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EXISTENCE OF NONTRIVIAL WEAK SOLUTIONS FOR SOME DISCRETE $p(.)$-LAPLACIAN BOUNDARY VALUE PROBLEMS IN n-DIMENSIONAL

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Abstract. This paper is devoted to the study of the existence results for weak solutions to some $p(.)$-Laplacian problem in n-dimensional space. Under suitable conditions, using the method of the critical point theory, we obtain the existence of nontrivial solutions for an energy functional. By unknown reasons this type of problems had not been studied previously for $n > 2$.

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1. Introduction

For $a, b \in \mathbb{N}$ with $a \leq b$, we define the discrete interval $\mathbb{N}[a,b] = \{a, a+1, \cdots, b\}$.
In what follows, for any $k = (k_1, \cdots, k_n) \in \prod_{i=1}^{n} \mathbb{N}[0, T_i + 1]$, let us put

$$\begin{aligned} k^{i+} &= (k_1, \cdots, k_{i-1}, k_i + 1, k_{i+1}, \cdots, k_n), \;\; k^{i-} = (k_1, \cdots, k_{i-1}, k_i - 1, k_{i+1}, \cdots, k_n), \\ k_0^i &= (k_1, \cdots, k_{i-1}, 0, k_{i+1}, \cdots, k_n) \text{ and } k_{T_i+}^i = (k_1, \cdots, k_{i-1}, T_i + 1, k_{i+1}, \cdots, k_n). \end{aligned}$$

In this manuscript, for any $\delta > 0$, we focus our attention on the existence of solutions for the following nonlinear discrete Dirichlet boundary value problem:

$$\begin{cases} -\sum_{i=1}^{n} \Delta_i \big(\phi_{p(k^{i-})}(\Delta_i u(k^{i-}))\big) + \alpha(k)|u(k)|^{q(k)-2}u(k) = \delta f(k, u(k)), \; k \in \prod_{i=1}^{n} \mathbb{N}[1, T_i] \\ \\ u(k_0^i) = 0 = u(k_{T_i+}^i), \quad \forall\, i \in \{1, 2, \cdots, n\}, \end{cases} \tag{1.1}$$

where, for any $i \in \{1, 2, \cdots, n\}$, $\Delta_i u(k) = u(k^{i+}) - u(k)$ is the forward difference operator and $q : k \in \prod_{i=1}^{n} \mathbb{N}[1, T_i] \longrightarrow [2; +\infty)$.

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The functions $\alpha(.) : \prod_{i=1}^{n} \mathbb{N}[1,T_i] \longrightarrow \mathbb{R}$ and $f(.,.) : \prod_{i=1}^{n} \mathbb{N}[1,T_i] \times \mathbb{R} \longrightarrow \mathbb{R}$ will be defined through assumptions and the $p(.)$-Laplacian operator $\phi_p(.)$ is given by $\phi_{p(k^{i-})}(s) = |s|^{p(k^{i-})-2}s$, with the function

$$p : k^{i-} \in \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \longrightarrow [2; +\infty)$$

where

$$\begin{cases} \mathcal{O}_1 = \left(\prod_{j=1}^{i-1} \mathbb{N}[0,T_j]\right) \times \mathbb{N}[1,T_i+1] \times \left(\prod_{j=i+1}^{n} \mathbb{N}[0,T_j]\right) & \text{for } i \in \{2,\cdots,n-1\}, \\ \mathcal{O}_2 = \mathbb{N}[1,T_i+1] \times \left(\prod_{j=i+1}^{n} \mathbb{N}[0,T_j]\right) & \text{for } i = 1, \\ \mathcal{O}_3 = \left(\prod_{j=1}^{i-1} \mathbb{N}[0,T_j]\right) \times \mathbb{N}[1,T_i+1] & \text{for } i = n. \end{cases}$$

The aim of this short note is to extend the study of the $p(.)$-Laplacian difference equations in n dimensions, $n \geq 2$. In the "one-dimensional" case, there has been significant growth around the study of difference equations in the recent years. For background and recent results, we refer to [1]- [7] and the references therein. For example, in [5] the authors proved by using critical point theory, the existence of a continuous spectrum of eigenvalues for the problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\right) = \lambda |u(k)|^{q(k)-2}u(k), \ k \in \mathbb{Z}[1,T], \\ \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

where $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator.

These models are of independent interest since their mathematical structure has a different nature and have applications in the mathematical modelling of certain physical and chemical processes. So, our goal is to contribute to the generalization of the study of difference equations in higher dimensions. The main obstacle to this generalization is related to the forward difference operator. In two dimensions, there are several ways to overcome this problem (see S. Du and Z. Zhou in [2], I. Ibrango and all in [3], $\cdots$). One of them is to use the definition in [2] where the authors considered the following problem:

$$\Delta_1(\phi_p(\Delta_1 x(i-1,j))) + \Delta_2(\phi_p(\Delta_2 x(i,j-1))) + \lambda f((i,j), x(i,j)) = 0 \tag{1.3}$$

for any $(i,j) \in \mathbb{N}[1,m] \times \mathbb{N}[1,n]$ with

$$\Delta_1 x(i,j) = x(i+1,j) - x(i,j) \quad \text{and} \quad \Delta_2 x(i,j) = x(i,j+1) - x(i,j).$$

Let us point out that in the literature, to our best knowledge, there were no such existence results for our problem in this situation (dimension $n > 2$) which is nevertheless discrete variants of the anisotropic or isotropic partial differential equations (see [4]).

The remaining part of this paper is organized as follows. In the next section, we give some basic preliminaries and an illustration. In section 3, we provide our main results that contains several theorem. We end with a conclusion in the last section.

2. Preliminary informations

In order to facilitate the manipulation of expressions we note

$$\sum_{k_1=1}^{T_1}\sum_{k_2=1}^{T_2}\cdots\sum_{k_n=1}^{T_n} u(k_1,k_2,\cdots,k_n)=\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} u(k),\quad X=\prod_{i=1}^{n}\mathbb{N}[0,T_i+1]\ \text{ and }\ W=\prod_{i=1}^{n}\mathbb{N}[1,T_i].$$

We will use the following notations

$$p^-=\min_{k\in\mathcal{O}} p(k),\quad q^-=\min_{k\in W} q(k),\quad \alpha^-=\min_{k\in W}\alpha(k),$$
$$p^+=\max_{k\in\mathcal{O}} p(k),\quad q^+=\max_{k\in W} q(k)\quad\text{and}\quad \alpha^+=\max_{k\in W}\alpha(k).$$

Let us introduce the following Hilbert space

$$E=\left\{u:X\longrightarrow\mathbb{R}\ \text{ such that }\ u(k_0^i)=0=u(k^i_{T_i+}),\ \forall\, i\in\mathbb{N}[1,n]\right\}$$

with the norm

$$\|u\|=\left(\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\neq i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^2\right)^{\frac{1}{2}}$$

and the equivalent norm

$$\|u\|_m=\left(\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}|u(k)|^m\right)^{1/m},\quad \forall\, m\ge 2.$$

Let the function $\varphi:E\longrightarrow\mathbb{R}$ given by the formula

$$\varphi(u)=\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\neq i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^{p(k^{i-})}+\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}|u(k)|^{q(k)}.$$

In the space E we can also introduce the Luxemburg norm

$$\|u\|_{p(.)}=\inf\{\lambda>0:\varphi(u/\lambda)\le 1\}.\tag{2.1}$$

Since E has a finite dimension, then all norms are equivalent. Therefore there exist positive constants C_1 and C_2 such that

$$C_1\|u\|_{p(.)}\le\|u\|\le C_2\|u\|_{p(.)}.\tag{2.2}$$

Proposition 2.1. *The function φ satisfies the following*

(1) $\varphi(u+v)\le\max\left(2^{p^+-1},2^{q^+-1}\right)(\varphi(u)+\varphi(v)),\ \ \forall\, u,v\in E.$

(2) *For $u\in E$*

(a) *if* $\lambda > 1$ *we have*

$$\min\left(\lambda^{p^-}, \lambda^{q^-}\right)\varphi(u) \le \varphi(\lambda u) \le \max\left(\lambda^{p^+}, \lambda^{q^+}\right)\varphi(u), \tag{2.3}$$

(b) *if* $0 < \lambda < 1$ *we have*

$$\min\left(\lambda^{p^+}, \lambda^{q^+}\right)\varphi(u) \le \varphi(\lambda u) \le \max\left(\lambda^{p^-}, \lambda^{q^-}\right)\varphi(u). \tag{2.4}$$

(3) *For any* $u \in E\backslash\{0\}$, $\varphi(\lambda u)$ *is a continuous convex even function in* λ, *and it increases strictly when* $\lambda \in [0, +\infty)$.

Proof.

(1) Let $u, v \in E$. We have

$$\begin{aligned}
\varphi(u+v) &= \sum_{i=1}^{n} \sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i(u+v)(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} |(u+v)(k)|^{q(k)}\\
&\le 2^{p^+-1}\left[\sum_{i=1}^{n} \sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i u(k^{i-})|^{p(k^{i-})} + \sum_{i=1}^{n} \sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i v(k^{i-})|^{p(k^{i-})}\right]\\
&\quad + 2^{q^+-1}\left[\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} |u(k)|^{q(k)} + \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} |v(k)|^{q(k)}\right]\\
&\le \max\left(2^{p^+-1}, 2^{q^+-1}\right)(\varphi(u)+\varphi(v)).
\end{aligned}$$

(2) Let $u \in E$

(a) for $\lambda > 1$, we have

$$\begin{aligned}
\varphi(\lambda u) &= \sum_{i=1}^{n} \sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i(\lambda u)(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} |(\lambda u)(k)|^{q(k)}\\
&\ge \lambda^{p^-} \sum_{i=1}^{n} \sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i u(k^{i-})|^{p(k^{i-})} + \lambda^{q^-} \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i} |u(k)|^{q(k)}\\
&\ge \min\left(\lambda^{p^-}, \lambda^{q^-}\right)\varphi(u).
\end{aligned}$$

We also have $\varphi(\lambda u) \le \max\left(\lambda^{p^+}, \lambda^{q^+}\right)\varphi(u)$.

(b) For $0 < \lambda < 1$, by mimicking the above approach we get

$$\min\left(\lambda^{p^+}, \lambda^{q^+}\right)\varphi(u) \le \varphi(\lambda u) \le \max\left(\lambda^{p^-}, \lambda^{q^-}\right)\varphi(u).$$

(3) (a) Let $\lambda \in (0,1)$. Since $x \longmapsto |x|^p,\ \ p>1$ is convex, then for any $u, v \in E$, we have

$$\varphi\left(\lambda u+(1-\lambda) v\right) \leq \lambda \varphi(u)+(1-\lambda) \varphi(v).$$

Namely, φ is convex.

(b) Let $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 < \lambda_2$. We have

$$\begin{aligned}
\varphi(\lambda_1 u) &= \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} |\Delta_i(\lambda_1 u)(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} |(\lambda_1 u)(k)|^{q(k)} \\
&= \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} \lambda_1^{p(k^{i-})} |\Delta_i u(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} \lambda_1^{q(k)} |u(k)|^{q(k)} \\
&< \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} \lambda_2^{p(k^{i-})} |\Delta_i u(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} \lambda_2^{q(k)} |u(k)|^{q(k)} = \varphi(\lambda_2 u).
\end{aligned}$$

Then, for any fixed $u \in E\backslash\{0\}$, $\varphi(\lambda u)$ increases strictly when $\lambda \in [0, +\infty)$.

(c) For the continuity of φ, let $(\lambda_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lambda_n \longrightarrow \lambda$ as $n \to +\infty$. We have

$$\varphi(\lambda_n u) = \sum_{i=1}^{n} \sum_{\substack{k_j=1 \\ 1\leq j\neq i\leq n}}^{T_j} \sum_{k_i=1}^{T_i+1} \lambda_n^{p(k^{i-})} |\Delta_i u(k^{i-})|^{p(k^{i-})} + \sum_{\substack{k_i=1 \\ 1\leq i\leq n}}^{T_i} \lambda_n^{q(k)} |u(k)|^{q(k)}.$$

So

$$\lim_{n\to+\infty} \varphi(\lambda_n u) = \varphi(\lambda u).$$

Consequently φ is continuous.

Thus, the proof is completed. □

According to the Proposition 2.1, for any $u \in E$ the following inequalities

$$\min\left(\|u\|_{p(.)}^{p^+}, \|u\|_{p(.)}^{p^-}\right) \leq \varphi(u) \leq \max\left(\|u\|_{p(.)}^{p^+}, \|u\|_{p(.)}^{p^-}\right) \tag{2.5}$$

hold.

For each $k \in X$, the function $f(k,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a function $\theta : X \longrightarrow [1, +\infty)$ such that

$$|f(k,\xi)| \leq \theta(k)|\xi|^{r(k)-1}, \tag{2.6}$$

where $1 < r\,(k) < p^-$ with the notations $r^- = \min\limits_{k\in X} r(k)$ and $r^+ = \max\limits_{k\in X} r(k)$ that will be used in the rest of the work.

We denote

$$F(k,\xi) = \int_0^{\xi} f(k,s)ds \ \text{ for } \ (k,\xi) \in X \times \mathbb{R} \tag{2.7}$$

and we deduce that

$$|F(k,\xi)| \leq \beta(k)|\xi|^{r(k)}, \tag{2.8}$$

with $\beta : X \longrightarrow (0,+\infty)$ is such that for all $k \in X$,

$$0 < \beta^- = \min_{k\in X}(\beta(k)) \leq \beta(k) \leq \beta^+ = \max_{k\in X}(\beta(k)) < +\infty. \tag{2.9}$$

We need the following assumption to show that the weak solution is non-trivial : there exist constants $C_1, C_2 > 0$ and $\mu > \max(p^+, q^+)$ such that

$$F(k,\xi) \geq C_1|\xi|^\mu - C_2. \tag{2.10}$$

Remark 2.2. (Illustration dimension $n = 3$)

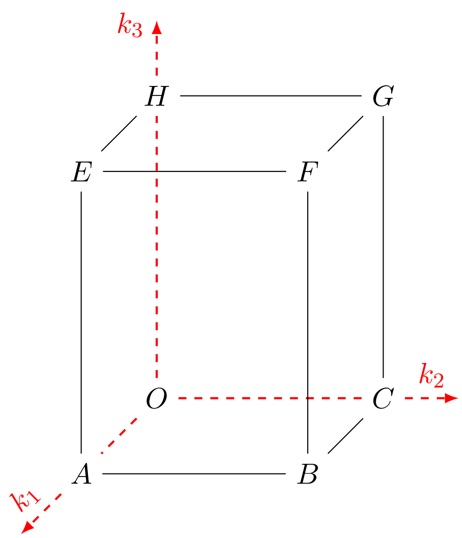

Figure 1. Domain $X = \prod_{i=1}^{3} \mathbb{N}[0, T_i + 1]$

with $O(0,0,0)$, $A(T_1+1,0,0)$, $C(0,T_2+1,0)$ and $H(0,0,T_3+1)$.
Let

$$X_1 = \text{Face}(ABFE) \cup \text{Face}(CGHO) = \{0, T_1+1\} \times \mathbb{N}[0, T_2+1] \times \mathbb{N}[0, T_3+1],$$

$$X_2 = \text{Face}(BCGF) \cup \text{Face}(AOHE) = \mathbb{N}[0, T_1+1] \times \{0, T_2+1\} \times \mathbb{N}[0, T_3+1]$$

and

$$X_3 = \text{Face}(ABCO) \cup \text{Face}(FGHE) = \mathbb{N}[0, T_1+1] \times \mathbb{N}[0, T_2+1] \times \{0, T_3+1\}.$$

We can write the Dirichlet condition as follow:

$$u(k) = 0, \ \ \forall\, k \in X_1 \cup X_2 \cup X_3.$$

3. Existence of nontrivial weak solutions

Throughout what follows all constants are positive.

Definition 3.1. By a weak solution for problem (1.1) we understand a function $u \in E$ such that

$$\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^{p(k^{i-})-2}\Delta_i u(k^{i-})\Delta_i v(k^{i-})+\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\alpha(k)|u(k)|^{q(k)-2}u(k)v(k)$$

$$-\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\delta f(k,u(k))v(k)=0 \tag{3.1}$$

for any $v \in E$.

Theorem 3.2. *Assume that conditions (2.6) -(2.10) are satisfied. Then, there exists a nontrivial weak solution of the problem (1.1).*

Proof. The energy functional $J : E \longrightarrow \mathbb{R}$, corresponding to problem (1.1), is given by $J = I - L$ where

$$I(u)=\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}\frac{1}{p(k^{i-})}|\Delta_i u(k^{i-})|^{p(k^{i-})}+\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\frac{\alpha(k)}{q(k)}|u(k)|^{q(k)}$$

and

$$L(u)=\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\delta F(k,u(k))$$

for any $u \in E$.

The functional J is well defined on E, it is of class $C^1(E,\mathbb{R})$ and for any $u, v \in E$ we have

$$\begin{aligned}\langle I'(u),v\rangle &= \sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^{p(k^{i-})-2}\Delta_i u(k^{i-})\Delta_i v(k^{i-})\\ &+ \sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\alpha(k)|u(k)|^{q(k)-2}u(k)v(k)\\ &= \sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}\phi_{p(k^{i-})}\left(\Delta_i u(k^{i-})\right)\Delta_i v(k^{i-})+\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\alpha(k)|u(k)|^{q(k)-2}u(k)v(k)\\ &= -\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i}\Delta_i\phi_{p(k^{i-})}\left(\Delta_i u(k^{i-})\right)v(k)+\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\alpha(k)|u(k)|^{q(k)-2}u(k)v(k)\end{aligned}$$

$$= -\sum_{i=1}^{n}\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\Delta_i\phi_{p(k^{i-})}\left(\Delta_i u(k^{i-})\right)v(k)+\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\alpha(k)|u(k)|^{q(k)-2}u(k)v(k)$$

$$= \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\left[-\sum_{i=1}^{n}\Delta_i\phi_{p(k^{i-})}\left(\Delta_i u(k^{i-})\right)+\alpha(k)|u(k)|^{q(k)-2}u(k)\right]v(k)$$

and

$$\langle L'(u),v\rangle = \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\delta f(k,u(k))v(k). \tag{3.2}$$

Then, we obtain that the functional J is differentiable in sense of Gâteaux and its Gâteaux derivative is as follows

$$\langle J'(u),v\rangle = \sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\left[-\sum_{i=1}^{n}\Delta_i\phi_{p(k^{i-})}\left(\Delta_i u(k^{i-})\right)+\alpha(k)|u(k)|^{q(k)-2}u(k)-f(k,u(k))\right]v(k).$$

For any fixed v in E, we see that the critical point u to J satisfies the problem (1.1). Also, note that, since E is a finite dimensional space, the weak solution coincide with the classical solution of the problem (1.1).

Now let us show that J in coercive on E and bounded from below. To prove the coercivity of J, we may assume that $\|u\|_{p(.)}>1$, and we have

$$\begin{aligned}
J(u) &= \sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}\frac{1}{p(k^{i-})}|\Delta_i u(k^{i-})|^{p(k^{i-})}+\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\frac{\alpha(k)}{q(k)}|u(k)|^{q(k)}-\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\delta F(k,u(k))\\
&\ge \frac{1}{p^+}\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\le j\ne i\le n}}^{T_j}\sum_{k_i=1}^{T_i+1}|\Delta_i u(k^{i-})|^{p(k^{i-})}+\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\frac{\alpha^+}{q^-}|u(k)|^{q(k)}-\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\delta\beta(k)|u(k)|^{r(k)}\\
&\ge \min\left(\frac{1}{p^+},\frac{\alpha^+}{q^-}\right)\varphi(u)-\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}\delta\beta(k)|u(k)|^{r(k)}\\
&\ge \min\left(\frac{1}{p^+},\frac{\alpha^+}{q^-}\right)\|u\|_{p(.)}^{p^-}-\delta\beta^+\left(\sum_{K_1}|u(k)|^{r^+}+\sum_{K_2}|u(k)|^{r^-}\right)\\
&\ge \min\left(\frac{1}{p^+},\frac{\alpha^+}{q^-}\right)\|u\|_{p(.)}^{p^-}-\delta\beta^+\sum_{K_1}|u(k)|^{r^+}-C_3\\
&\ge \min\left(\frac{1}{p^+},\frac{\alpha^+}{q^-}\right)\|u\|_{p(.)}^{p^-}-\delta\beta^+\sum_{\substack{k_i=1\\1\le i\le n}}^{T_i}|u(k)|^{r^+}-C_3
\end{aligned}$$

$$\begin{aligned} &\geq \min\left(\frac{1}{p^+},\frac{\alpha^+}{q^-}\right)\|u\|_{p(.)}^{p^-}-\delta\beta^+\left(C(T,p)\right)^{r^+}\left(\prod_{i=1}^{n}T_i\right)\|u\|_{p(.)}^{r^+}-C_3\\ &\geq C_5\|u\|_{p(.)}^{p^-}-C_4\|u\|_{p(.)}^{r^+}-C_3. \end{aligned}$$

Since $p^- > r^+$, we have $\lim\limits_{\|u\|_{p(.)}\to+\infty} J(u)=+\infty$. Thus, J is coercive on E and bounded from below. Besides, for $||u||\leq 1$, we have

$$J(u) \geq \frac{1}{p^+}\|u\|_{p(.)}^{p^+}-\delta\beta^+\left(C(T,p)\right)^{r^+}\left(\prod_{i=1}^{n}T_i\right)\|u\|_{p(.)}^{r^+}-C_3\geq -C_6-C_3>-\infty$$

namely, J is bounded from below.

Since J is continuous, bounded from below and coercive on E, using the relation between critical points of J and problem (1.1), we deduce that J has a minimizer which is a weak solution of problem (1.1).

In what follows, we prove that the solution u is nontrivial. For $u\in E\setminus\{0\}$ and $t>1$, we have

$$\begin{aligned} J(tu) &= \sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}\frac{1}{p(k^{i-})}\left|\Delta_i\left(tu(k^{i-})\right)\right|^{p(k^{i-})}+\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\frac{\alpha(k)}{q(k)}|tu(k)|^{q(k)}\\ &- \sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\delta F(k,tu(k))\\ &\leq \frac{t^{p^+}}{p^-}\sum_{i=1}^{n}\sum_{\substack{k_j=1\\1\leq j\neq i\leq n}}^{T_j}\sum_{k_i=1}^{T_i+1}\left|\Delta_i\left(u(k^{i-})\right)\right|^{p(k^{i-})}+\frac{\alpha^+t^{q^+}}{q^-}\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}|u(k)|^{q(k)}\\ &- \delta\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}\left(C_1|tu(k)|^{\mu}-C_2\right)\\ &\leq \max\left(\frac{t^{p^+}}{p^-},\frac{\alpha^+t^{q^+}}{q^-}\right)\varphi(u)-C_1t^{\mu}\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}|u(k)|^{\mu}+C_2\prod_{i=1}^{n}T_i\\ &\leq \max\left(\frac{t^{p^+}}{p^-},\frac{\alpha^+t^{q^+}}{q^-}\right)\max\left(\|u\|_{p(.)}^{p^-},\|u\|_{p(.)}^{p^+}\right)-C_1t^{\mu}\sum_{\substack{k_i=1\\1\leq i\leq n}}^{T_i}|u(k)|^{\mu}+C_2\prod_{i=1}^{n}T_i. \end{aligned}$$

Since $\mu>\max\left(p^+,q^+\right)$, for sufficiently large $t>1$ we assert that $J(tu)<0$. □

4. Conclusion

In the present paper, we have investigated the existence of nontrivial weak solutions for discrete nonlinear problems in an n-dimensional space. The minimization technique allows us to show that the energy functional admits at least one nontrivial critical point which is a weak solution of the associated problem. The originality of this work lies in the generalization of the space (dimension $n > 2$). Some interesting topics for further research remain. The results obtained in this paper will allow authors to reflect on the generalization of difference equations with Leray-Lions type operators.

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