OPTIMAL CONTROL OF DROUGHTS, DISEASES AND RETALIATORY KILLING IN PREY-PREDATOR SYSTEMS

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Abstract. In this article, we describe an application of optimal control theory to evaluate the success of controls on the dynamics of prey-predator systems of the Serengeti ecosystem, which includes wildebeest, zebra, and lion. This is accomplished by suggesting control variables like education (to prevent retaliatory killing), dam construction (to prevent drought), and treatment (to combat infections). Maximizing population density is the major objective. For this aim, the Pontryagin's maximum principle has been applied. The optimal control are characterized in terms of optimality system and solved numerically for several scenarios. Results shows that multiple optimal control measures is the most effective strategy in management of wildlife populations. Results also shows that, if the ecosystem management decide to use a single control, the construction of dams is the best control in maximizing the objective function.

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1. Introduction

The core of mathematical ecology is the dynamics and interaction of species in the ecosystem, with their complex behavior being a problem for many biological and ecological processes. All organism in the ecosystem are interdependent [1], hence studying their behaviour and dynamics is vital for scientific management of the ecosystem [9]. Serengeti ecosystem is extraordinary in species and is one of the most well-known and remarkable wildlife reserves in the world [7]. However, numerous biological species in the Serengeti ecosystem have perished as a result of outside factors like extinction, overexploitation, illness, droughts, and poor ecosystem management. [11].

The ecosystem also has been the subject of various hazards such as pollution, fire, drought and

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catastrophes that lead to perturbations of the ecosystem. Some of the strong studied perturbation of Serengeti ecosystem are the dry season drought of 1993 [11] and the eruption of rinderpest disease in wildebeest as well as canine dispenser virus of 1994 [12]. Poaching also is considered by many to threaten the population viability of prey-predator species in Serengeti ecosystem. Poaching and trophy hunting have always contributed to the off-take of many species of the ecosystem. [5] mention 55% of household around the ecosystem to use bush meat atleast once a year. Also lion killing due to cultural practices [5] by Maasai, has been threatening the population of lion.
Hence if Serengeti ecosystem is to retain its diverse and abundant fauna, more efficient long-term conservation is needed. Based on this understanding, formulating a mathematical model to study how the ecosystem behave may well provide a way to achieve this. Prey-predator model has been widely studied in literature. However very few studies are on optimal control of prey-predator systems. Recently for example [11] studied the threat to lion-wildebeest prey-predator dynamics with optimal control in Serengeti ecosystem. Other studies are [2], [3], [1] and [5]. But none of these has considered the aspect of disease, drought and retaliatory killing as the threat to be controlled for the survival of prey-predator system particularly wildebeest, zebra and lion in the Serengeti ecosystem. This study intends to apply optimal control theory to maximize wildebeest, zebra and lion which the study regard as keystone species of Serengeti ecosystem.

2. The Optimal Control Model Formulation

It is assumed that lion depends completely on wildebeest and zebra as the source of food where wildebeest and zebra has unlimited sources of food. The dynamics therefore will follow the Holling type II function response. In this case $x(t)$, $y(t)$ and $z(t)$ represents the population of wildebeest, zebra and lion respectively. In the absence of predator, droughts and disease, prey species are assumed to grow logistically with carrying capacities k and l respectively. However the inter-specific competition among wildebeest and zebra is exploitative. From the above assumptions we formulate the system of model equation:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - b_{12}xy - \frac{b_{13}xz}{1+ax} - f_1x - w_1x \\
\frac{dy}{dt} &= sy(1-\frac{s}{l}) - b_{21}xy - \frac{b_{23}yz}{1+dy} - f_2y - w_2y \\
\frac{dz}{dt} &= -cz + \frac{b_{31}b_{13}xz}{1+ax} + \frac{b_{32}b_{23}yz}{1+dy} - ez - w_3z
\end{aligned} \tag{1}$$

where $x(0) \geq 0$, $y(0) \geq 0$ and $z(0) \geq 0$.
Then we introduce in the system (1) the time dependent control variables which are $u_1(t)$, $u_2(t)$ and $u_3(t)$. Therefore the model become:

$$
\begin{aligned}
\frac{dx}{dt} &= rx(1-\frac{x}{k}) - b_{12}xy - \frac{b_{13}xz}{1+ax} - (1-u_1(t))f_1x - (1-u_3)(t)w_1x \\
\frac{dy}{dt} &= sy(1-\frac{y}{l}) - b_{21}xy - \frac{b_{23}yz}{1+dy} - (1-u_1(t))f_2y - (1-u_3(t))w_2y \\
\frac{dz}{dt} &= -cz + \frac{b_{31}b_{13}xz}{1+ax} + \frac{b_{32}b_{23}yz}{1+dy} - (1-u_2(t)ez) - (1-u_3(t))w_3z
\end{aligned}
\tag{2}
$$

where, $u_1(t)$, $u_2(t)$ and $u_3(t)$ are control variables, $u_1(t)$ is the construction of dams for drought, $u_2(t)$ is education campaign for retaliatory killing and $u_3(t)$ is treatment for diseases. w_1, w_2 and w_3 are the disease induced death rates to wildebeest, zebra and lion respectively where as f_1 and f_2 are the death rates of wildebeest and zebra respectively due to drought.

3. Analysis of Optimal Control

An objective function J is formulated and maximized subject to the number of affected species:

$$
J = max\left[(A_1x + A_2y + A_3z) - \int_0^T (B_1\frac{u_1^2}{2} + B_2\frac{u_2^2}{2} + B_3\frac{u_3^2}{2})dt\right] \tag{3}
$$

where A_1, A_2 and A_3 are positive constant weights for wildebeest, zebra and lion respectively. B_1, B_2 and B_3 are positive constant weights balancing the cost elements attached to the control parameters u_1, u_2 and u_3. The weights used here are intended only for theoretical purpose to investigate the effect of various control practices. Importantly the cost associated to any control scenario is presumed to be non-linear and takes a quadratic form which is $\frac{B_1u_1^2}{2}$ refers to the cost of control efforts on construction of dam to reduce the effects of drought, $\frac{B_2u_2^2}{2}$ refers to the cost of control efforts of education campaign to reduce the effects of retaliatory killing, $\frac{B_3u_3}{2}$ is the cost of treatment for mitigating the effects of diseases.

In the light of the objective function $J(u_1, u_2, u_3)$, the intention is to maximize J. Therefore it is required to find numerically the optimal control u_1^*, u_2^*, u_3^* such that:

$$
J(u_1^*, u_2^*, u_3^*) = \max_{u_1^*, u_2^*, u_3^* \in U} J(u_1, u_2, u_3) \tag{4}
$$

for $U = (u_1^*, u_2^*, u_3^*)$ such that u_1^*, u_2^*, u_3^* are measurable with $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$ and $0 \leq u_3 \leq 1$, for $t \in [0, T]$.

3.1. **Existence of Optimal control.**

Theorem 1. *An optimal control set $(u_1^*, u_2^*, u_3^*) \in U$ with corresponding non-negativity states (x, y, z) that maximize the objective function $J(u_1(t), u_2(t), u_3(t))$ exists.*

Proof: The positiveness and consistent boundness of the state variables alongside the controls on $[0, T]$ suggests that the existence of a maximizing sequence $J(u_1^n(t), u_2^n(t), u_3^n(t))$ such that:

$$\lim_{n\to\infty} J(u_1^n(t), u_2^n(t), u_3^n(t)) = \inf_{(u_1^n(t), u_2^n(t), u_3^n(t))\in U} J(u_1^n(t), u_2^n(t), u_3^n(t)) \tag{5}$$

The boundness of all the state and control parameters insinuates that all derivatives of the state variables are bounded as well. Supposing the respective sequence of the state variables denoted by (x, y, z), subsequently, all the state variable are Lipschitz continous with the same Lipschitz constant. This means that, the sequence (x, y, z) is consistently equicontinous in $[0, T]$. In accordance with the method by [8], the state sequence has subsequence that converges steadly to (x, y, z) in $[0, T]$. Also, it can be taken that, the control sequence $u_n^n = (x^n, y^n, z^n)$ has sequence that weakly converges in $L^2(0, T)$. Let $(u_1^*, u_2^*, u_3^*) \in U$ be in the form of $u_i^n \to u_i^*$ weakly in $L^2(0, T)$ for $i = 1, 2, 3, ...$ Implementing the lower semi-continuity of norms in weak L^2:

$$\parallel u_i^* \parallel^2 L^2 \leq \lim_{n\to\infty} \inf \parallel u_i^n(t) \parallel^2 L^2, i = 1, 2, 3... \tag{6}$$

This means that:

$$J(u_1^*, u_2^*, u_3^*) \geq \lim_{n\to\infty} \int_0^T (A_1x + A_2y + A_3z) - \int_0^T \left(\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}\right) \tag{7}$$

Therefore; the control set (u_1^*, u_2^*, u_3^*) that maximize the objective function $J(u_1, u_2, u_3)$ exists.

3.2. **Characterization of optimal control.** In this section, we derive conditions required for optimal control, characterizing optimal control using upper and lower bound technique and formulating optimality system that characterize the optimal control. The asential requirement is that; the optimal pair should satisfy the necessary conditions that come from Pontryagin Maximum Principle [8] and which are also discussed in [4]. This principle converts state (2), objective function (3) and control (4) into minimal value of Lagrangian of optimal problem. The Lagrangian of the optimal problem is given by:

$$\Gamma = (A_1x + A_2y + A_3z) - \left(\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}\right) \tag{8}$$

In order to seek for maximum Lagrangian of optimal problem, we define the Hamilitonian H for the control problem with respect to u_1 and u_2 as:

$$H = (A_1x + A_2y + A_3z) - \left(\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2} + \sum_{i=1}^{3} \lambda_i f_i\right) \tag{9}$$

where f_i is the right hand side of the differential equation of ith state variable in system (2) and λ_i for $i = 1, 2, 3$ is the set of adjoint functions. That means the Hamilitonian consists of integrand of objective functional and the inner product of right hand side of state equations and corresponding adjoint variables $\lambda_1, \lambda_2, \lambda_3$. The expanded expression form of Hamilitonian H in (9) is given by:

$$
\begin{aligned}
H &= A_1x(T) + A_2y(T) + A_3z(T) - (\frac{B_1u_1^2}{2} + \frac{B_2u_2^2}{2} + \frac{B_3u_3^2}{2}) \\
&+ \lambda_1\left[\frac{dx}{dt} = rx(1-\frac{x}{l}) - b_{12}xy - \frac{b_{13}xz}{1+ax} - (1-u_1(t))f_1x - (1-u_3)(t)w_1x\right] \\
&+ \lambda_2\left[\frac{dy}{dt} = sy(1-\frac{y}{l}) - b_{21}xy - \frac{b_{23}yz}{1+dy} - (1-u_1(t))f_2y - (1-u_3(t))w_2y\right] \\
&+ \lambda_3\left[\frac{dz}{dt} = -cz + \frac{b_{31}b_{13}xz}{1+ax} + \frac{b_{32}b_{23}yz}{1+dy} - (1-u_2(t)ez - (1-u_3(t))w_3z\right]
\end{aligned}
\tag{10}
$$

Where λ_1, λ_2 and λ_3 are adjoint co-state variables. Applying pontryagin maximum principle (cite) and the existence results for the optimal control from (cite), the following preposition is obtained.

Theorem 2. *For the optimal control u_1^*, u_2^* and u_3^* that maximizes $j(u_1, u_2, u_3)$ over U, there exists adjoint variables L_1, L_2, L_3 satisfying:*

$$
\begin{aligned}
\frac{-\partial H}{\partial x} &= \frac{\partial \lambda_1}{dt} = -A_1 - \lambda_1 r + \frac{2\lambda_1 rx}{k} + \lambda_1 b_{12}y + \lambda_2 b_{21}y + \frac{\lambda_1 b_{13}z}{(1+ax)^2} \\
&\quad + \lambda_1(1-u_1)f_1 + \lambda_1(1-u_2)w_1 - \frac{\lambda_3 b_{31}b_{13}z}{(1+ax)^2} \\
\frac{-\partial H}{\partial y} &= \frac{\partial \lambda_2}{dt} = -A_2 - \lambda_2 s + \frac{2\lambda_2 sy}{l} + \lambda_1 b_{12}x + b_{21}\lambda_2 x + \frac{\lambda_2 b_{23}z}{(1+dy)^2} \\
&\quad + \lambda_2(1-u_1)f_2 + \lambda_2(1-u_2)w_2 - \frac{\lambda_3 b_{32}b_{23}z}{(1+ax)^2} \\
\frac{-\partial H}{\partial z} &= \frac{\partial \lambda_3}{dt} = -A_3 + \frac{\lambda_1 b_{13}x}{(1+ax)^2} + \frac{\lambda_2 b_{23}y}{(1+dy)^2} - \lambda_3 c - \frac{\lambda_3 b_{31}b_{13}x}{(1+ax)^2} \\
&\quad - \frac{\lambda_3 b_{32}b_{23}y}{(1+ax)^2} - \lambda_3(1-u_3)e + \lambda_3(1-u_2)w_3
\end{aligned}
$$

with transversality conditions $\lambda_1(T) = A_1$, $\lambda_2(T) = A_2$ and $\lambda_3(T) = A_3$. The following characterization holds on the interior of the control set:

$$
\begin{aligned}
u_1^* &= \min\left\{1, \max\left(0, \frac{\lambda_1 f_1 x + \lambda_2 f_2 y}{B_1}\right)\right\} \\
u_2^* &= \min\left\{1, \max\left(0, \frac{\lambda_1 w_1 x + \lambda_2 w_2 y + \lambda_3 w_3 z}{B_2}\right)\right\} \\
u_3^* &= \min\left\{1, \max\left(0, \frac{\lambda_3 ez}{B_3}\right)\right\}
\end{aligned}
$$

where λ_1, λ_2, λ_3 are solutions of the equation (10).

Proof: To prove this, the function (10) is differentiated partially w.r.t its state variables which gives the adjoint system. With the Pontryagin's Maximum Principle, we get the following adjoint system evaluated at the optimal control pair corresponding to the state variables.

$H = A_1x + A_2y + A_3z - \frac{B_1u_1^2}{2} - \frac{B_2u_2^2}{2} - \frac{B_3u_3^2}{2} + \lambda_1 rx - \lambda_1 x^2 - \alpha_1\lambda_1 xy - \frac{\beta_1\lambda_1(1-m)xz}{1+\lambda_1(1-m)x}$

$-\lambda_1(1-u_1(t))P_1x-\lambda_1(1-u_3(t))w_1x+\lambda_2sy-\lambda_2s^2-\alpha_2\lambda_2xy-\frac{\beta_2\lambda_2(1-m)yz}{1+\lambda_2(1-m)y}$
$-\lambda_2(1-u_1(t))P_2y-\lambda_2(1-u_3(t))w_2y-\lambda_3zc+\frac{\lambda_3\mu_1(1-m)xz}{1+\lambda_1(1-m)x}+\frac{\lambda_3\mu_2(1-m)yz}{1+\lambda_2(1-m)y}$
$+(1-u_2(t))\lambda_3qz-(1-u_3(t))w_3\lambda_3z$
$\frac{-\partial H}{\partial x}=\frac{\partial\lambda_1}{dt}=-A_1+\lambda_1r_1+2\lambda_1x+\alpha_1\lambda_1y+\frac{\beta_1\lambda_1(1-m)z}{\lambda_1^2(1-m)^2}$
$+(1-u_1(t))\lambda_1P_1+(1-u_3(t))\lambda_1w_1-\frac{\mu_1\lambda_3(1-m)z}{\lambda_1^2(1-m)^2}$
$\frac{-\partial H}{\partial y}=\frac{\partial\lambda_2}{dt}=-A_2-\lambda_2s+2\lambda_2s+\alpha_2\lambda_2x+\frac{\beta_2\lambda_2(1-m)z}{\lambda_2^2(1-m)^2}$
$+(1-u_1(t))\lambda_2P_2+(1-u_3(t))\lambda_2w_2-\frac{\mu_2\lambda_3(1-m)z}{\lambda_2^2(1-m)^2}$
$\frac{-\partial H}{\partial z}=\frac{\partial\lambda_3}{dt}=-A_3+\frac{\beta_1\lambda_1(1-m)x}{1+\lambda_1(1-m)x}+\frac{\beta_2\lambda_2(1-m)y}{1+\lambda_2(1-m)y}+\lambda_3c-\frac{\mu_1\lambda_3(1-m)x}{1+\lambda_1(1-m)x}$
$+\frac{\mu_2L_3(1-m)y}{1+\lambda_2(1-m)y}-(1-u_2(t))\lambda_3q-(1-u_3(t))\lambda_3w_3$

Now to obtain the optimal control solution $(u_i, i=1,2,3...)$ of our Lagrangian we differentiate partially the Lagrangian L, with respect to u_1, u_2, u_3 and set it to zero as follows;

$\frac{\partial H}{\partial u_1}=-B_1u_1+\lambda_1f_1x+\lambda_2f_2y$
$\frac{\partial H}{\partial u_2}=-B_2u_2+\lambda_1w_1x+\lambda_2w_2y-\lambda_3w_3z$
$\frac{\partial H}{\partial u_3}=-B_3u_3+\lambda_3ez$

Setting $\frac{\partial\lambda}{\partial u_i}=0$ for $i=1,2,3$ and solving for the optimal control;

$u_1^*=\frac{\lambda_1f_1x+\lambda_2f_2y}{B_1}$, $u_2^*=\frac{\lambda_1w_1x+\lambda_2w_2y+\lambda_3w_3z}{B_2}$, $u_3^*=\frac{\lambda_3ez}{B_3}$

4. Numerical Simulation

In this section, we study numerically an optimal control of prey-predator system of Serengeti ecosystem. By the virtue of the fact that, the Serengeti ecosystem is complex and extraordinary and that a single control measure can not present all the threats, an investigation on the impacts of merging a minimum of three control parameters over eight years period is done. Further more, estimation of the objective functions real weight is extremely demanding and requires a bunch of information. In that regard, the objective weights are chosen on theoretical basis as $A_1=60$, $A_2=10$, $A_3=90$, $B_1=100$, $B_2=150$, $B_3=80$ just to grant the control parameters proposed in the paper and the initial state variable are chosen as; $x(0)=40$, $y(0)=30$ and $z(0)=20$. Other parameters are $r=2.09$, $s=2.09$, $k=200$, $l=100$, $b_{12}=0.001$, $b_{21}=0.002$, $b_{13}=0.02$, $b_{23}=0.03$, $c=1$, $b_{31}=1.5$, $a=0.1$, $d=0.2$, $f_1=0.15$, $f_2=0.1$, $e=0.05$, $w_1=0.05$, $w_2=0.05$, $w_3=0.25$. Next we investigate the of the following optimal control strategies on the wildebeest, zebra and lion prey-predator population under threats.

4.1. **Strategy A: Application of construction of dams to control drought.**

In this control scenario, the construction of dams u_1 is utilized to optimize the objective function J while treatment u_2 and education u_3 are not practiced. Figure 1 indicates the significant difference in all populations with optimal control strategy compared to prey and predator populations without control.

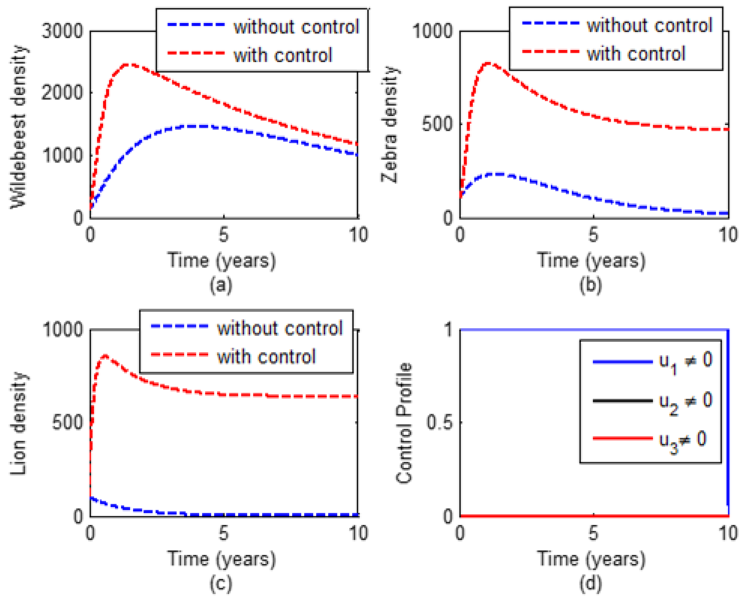

FIGURE 1. Simulation of the system (2) showing the effect of optimal application of construction of dams

4.2. **Strategy B: Treatment to Control Infections.** Under this scheme, the application of treatment u_2 is utilized to optimize the objective function J while construction of dams u_1 and education u_3 are not implemented. Results in figure (2) shows a significant difference in the prey and predator populations with optimal strategy compared to those without control. With this strategy, treatment u_2 should full be applied the entire period (10 years) as shown in the control profile in Figure 2(d).

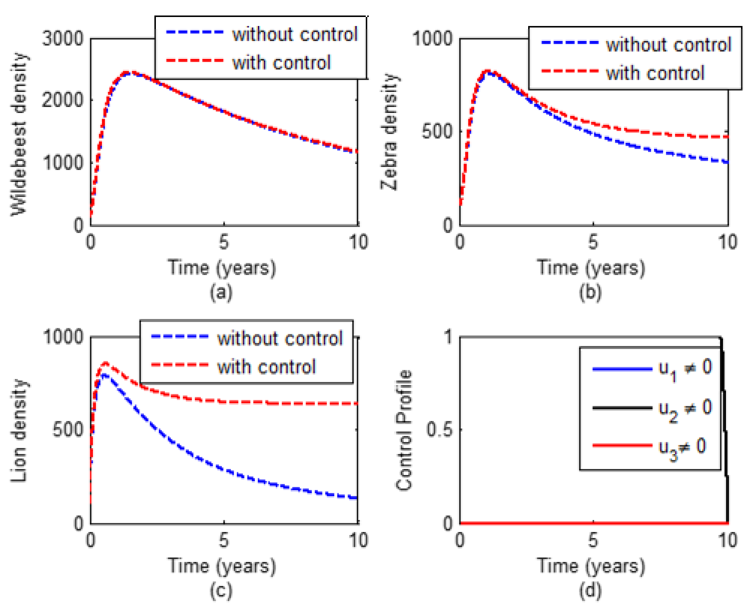

FIGURE 2. Simulation of the system (2) showing the effect of optimal application of treatment

4.3. **Strategy C: Education to Control retaliatory killing.** In this strategy, education, u_3 is used to optimize the objective function J while the set of application of treatment u_2 and construction of dams u_1 are not taken into account. In figure (3), the results show a significant difference in the predator population with optimal strategy compared to predator without control. With this strategy, education u_3 should full be applied the entire period (10 years) as shown by the control profile in Figure 3(d).

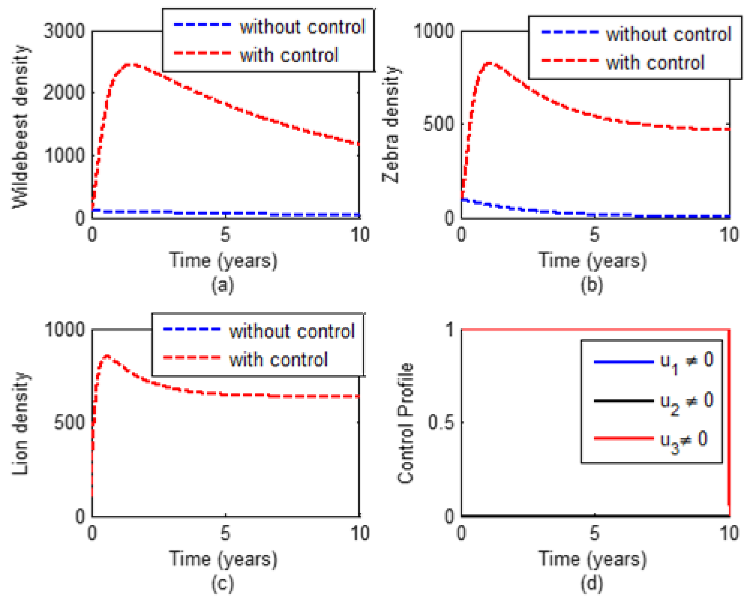

FIGURE 3. Simulation of the system (2) showing the effect of optimal application of education

4.4. **Strategy D: Combination of contruction of dams and treatment.** In this aspect, the application of construction of dams u_1 and treatment, u_2 are utilized in optimization of the objective function whilst education u_3 is not taken into account. Figure 4 shows the significant difference between prey and predator before and after control. With this strategy, construction of dams u_1 should be full applied throughout the entire period (10 years) while treatment u_3 is at the peak for 6 years and then decline to zero as shown by the control profile in Figure 5(d).

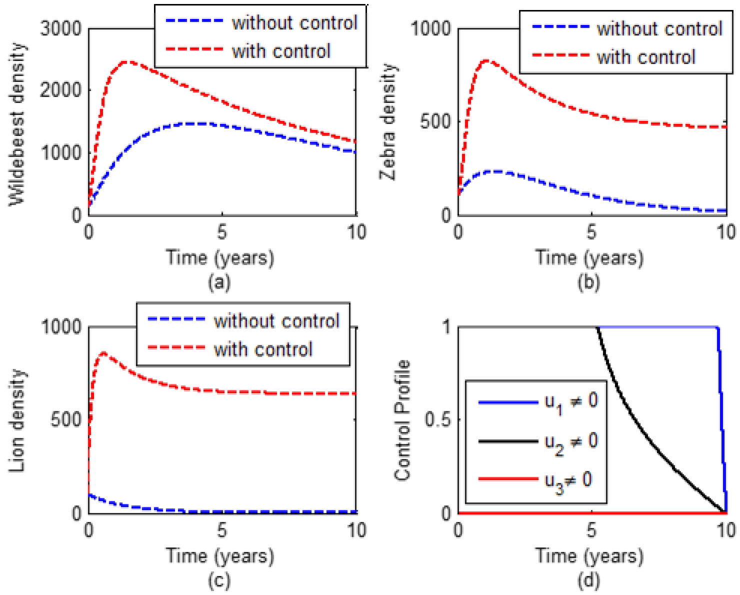

FIGURE 4. Simulation of the system (2) showing the effect of optimal application of construction of dams and treatment

4.5. **Strategy E: Combination of Construction of dams and Education.** The application of construction of dams u_1 and education u_3 are implemented in optimization of the objective function whilst treatment u_2 is set to be zero. In figure 5, the results shows significant difference between prey and predator population before and after control. With this strategy, construction of dams u_1 and education u_3 should be full applied throughout the entire period (10 years) as shown by the control profile in Figure 6(d).

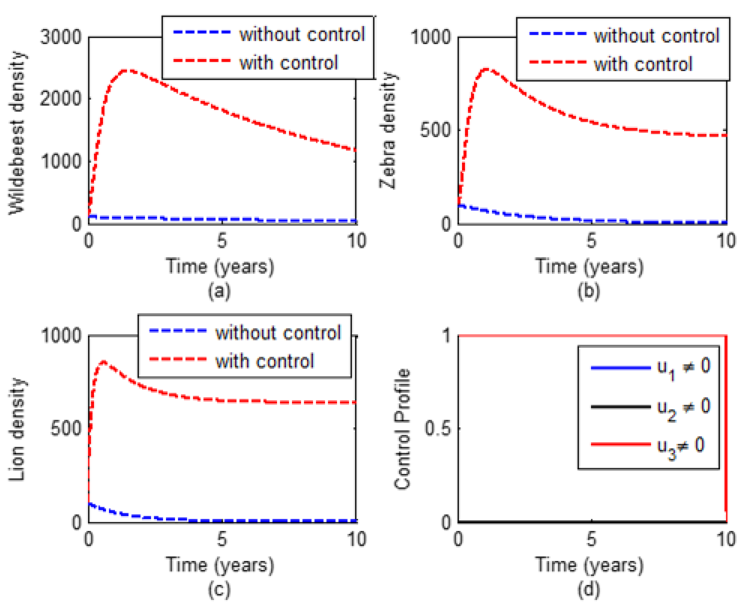

FIGURE 5. Simulation of the system (2) showing the effect of optimal application of construction of dams and education

4.6. **Strategy F: Combination of Treatment and Education.** The application of treatment u_2 and education u_3 are used in optimization of the objective function J while the construction of dams is set to zero. In Figure 6, the results shows a significant difference in prey and predator populations before and after control. With this strategy, treatment u_2 and education u_3 should be full applied throughout the entire period (10 years) as shown by the control profile in Figure 6(d).

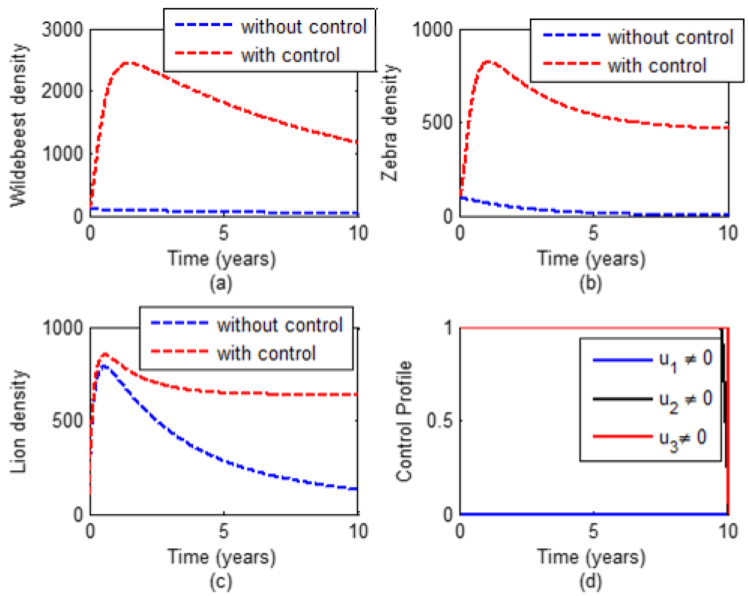

FIGURE 6. Simulation of the system (2) showing the effect of optimal application of treatment and education

4.7. **Strategy G: Combination of construction of dams, treatment and education.** Here, the combination of all controls; construction of dams u_1, treatment, u_2 and education, u_3 are utilized in optimization of the objective function J. In Figure 7, the results show significant difference in prey and predator populations before and after the control. With this practices, the contraction of dams u_1 and education u_3 should be full applied throughout the entire period (10 years) while treatment u_2 is at peak for 5 years and decrease to zero as shown by control profile in Figure 7(d).

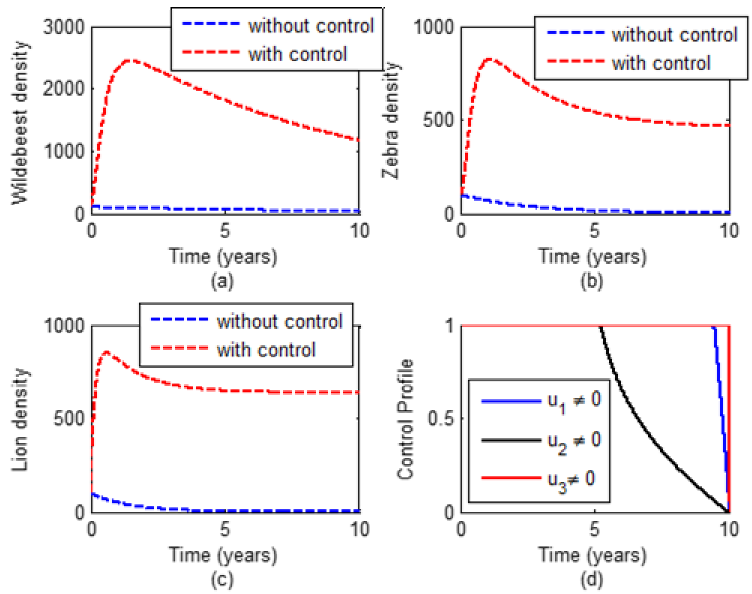

FIGURE 7. Simulation of the system (2) showing the effect of optimal application of construction of dams, treatment and education

4.8. **Comparison of Optimal control strategies.** Figure 8 shows the comparison of various combinations of the controls on wildebeest, zebra and lion populations. The combination of construction of dams, treatment and education campaign seem to be the best combination in optimization of the objective function. However if the ecosystem management decide to use a single control, the construction of dams is the best control in maximizing the objective function.

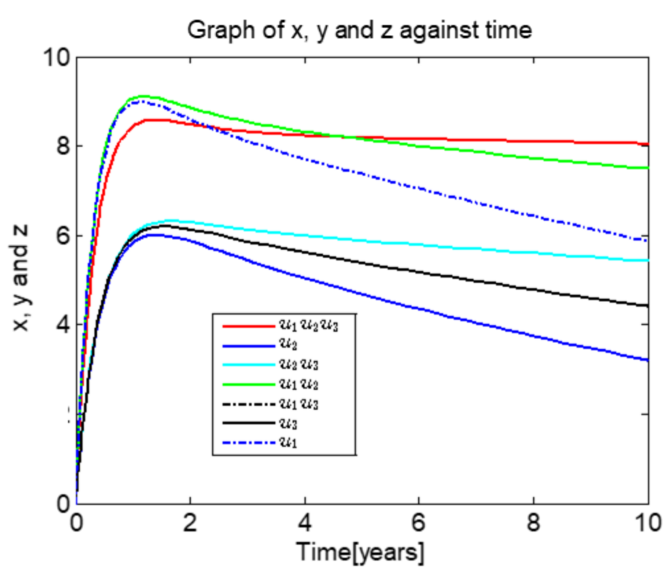

FIGURE 8. Comparison of optimal control strategies on wildebeest, zebra and lion populations

It is also observed from figure 8 that, the optimal solution of construction of dams is the same as the optimal solution of the combination of construction of dams and treatment. Hence, for the purpose of minimizing cost, the combination of construction of dams and treatment is not recommended in this study.

5. Discussion and Conclusion

In this paper we presented a prey-predator model with optimal control and control variables was introduced in the model where construction of dam was introduced to control drought, treatment was introduced to control infections and education was introduced to control retaliatory killing in Serengeti ecosystem. In investigating the use of optimal control, we use one control at a time, combination of two at a time while setting others to zero to compare the effect of optimal strategies on elimination those threats of the ecosystem. The use of all control were also considered. Numerical results suggests that, the use of all control has the greatest impact in the control of the ecosystem. However, if the ecosystem management decide to use a single control, the construction of dams is the best control in maximizing the objective function.

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