabla u))(\nabla u_n-\nabla u)\to 0.$$

Under our assumption on the function $a(x,s,\xi)$ and the previous limit, we conclude (4.62) as in [9]. □

Let $v\in\mathcal{K}_\psi\cap L^\infty(\Omega)$, and choose $u_n-T_k(u_n-v)$ as test function in (3.1). We get

$$\int_\Omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla(u_n-v)dx\le\int_\Omega F\cdot\nabla(u_n-v)dx. \tag{4.69}$$

Finally, we can pass to the limit in (4.69), this completes the proof of theorem 2.5.

References

[1] A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, Ann. Mat. Pura Appl. IV. Ser. 182 (2003), 53–79. https://doi.org/10.1007/s10231-002-0056-y.

[2] H. Ayadi, R. Souilah, Existence and regularity results for unilateral problems with degenerate coercivity, Math. Slovaca. 69 (2019), 1351–1366. https://doi.org/10.1515/ms-2017-0313.

[3] P. Bélinan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vasquez, An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 22 (1995), 241–273.

[4] A. Benkirane, A. Youssfi, Regularity for solutions of nonlinear elliptic equations with degenerate coercivity, Ricerche Mat. 56 (2007), 241–275. https://doi.org/10.1007/s11587-007-0019-0.

[5] L. Boccardo, R. Cirmi, Existence and uniqueness of solution of unilateral problems with L^1 data, J. Convex Anal. 6 (1999), 195–206.

[6] L. Boccardo, R. Cirmi, Unilateral problems with degenerate coercivity, Le Matematiche, 54 (1999), 61-73.

[7] L. Boccardo, D. Giachetti, existence results via regularity for some nonlinear elliptic problems, Commun. Part. Diff. Equ. 14 (1989), 663–680. https://doi.org/10.1080/03605308908820624.

[8] L. Boccardo, D. Giachetti, Alcune osservazioni sulla regolarita delle soluzioni di problemi fortemente non linearie applicazioni, Ricerche Mat. 34 (1985), 309–323.

[9] L. Boccardo, F. Murat, J.P. Puel, Existence of bounded solutions for non linear elliptic unilateral problems, Ann. Mat. Pura Appl. 152 (1988), 183–196. https://doi.org/10.1007/bf01766148.

[10] L. Boccardo, A. Dall'Aglio, L. Orsina, Existence and regularity results for some nonlinear equations with degenerate coercivity, Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 51–81.

[11] J.L. Lions, Quelques méthodes de resolution de problèmes aux limites non linéaires, Dunod Gauthier-Villars, Paris, 1969.