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REGULATOR PROBLEM FOR LINEAR CONTINUOUS-TIME DELAY DYNAMICAL SYSTEMS WITH SYMMETRICAL CONSTRAINED CONTROL

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Abstract. This paper delves into the analysis of the linear constrained regulation problem (LCRP) in continuous-time delay dynamical systems. We assume that the control variable is subjected to symmetrical constraints, with the equilibrium positioned on the boundary of the constraint domain. The objective is to examine the conditions under which a state feedback law exists, ensuring constraint satisfaction and asymptotic convergence to the origin for the system's state when the initial states belong to the largest positively invariant set. Both delay-independent and delay-dependent conditions are considered during the investigation.

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1. Introduction

The regulator problem for linear continuous-time delay dynamical systems with non-symmetrical constrained control was extensively investigated by many authors, such as [20], [16], [6], [8], [26], [18] and [2]. In all these publications the regulation is made around an equilibrium situated in the interior of a domain of attraction. So, for the regulation around an equilibrium situated on the boundary of the domain of constraints results are needed. Recently, [4], [10], [11] and [24] have considered the linear constrained regulation problem, for discrete system and continuous systems with the equilibrium on the boundary of the domain of constraints. In this paper, we investigate the linear constrained regulation problem for a continuous-time systems with delay and the equilibrium on the boundary of the domain of attraction.

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We consider linear continuous-time systems with time delay described by the difference equation:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-r) + Bu(t),\ t \geq t_0 \\ x(\theta) = \varphi(\theta),\ \theta \in [-r, 0] \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ represents the input vector, $k \in T$ represents the time variable, and $r \in \mathbb{N}$ represents the time delay.

The system is characterized by constant matrices $A_0 \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$, which satisfy the following conditions:

(a) In the case of independent delay, the pair (A_0, B) is stabilizable.

(b) In the case of dependent delay, the pair $(A_0 + A_1, B)$ is stabilizable.

The control vector, denoted by u, must satisfy linear constraints of the form:

$$-q \leq u \leq q, \quad \text{where} \quad q \in \mathbb{R}+^m \tag{2}$$

The Linear Constrained Regulation Problem (LCRP) involves finding a linear state feedback control law, represented as $u = Fx$, where $F \in \mathbb{R}^{m\times n}$, and determining a domain of attraction, denoted by D. The objective is to ensure that for all initial states $x(\theta) \in D$ with $\theta \in [-r, 0]$, the trajectories $x(k; x(\theta))$ of the closed-loop system defined as:

$$\dot{x}(t) = (A0 + BF)x(t) + A_1 x(t-r), \quad t \geq 0 \tag{3}$$

converge asymptotically to the equilibrium $x_e = 0$, while respecting the control constraints. The set D is said to be an admissible domain of attraction. Generally, the matrix F is chosen to satisfy the following conditions:

(i) In the case of independent delay, the system defined in equation (3), which is asymptotically stable when $A_1 = 0$, i.e., satisfies:

$$\Re(\lambda_i(A_0 + BF)) < 0, \quad i = 1, ..., n \tag{4}$$

additionally, is asymptotically stable independent of delay:

$$\det(sI - A_0 - BF - e^{-rs}A_1) \neq 0 \quad \text{for} \quad \Re(s) \geq 0, \quad \forall r \geq 0 \tag{5}$$

(ii) In the case of dependent delay, the system defined in equation (3), which is asymptotically stable when $r = 0$, i.e., satisfies:

$$\Re(\lambda_i(A_0 + A_1 + BF)) < 0, \quad i = 1, ..., n \tag{6}$$

additionally, is asymptotically stable with delay dependence:

$$\det(sI - A_0 - BF - e^{-rs}A_1) \neq 0 \quad \text{for} \quad \Re(s) \geq 0, \quad \text{with fixed} \quad r > 0 \tag{7}$$

According to what we mentioned above, the main results of this paper is to give necessary and sufficient conditions under which the symmetrical region of initial states $D(F, q, q) = \{x \in \mathbb{R}^n \mid -q \leq Fx \leq q\}$

with the equilibrium state $x_e = 0$ on its boundary is a domain of attraction for system (3). Since $x_e = 0$ is on the boundary of $D(F, q, q)$ this implies that at least one component of q is null. Without loss of generality, we assume that

$$q_j > 0,\ j = 1, ..., s \ and\ q_j = 0\ \ j = s + 1, ..., m \tag{8}$$

Indeed, if this is not the case, a suitable change in input variable coordinate will transform the system (1) and constraints (2) to

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r) + B^{'} v(t)$$

with $-q^{'} \le v \le q^{'}$ where $q^{'}$ satisfies relation (8).

In the following, we will denote $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$ with $q^* = \begin{pmatrix} q_1 \\ \vdots \\ q_s \end{pmatrix}$ and $q^* > 0$.

The paper is organized as follows: In Section 2, we present some definitions and useful results for the following. In Section 3, we establish sufficient conditions for $u = Fx$ with $F \in \mathbb{R}^{m\times n}$, $rankF = m$, and F verify (4) and (5) (or (6) and (7)), to be a solution to the linear constrained regulation problem. Finally, an algorithm and example are given in section 4.

Notation

In this paper, capital letters are used to represent real matrices, while lowercase letters represent column vectors or scalars. The notation $\mathbb{R}^n$ refers to the real n-space, $\mathbb{R}^n_+$ denotes the nonnegative orthant of the real n-space, and $\mathbb{R}^{n\times p}$ represents the set of real $n \times p$ matrices. For two real vectors $x = [x_1\ x_2\ ...\ x_n]^T$ and $y = [y_1\ y_2\ ...\ y_n]^T$, the notation $x < y$ $(x \le y)$ is equivalent to $x_i < y_i$ $(x_i \le y_i)$ for $i = 1, 2, ..., n$. Similar notation is applied to real matrices. For a real matrix $H = (h_{ij})$, $\mid H \mid$ represents the matrix obtained by taking the absolute values of its components, i.e., $\mid H \mid = (\mid h_{ij} \mid)$. The symbol $\rho(H)$ denotes the spectral radius of H, and H^+ (or H^-) is a matrix whose elements are defined as $h^+ij = \max(hij, 0)$ (or $h^-ij = \max(-hij, 0)$, respectively).

2. Conditions of positive invariance

We consider linear continuous-time systems with time delay described by the difference equation:

$$\begin{cases} \dot{z}(t) = Hz(t) + Gz(t - r),\ t \ge 0 \\ z(\theta) = \psi(\theta),\ \theta \in [-r, 0] \end{cases} \tag{9}$$

with $z \in \mathbb{R}^m$, $H \in \mathbb{R}^{m\times m}$ and $G \in \mathbb{R}^{m\times m}$.

Let us define the domain

$$D(I_m, q, q) = \{z \in \mathbb{R}^m \mid -q \le z \le q\}$$

with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ and $q^* \in \mathbb{R}^s$.

Definition 2.1. A set D of $\mathbb{R}^m$ is said to be positively invariant with respect to motions of system (9), if for every $\psi(\theta) \in D(\theta \in [-r, 0])$ the motion $z(t;\psi) \in D$ for every $t \geq 0$.

2.1. **Positively invariant conditions independent of delay.**

In [17] the author has considered the case where $x_e = 0$ is in the interior of $D(I_m, q, q)$, that is $q > 0$ and has proved that $D(I_m, q, q)$ is positively invariant with respect to system (9) if and only if

$$(\overline{H} + \mid G \mid) q \leq 0 \tag{10}$$

with

$$\begin{cases} \mid G \mid = (\mid g_{ij} \mid)_{1 \leq i,j \leq m} \\ \overline{H} = (\overline{h}_{ij})_{1 \leq i,j \leq m}, \ (\overline{h}_{ij} = h_{ii} \ if \ i = j \ and \ \overline{h}_{ij} = \mid h_{ij} \mid \ \ if \ i \neq j) \end{cases}$$

In this paper, where $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$, we prove the following result.

Theorem 2.2. *The polyhedral set $D(I_m, q, q)$ with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ and $q^* \in \mathbb{R}^s$ is positively invariant independent of delay with respect to system (9) if and only if*

$$\begin{cases} (\overline{H_{11}} + \mid G_{11} \mid) q^* \leq 0 \\ H_{21} = G_{21} = 0 \end{cases}$$

with

$$\begin{cases} H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \ and \ G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\ H_{11}, G_{11} \in \mathbb{R}^{s \times s}, \ H_{12}, G_{12} \in \mathbb{R}^{s \times (m-s)}, \ H_{21}, G_{21} \in \mathbb{R}^{(m-s) \times s} \ and \ H_{22}, G_{22} \in \mathbb{R}^{(m-s) \times (m-s)} \end{cases}$$

Proof.

If:) Let assume that $(\overline{H_{11}} + \mid G_{11} \mid) q^* \leq 0$ and $H_{21} = G_{21} = 0$. Let $z(.)$ the solution of system (9) with $z(t) \in D(I_m, q, q)$, $\forall t \in [-r, 0]$, that means

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leq z(t) \leq \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall t \in [-r, 0] \tag{11}$$

we can decompose $z(t)$ into two vectors $z_1(t)$ and $z_2(t)$ with $-q^* \leq z_1(t) \leq q^*$ and $z_2(t) = 0$ for all $t \in [-r, 0]$. According to the above, we can decompose system (9) into two systems, so we have

$$\begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} + \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} z_1(t-r) \\ z_2(t-r) \end{pmatrix}$$

with $H_{11}, G_{11} \in \mathbb{R}^{s\times s}$, $H_{21}, G_{21} \in \mathbb{R}^{(m-s)\times s}$, $H_{12}, G_{12} \in \mathbb{R}^{s\times(m-s)}$ and $H_{22}, G_{22} \in \mathbb{R}^{(m-s)\times(m-s)}$

Then

$$\begin{cases} \dot{z}_1(t) = H_{11}z_1(t) + H_{12}z_2(t) + G_{11}z_1(t-r) + G_{12}z_2(t-r) \\ \dot{z}_2(t) = H_{21}z_1(t) + H_{22}z_2(t) + G_{21}z_1(t-r) + G_{22}z_2(t-r) \end{cases} \tag{12}$$

Using $H_{21} = G_{21} = 0$ we obtain

$$\dot{z}_1(t) = H_{11}z_1(t) + H_{12}z_2(t) + G_{11}z_1(t-r) + G_{21}z_2(t-r) \tag{13}$$

and

$$\dot{z}_2(t) = H_{22}z_2(t) + G_{21}z_2(t-r) \tag{14}$$

The solution of system (14) can be written under the form

$$\begin{aligned} z_2(t) &= e^{H_{22}t}z_2(0) + \int_0^t e^{H_{22}(t-\tau)}G_{22}z_2(\tau - r)d\tau \\ &= e^{H_{22}t}z_2(0) + \int_0^t e^{H_{22}\mu}G_{22}z_2(t-\mu-r)d\mu \end{aligned}$$

For $0 \le \mu \le t$ and $0 \le t \le r$ we obtain $-r \le t-\mu-r \le 0$ from $z_2(t) = 0, \forall t \in [-r, 0]$ we obtain $z_2(t) = 0, \forall t \in [0, r]$. Following the same reasoning we obtain $z_2(t) = 0$ on the intervals $[r, 2r]$, ..., finally $z_2(t) = 0$, $\forall t \ge -r$. To complete the proof we shall prove that $z_1(t) \le q^*$ for all $t \ge -r$. By using $z_2(t) = 0, \forall t \ge -r$, we deduce from system (13) that

$$\dot{z}_1(t) = H_{11}z_1(t) + G_{11}z_1(t-r) \tag{15}$$

with $-q^* \le z_1(t) \le q^*$, $\forall t \ge -r$ and $q^* > 0$.

By replacing q by q^* in (10), and system (9) by system (15), we deduce from [17] that $D(I_s, q^*, q^*)$ is positively invariant with respect to system (15), that is

$$-q^* \le z_1(t) \le q^*, \ \forall t \ge -r$$

finally

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \le z(t) \le \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall t \ge -r$$

this implies that the polyhedral set $D(I_m, q, q)$ is positively invariant independent of delay with respect to system (9).

Only If:)

Assume that the polyhedral set $D(I_m, q, q)$ is positively invariant with respect to system (9). Let $z(.)$ be the solution of system (9) with

$$-q \le z(t) \le q, \ \forall t \in [-r, 0]$$

The positive invariance of the set $D(I_m, q, q)$ implies that

$$-q \leq z(t) \leq q,\ \forall t \geq -r$$

Therefore

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \leq \begin{pmatrix} q^* \\ 0 \end{pmatrix},\ \forall t \geq -r$$

then $z_2(t) = 0$ for all $t \geq -r$.

From system (12) we obtain

$$\dot{z}_2(t) = H_{21}z_1(t) + G_{21}z_1(t-r) = 0\ ,\ \forall t \geq 0$$

then

$$H_{21}z_1(t) + G_{21}z_1(t-r) = 0\ ,\ \forall z_1(t), z_1(t-r) \in D(I_s, q^*, q^*)$$

which implies $H_{21} = G_{21} = 0$.

From system (12) and $H_{21} = G_{21} = 0$ we deduce that

$$\dot{z}_1(t) = H_{11}z_1(k) + G_{11}z_1(k-r) \tag{16}$$

with

$$-q^* \leq z_1(t) \leq q^*\ ,\ \forall t \geq -r,\ q^* > 0$$

this implies that the domain $D(I_s, q^*, q^*)$, with $q^* > 0$, is a positively invariant set respectively to system (16). As mentioned at the beginning of the subsection we deduce from [17] that $(\overline{H_{11}} + \mid G_{11} \mid)q^* \leq 0$. □

2.2. **Positively invariant conditions dependent of delay.** It is easy to prove the solution of system (14) verifies

$$\begin{aligned} z(t-r) &= z(t) - \textstyle\int_{-r}^{0} \dot{z}(t+s)ds \\ &= z(t) - \textstyle\int_{-r}^{0} [Hz(t+s) + Gz(t-r+s)]ds \end{aligned} \tag{17}$$

If we return to system (14) using this expression for $z(t-r)$, we obtain the equation

$$\dot{z}(t) = (H+G)z(t) - G\int_{-r}^{0} [Hz(t+s) + Gz(t-r+s)]ds,\ \forall t > 0 \tag{18}$$

Then

$$\begin{cases} \dot{z}(t) = Mz(t) - \int_{-r}^{0} [Vz(t+s) + Wz(t-r+s)]ds,\ \forall t > 0 \\ z(\theta) = \phi(\theta),\ \theta \in [-2r, 0] \end{cases} \tag{19}$$

with

$$M = H + G,\ V = GH,\ W = G^2. \tag{20}$$

For arbitrary initial data on $[-2r, 0]$. If the zero solution of (19) is asymptotically stable, then the zero solution of (9) is asymptotically stable since (9) is a special case of (19)). Therefore, for simplicity, we shall use the system dynamics in (19) to obtain stability or positive invariance conditions for system (9).

In the following, we will give necessary and sufficient condition to have $D(I_m, q, q)$ positively invariant with respect to motions of system (9) with delay dependence.

Remark 1. In [2] the author has proved that $D(I_m, q, q)$, with $q > 0$, is positively invariant dependent of delay with respect to system (9) if and only if

$$(\overline{M} + r(\mid V \mid + \mid W \mid))q \leq 0 \tag{21}$$

In this paper, where $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$, we prove the following result.

Theorem 2.3. *The polyhedral set $D(I_m, q, q)$ with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, $q^* > 0$ and $q^* \in \mathbb{R}^s$ is positively invariant dependent of delay with respect to system (9) if and only if*

$$\begin{cases} (\overline{M_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid))q^* \leq 0 \\ M_{21} = V_{21} = W_{21} = 0 \end{cases}$$

with

$$\begin{cases} M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \ V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \ \text{and} \ W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \\ M_{11}, V_{11}, W_{11} \in \mathbb{R}^{s\times s} \ \text{and} \ M_{12}, V_{12}, W_{12} \in \mathbb{R}^{s\times(m-s)} \\ M_{21}, V_{21}, W_{21} \in \mathbb{R}^{(m-s)\times s} \ \text{and} \ M_{22}, V_{22}, W_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

Proof.

If:)

Let assume that

$$\begin{cases} (\overline{M_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid))q^* \leq 0 \\ M_{21} = V_{21} = W_{21} = 0 \end{cases}$$

Let be $z(.)$ be a solution of system (19) with

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leq z(t) \leq \begin{pmatrix} q^* \\ 0 \end{pmatrix}, \ \forall t \in [-2r, 0] \tag{22}$$

we can decompose $z(t)$ into two vectors $z_1(t)$ and $z_2(t)$ with $-q^* \leq z_1(t) \leq q^*$ and $z_2(t) = 0$ for all $t \in [-2r, 0]$.

According to the above we can decompose system (19) into two systems, so we have

$$\begin{cases} \begin{pmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} - \\ \int_{-r}^{0} \left[\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \begin{pmatrix} z_1(t+s) \\ z_2 t+s) \end{pmatrix} + \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} z_1(t-r+s) \\ z_2(t-r+s) \end{pmatrix} \right] ds \\ \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \end{cases}$$

with

$$\begin{cases} M_{11}, V_{11}, W_{11} \in \mathbb{R}^{s\times s} \; and \; M_{12}, V_{12}, W_{12} \in \mathbb{R}^{s\times(m-s)} \\ M_{21}, V_{21}, W_{21} \in \mathbb{R}^{(m-s)\times s} \; and \; M_{22}, V_{22}, W_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

Thus, we have

$$\begin{cases} \dot{z}_1(t) &= M_{11}z_1(t) + M_{12}z_2(t) - \int_{-r}^{0}[V_{11}z_1(t+s) + V_{12}z_2(t+s) \\ & \quad +W_{11}z_1(t-r+s) + W_{12}z_2(t-r+s)]ds \\ z_1(\theta) &= \phi_1(\theta), \; \theta \in [-2r, 0] \end{cases} \tag{23}$$

and

$$\begin{cases} \dot{z}_2(t) &= M_{21}z_1(t) + M_{22}z_2(t) - \int_{-r}^{0}[V_{21}z_1(t+s) + V_{22}z_2(t+s) \\ & \quad +W_{21}z_1(t-r+s) + W_{22}z_2(t-r+s)]ds \\ z_2(\theta) &= \phi_2(\theta), \; \theta \in [-2r, 0] \end{cases} \tag{24}$$

Using $M_{21} = V_{21} = W_{21} = 0$ we obtain

$$\begin{cases} \dot{z}_2(t) = M_{22}z_2(t) - \int_{-r}^{0}[V_{22}z_2(t+s) + W_{22}z_2(t-r+s)]ds \\ z_2(\theta) = \phi_2(\theta), \; \theta \in [-2r, 0] \end{cases} \tag{25}$$

the solution of system (25) can be written as

$$\begin{aligned} z_2(t) &= e^{H_{22}t}z_2(0) + \int_0^t e^{M_{22}(t-\tau)} \int_{-r}^{0}[V_{22}z_2(\tau+s) + W_{22}z_2(\tau-r+s)]dsd\tau \\ &= e^{H_{22}t}z_2(0) + \int_0^t e^{M_{22}\mu} \int_{-r}^{0}[V_{22}z_2(t-\mu+s) + W_{22}z_2(t-\mu-r+s)]dsd\mu \end{aligned}$$

For $0 \le \mu \le t$, $0 \le t \le r$ and $-r \le s \le 0$ we obtain $-r \le t-\mu+s \le 0$ and $-2r \le t-\mu-r+s \le -r$, by using $z_2(t) = 0$ for all $t \in [-2r, 0]$ then $z_2(t-\mu+s) = 0$ and $z_2(t-\mu-r+s) = 0$ thus, we have $z_2(t) = 0$, $\forall t \in [0, r]$. In the same way we obtain $z_2(t) = 0$ on the intervals $[r, 2r]$, $[2r, 3r]$,..., then $z_2(t) = 0$, $\forall t \ge 0$. To complete the proof we shall prove that $z_1(t) \le q^*$, $\forall t \ge 0$. By using $z_2(t) = 0$, $\forall t \ge -r$, we deduce from system (23) that

$$\begin{cases} \dot{z}_1(t) = M_{11}z_1(t) - \int_{-r}^{0}[V_{11}z_1(t+s) + W_{11}z_1(t-r+s)]ds \\ z_1(\theta) = \phi_1(\theta), \; \theta \in [-2r, 0] \end{cases} \tag{26}$$

with

$$-q^* \leq z_1(t) \leq q^*,\ \forall t \in [-2r, 0]\ and\ q^* > 0$$

since $(\overline{M_{11}} + r(|\ V_{11}\ | + |\ W_{11}\ |))q^* \leq 0$, and by virtue of Remark 1 we deduce that

$$-q^* \leq z_1(t) \leq q^*,\ \forall t \geq -2r$$

finally we have $-q \leq z(t) \leq q,\ \forall t \geq -2r$. This implies that the polyhedral set $D(I_m, q, q)$ is positively invariant dependent of delay with respect to system (9).

Only If:)

Assume that the polyhedral set $D(I_m, q, q)$ is positively invariant dependent of delay with respect to system (9). Let $z(.)$ the solution of system (19) with

$$-q \leq z(t) \leq q,\ \forall t \in [-2r, 0]$$

The positive invariance of the set $D(I_m, q, q)$ implies that

$$-q \leq z(t) \leq q,\ \forall k \geq -2r$$

Therefore

$$-\begin{pmatrix} q^* \\ 0 \end{pmatrix} \leq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \leq \begin{pmatrix} q^* \\ 0 \end{pmatrix},\ \forall t \geq -2r$$

then $z_2(t) = 0$, $\forall t \geq -2r$. From system (23) we obtain

$$\dot{z}_2(t) = M_{21} z_1(t) - \int_{-r}^{0} [V_{21} z_1(t+s) + W_{21} z_1(t-r+s)] ds = 0,\ t \geq -2r$$

for all $z_1(t)$, $z_1(t+s)$ and $z_1(t-r+s)$ in $D(I_s, q^*, q^*)$ with $-r \leq s \leq 0$, this implies that $M_{21} = V_{21} = W_{21} = 0$.

From system (24) and $M_{21} = V_{21} = W_{21} = 0$ we deduce that

$$\begin{cases} \dot{z}_1(t) = M_{11} z_1(t) - \int_{-r}^{0} [V_{11} z_1(t+s) + W_{11} z_1(t-r+s)] ds \\ z_1(\theta) = \phi_1(\theta),\ \theta \in [-2r, 0] \end{cases} \quad (27)$$

with

$$-q^* \leq z_1(t) \leq q^*,\ \forall t \geq -2r$$

By virtue of Remark 1 we obtain $(\overline{M_{11}} + r(|\ V_{11}\ | + |\ W_{11}\ |))q^* \leq 0$. □

3. Main results

In this section, we will establish sufficient conditions for a linear state feedback control law $u = Fx$ with $F \in \mathbb{R}^{m\times n}$, $rankF = m$, and F verify (4) and (5) (or (6) and (7)) to be a solution to the linear constrained regulation problem. For that, we need the two lemmas below.

Lemma 3.1 ([16], Lemma 4.1)**.** *The set $KerF$ with $F \in \mathbb{R}^{m\times n}$, and $rankF = m$ is positively invariant with respect to motions of system (3) if and only if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \end{cases}$$

Lemma 3.2 ([16], Lemma 4.2)**.** *If domain $D(F, q, q)$ is positively invariant with respect to system (3), then $kerF$ is also positively invariant with respect to system (3).*

Remark 2. The strict positivity of q in Lemma 3.2 is not necessary.

In the following, we apply the results established in section 2 and the results of Lemma 3.1 and Lemma 3.2 to the problem of the constrained regulator described in section 1, we obtain the following results.

3.1. Independent of delay case.

Theorem 3.3. *The polyhedral set $D(F, q, q)$ with $F \in \mathbb{R}^{m\times n}$, and $rankF = m$ is positively invariant independent of delay with respect to system (3) if and only if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \\ (\overline{H_{11}}+ \mid G_{11} \mid)q^* \leq 0 \text{ and } H_{21} = G_{21} = 0 & (c3) \end{cases}$$

with

$$\begin{cases} H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \text{ and } G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\ H_{11}, G_{11} \in \mathbb{R}^{s\times s}, \\ H_{12}, G_{12} \in \mathbb{R}^{s\times(m-s)},\ H_{21}, G_{21} \in \mathbb{R}^{(m-s)\times s} \text{ and } H_{22}, G_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

Proof.

Necessity:

Let domain $D(F, q, q)$ be positively invariant with respect to system (3). By virtue of Lemma 3.2, $KerF$

is also positively invariant with respect to system (3) and by Lemma 3.1 there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \end{cases}$$

Consider the change in variables $z(t) = Fx(t)$. By conditions $(c1)$-$(c2)$, system (3) can be transformed to system (9) and $D(F, q, q)$ to domain $D(I_m, q, q)$ which is also positively invariant with respect to system (9) and by virtue of Theorem 2.2 is equivalent to condition $(c3)$.

Sufficiency:

Consider the change in variables $z(t) = Fx(t)$. By conditions $(c1)$-$(c2)$, system (3) can be transformed to system (9) and domain $D(F, q, q)$ to domain $D(I_m, q, q)$. By the use of Theorem 2.2, condition $(c3)$ ensures the domain $D(I_m, q, q)$ is positively invariant with respect to system (9), or equivalently $D(F, q, q)$ is positively invariant with respect to system (3). □

We are now in a position to establish conditions for a linear state feedback control law $u = Fx$ to be a solution to the linear constrained regulation problem.

Theorem 3.4. *For a matrix $F \in \mathbb{R}^{m\times n}$ with $rankF = m$ if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \\ (\overline{H_{11}} + \mid G_{11} \mid)q^* < 0 \text{ and } H_{21} = G_{21} = 0 & (c3) \end{cases}$$

with

$$\begin{cases} H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \text{ and } G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \\ H_{11}, G_{11} \in \mathbb{R}^{s\times s},\ H_{12}, G_{12} \in \mathbb{R}^{s\times(m-s)},\ H_{21}, G_{21} \in \mathbb{R}^{(m-s)\times s} \text{ and } H_{22}, G_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

then $u = Fx$ is a stabilizing control and $D(F, q, q)$ is an admissible domain of attraction for the system (3).

Proof. By virtue of Theorem 3.3, the first conditions $(c1)$-$(c2)$ and $(c3)$ imply the positive invariance of the set $D(F, q, q)$. To complete the proof we shall prove that $u = Fx$ is a stabilizing control in $D(F, q, q))$. Let us make the change in variables $z(t) = Fx(t)$ and $rankF = m$, it follows that

$$\begin{cases} \dot{z}(t) = Hz(t) + Gz(t - r) \\ z(\theta) = \psi(\theta),\ \theta \in [-r, 0] \end{cases} \tag{28}$$

Let $x(\theta) \in D(F, q, q)$ for $\theta \in [-r, 0]$, by positive invariance of $D(F, q, q)$ we deduce $x(t) \in D(F, q, q)$, $\forall t \geq 0$ thus, we have $z(t) \in D(I_m, q, q)$, then we decompose the system (28) into two systems that we have already established in section 1. So, we have $z_1(t) \in D(I_s, q^*, q^*)$ and $z_2(t) \in D(I_{m-s}, 0, 0)$ with $z_1 \in \mathbb{R}^s$ and $z_2 \in \mathbb{R}^{m-s}$.

We are interested in the system

$$\begin{cases} \dot{z}_1(t) = H_{11}z_1(t) + G_{11}z_1(t-r) \\ z_1(\theta) = \psi_1(\theta),\ \theta \in [-r, 0] \end{cases} \tag{29}$$

with $z_1(t) \in D(I_s, q^*, q^*)$ for all $t \geq 0$.

Let

$$V(z_1(t)) = \max_{0 \leq i \leq s}(\max(\frac{(z_1)_i(t)}{q_i^*}, \frac{-(z_1)_i(t)}{q_i^*}))$$

where $z_1(t)$ denotes the trajectory of system (29), $(z_1)_i$ the i-th component of z_1 and q_i^* the i-th component of q^*. We shal prove that under condition $(\overline{H_{11}} + \mid G_{11} \mid)q^* < 0$, the positive definite function V is a Lyapunov function.

Let $z_1(.)$ be a solution of system (29) such that at time t the following inequality holds [14], [12] and [21]

$$V(z_1(t-r)) \leq V(z_1(t)) \tag{30}$$

there exist $i \in 1, ..., s$ such that

$$V(z_1(t)) = \frac{(z_1)_i(t)}{q_i^*} \ or \ V(z_1(t)) = \frac{-(z_1)_i(t)}{q_i^*}$$

Let assume that $V(z_1(t)) = \dfrac{(z_1)_i(t)}{q_i^*}$, then

$$\dot{V}(z_1(t)) = \frac{1}{q_i^*}[\sum_{j=1}^{s} h_{ij}(z_1)_j(t) + \sum_{j=1}^{s} g_{ij}(z_1)_j(t-r)]$$

with h_{ij} the components of i-th row of matrix H_{11} and with g_{ij} the components of i-th row of matrix G_{11}.

Using

$$\begin{cases} h_{ij} = h_{ij}^+ - h_{ij}^- \\ g_{ij} = g_{ij}^+ - g_{ij}^- \end{cases}$$

we obtain

$$\begin{aligned} \dot{V}(z_1(t)) \quad = \quad & \frac{1}{q_i^*}[h_{ii}(z_1)_i(t) + \textstyle\sum_{j \neq i} h_{ij}^+(z_1)_j(t) + \sum_{j \neq i} h_{ij}^-(-(z_1)_j(t)) \\ & + \textstyle\sum_{j=1}^{s} g_{ij}^+(z_1)_j(t-r) + \sum_{j=1}^{s} g_{ij}^-(-(z_1)_j(t-r))] \end{aligned}$$

According to the definition of V:

$$(z_1)_j(t) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t) \ and \ -(z_1)_j(t) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t)$$

and from inequality 30:

$$(z_1)_j(t-r) \le \frac{q_j^*}{q_i^*}(z_1)_i(t) \; and \; -(z_1)_j(t-r) \le \frac{q_j^*}{q_i^*}(z_1)_i(t)$$

it follows that

$$\dot V(z_1(t)) \le \frac{1}{q_i^*}[h_{ii}q_i^* + \sum_{j\neq i} h_{ij}^+ q_j^* + \sum_{j\neq i} h_{ij}^- q_j^* + \sum_{j=1}^{s} g_{ij}^+ q_j^* + \sum_{j=1}^{s} g_{ij}^- q_j^*]V(z_1(t))$$

By using

$$\begin{cases} \mid g_{ij} \mid = g_{ij}^+ + g_{ij}^- \\ \overline{h}_{ij} = h_{ii} \; if \; i=j \; and \; \overline{h}_{ij} = \mid h_{ij} \mid \; if \; i \neq j \end{cases}$$

we have

$$\dot V(z_1(t)) \le \frac{1}{q_i^*}[\sum_{j=1}^{s} \overline{h}_{ij} q_j^* + \sum_{j=1}^{s} \mid g_{ij} \mid q_j^*]V(z_1(t))$$

vectorial form is

$$\dot V(z_1(t)) \le \frac{1}{q_i^*}[\overline{H_{11}} + \mid G_{11} \mid]_i q^* V(z_1(t))$$

where, denotes $[\overline{H_{11}} + \mid G_{11} \mid]_i$ the i-th row of the matrix $\overline{H_{11}} + \mid G_{11} \mid$. From condition $(\overline{H_{11}} + \mid G_{11} \mid)q^* < 0$, we have $\dot V(z_1(t)) < 0$.
If $V(z_1(t)) = \dfrac{-(z_1)_i}{q_i^*}$, then a similar argument leads to the same conclusion. So, $\lim_{t\to+\infty}(z_1(t;\psi_1(\theta))) = 0$ and $z_2(k) = 0, \; \forall t \ge 0$, therefore $\lim_{t\to+\infty}(z(t;\psi(\theta))) = 0$. By $rankF = m$ we deduce that $\lim_{t\to+\infty}(x(t;\varphi(\theta))) = 0$. □

3.2. **Dependent of delay case.**

Theorem 3.5. *The polyhedral set $D(F,q,q)$ with $F \in \mathbb{R}^{m\times n}$ and $rankF = m$ is positively invariant dependent of delay with respect to system (3) if and only if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \\ (\overline{M_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid))q^* \le 0 \; and \; M_{21} = V_{21} = W_{21} = 0 & (c3) \end{cases}$$

with

$$\begin{cases} M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \; V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \; and \; W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \\ M_{11}, V_{11}, W_{11} \in \mathbb{R}^{s\times s} \; and \; M_{12}, V_{12}, W_{12} \in \mathbb{R}^{s\times(m-s)} \\ M_{21}, V_{21}, W_{21} \in \mathbb{R}^{(m-s)\times s} \; and \; M_{22}, V_{22}, W_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

Proof. This follows readily from Theorem 2.3 and Theorem 3.3. □

Theorem 3.6. *For a matrix $F \in \mathbb{R}^{m\times n}$ with $rank F = m$ if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ satisfying:*

$$\begin{cases} F(A_0 + BF) = HF & (c1) \\ FA_1 = GF & (c2) \\ (\overline{M_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid))q^* < 0 \text{ and } M_{21} = V_{21} = W_{21} = 0 & (c3) \end{cases}$$

with

$$\begin{cases} M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \text{ and } W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \\ M_{11}, V_{11}, W_{11} \in \mathbb{R}^{s\times s} \text{ and } M_{12}, V_{12}, W_{12} \in \mathbb{R}^{s\times(m-s)} \\ M_{21}, V_{21}, W_{21} \in \mathbb{R}^{(m-s)\times s} \text{ and } M_{22}, V_{22}, W_{22} \in \mathbb{R}^{(m-s)\times(m-s)} \end{cases}$$

Then $u = Fx$ is a stabilizing control and $D(F, q, q)$ is an admissible domain of attraction for the system (3).

Proof. By virtue of Theorem 3.5, the first conditions $(c1)$-$(c2)$ and $(c3)$ imply the positive invariance of the set $D(F, q, q)$. To complete the proof, we shall prove that $u = Fx$ is a stabilizing control in $D(F, q, q)$. The change in variable $z(t) = Fx(t)$ transform the system (3) to

$$\begin{cases} \dot{z}(t) = Hz(t) + Gz(t-r) \\ z(\theta) = \psi(\theta), \ \theta \in [-r, 0] \end{cases} \tag{31}$$

The usual scheme used in the literature for obtaining delay-dependent stability is to use system (19) instead of system (31) that is

$$\begin{cases} \dot{z}(t) = Mz(t) - \int_{-r}^{0} [Vz(t+s) + Wz(t-r+s)]ds, \ \forall t > 0 \\ z(\theta) = \phi(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{32}$$

The asymptotic stability of (32) guarantees the asymptotic stability of (31) .

Let $x(\theta) \in D(F, q, q)$ for $\theta \in [-r, 0]$, by positive invariance of $D(F, q, q)$ we deduce that $x(t) \in D(F, q, q)$, $\forall t \geq 0$ thus, we have $z(k) \in D(I_m, q, q)$, $\forall t \geq 0$ with $q = \begin{pmatrix} q^* \\ 0 \end{pmatrix}$, then we decompose the system (32) into two systems that we have already established in section 1. So, we have $z_1(t) \in D(I_s, q^*, q^*)$ and $z_2(t) \in D(I_{m-s}, 0, 0)$, $\forall t \geq 0$ with $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}$. We are interested in the system

$$\begin{cases} \dot{z}_1(t) = M_{11}z_1(t) - \int_{-r}^{0} [V_{11}z_1(t+\mu) + W_{11}z_1(t-r+\mu)]d\mu \\ z_1(\theta) = \phi_1(\theta), \ \theta \in [-2r, 0] \end{cases} \tag{33}$$

with $z_1(t) \in D(I_s, q^*, q^*)$ for all $t \geq 0$.

Let

$$V(z_1(t)) = \max_{0\leq i\leq s} (\max(\frac{(z_1)_i(t)}{q_i^*}, \frac{-(z_1)_i(t)}{q_i^*}))$$

where $z_1(t)$ denotes the trajectory of system (33), $(z_1)_i$ the i-th component of z_1 and q_i^* the i-th component of q^*. We shal prove that under condition $(\overline{M_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid))q^* < 0$, the positive definite function V is a Lyapunov function.

Let $z_1(.)$ be a solution of system (33) such that at time t the following inequality holds [14], [12] and [21]

$$V(z_1(t+l)) \leq V(z_1(t)),\ for\ l \in [-2r, 0] \tag{34}$$

there exist $i \in 1, ..., s$ such that $V(z_1(t)) = \dfrac{(z_1)_i}{q_i^*}$ or $V(z_1(t)) = \dfrac{-(z_1)_i}{q_i^*}$.

Let assume that $V(z_1(t)) = \dfrac{(z_1)_i}{q_i^*}$, then

$$\dot{V}(z_1(t)) = \frac{1}{q_i^*}[\sum_{j=1}^{s} m_{ij}(z_1)_j(t) - \int_{-r}^{0}\sum_{j=1}^{s}(v_{ij}(z_1)_j(t+\mu) + w_{ij}(z_1)_j(t-r+\mu))d\mu]$$

with m_{ij} the components of i-th row of matrix M_{11}, v_{ij} the components of i-th row of matrix V_{11} and w_{ij} the components of i-th row of matrix W_{11}.

Using

$$\begin{cases} m_{ij} = m_{ij}^+ - m_{ij}^- \\ v_{ij} = v_{ij}^+ - v_{ij}^- \\ w_{ij} = w_{ij}^+ - w_{ij}^- \end{cases}$$

we obtain

$$\begin{aligned} \dot{V}(z_1(t)) \quad = \quad & \frac{1}{q_i^*}[m_{ii}(z_1)_i(t) + \textstyle\sum_{j\neq i}(m_{ij}^+(z_1)_j(t) + m_{ij}^-(-(z_1)_j(t))) + \int_{-r}^{0}\sum_{j=1}^{s}(v_{ij}^+(z_1)_j(t+\mu) \\ & +v_{ij}^-(-(z_1)_j(t+\mu)) + w_{ij}^+(z_1)_j(t-r+\mu) + w_{ij}^-(-(z_1)_j)(t-r+\mu))d\mu] \end{aligned}$$

According to the definition of V:

$$(z_1)_j(t) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t)\ and\ -(z_1)_j(t) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t)$$

and from inequality 34:

$$(z_1)_j(t+l) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t)\ and\ -(z_1)_j(t+l) \leq \frac{q_j^*}{q_i^*}(z_1)_i(t),\ for\ l \in [-2r, 0]$$

it follows that

$$\dot{V}(z_1(t)) \leq \frac{1}{q_i^*}[m_{ii}q_i^* + \sum_{j\neq i}(m_{ij}^+q_j^* + m_{ij}^-q_j^*) + \int_{-r}^{0}\sum_{j=1}^{s}(v_{ij}^+q_j^* + v_{ij}^-q_j^* + w_{ij}^+q_j^* + w_{ij}^-q_j^*)d\mu]V(z_1(t))$$

then

$$\dot{V}(z_1(t)) \leq \frac{1}{q_i^*}[m_{ii}q_i^* + \sum_{j\neq i}(m_{ij}^+q_j^* + m_{ij}^-q_j^*) + r\sum_{j=1}^{s}(v_{ij}^+q_j^* + v_{ij}^-q_j^* + w_{ij}^+q_j^* + w_{ij}^-q_j^*)]V(z_1(t))$$

By using

$$\begin{cases} \mid v_{ij} \mid= v_{ij}^{+} + v_{ij}^{-} \\ \mid w_{ij} \mid= w_{ij}^{+} + w_{ij}^{-} \\ \overline{m}_{ij} = m_{ii} \; if \; i = j \; and \; \overline{m}_{ij} =\mid m_{ij} \mid \; if \; i \neq j \end{cases}$$

we have

$$\dot{V}(z_1(t)) \leq \frac{1}{q_i^*}[\sum_{j=1}^{s} \overline{m}_{ij} q_j^* + r \sum_{j=1}^{s}(\mid v_{ij} \mid q_j^* + \mid w_{ij} \mid q_j^*)] V(z_1(t))$$

vectorial form is

$$\dot{V}(z_1(t)) \leq \frac{1}{q_i^*}[\overline{H_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid)]_i q^* V(z_1(t))$$

where, denotes $[\overline{H_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid)]_i$ the i-th row of the matrix $\overline{H_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid)$. From condition $[\overline{H_{11}} + r(\mid V_{11} \mid + \mid W_{11} \mid)]q^* < 0$, we have $\dot{V}(z_1(t)) < 0$. If $V(z_1(t)) = \frac{-(z_1)_i}{q_i^*}$, then a similar argument leads to the same conclusion. So, $\lim_{t \to +\infty}(z_1(t; \phi_1(\theta))) = 0$ and $z_2(t) = 0, \; \forall t \geq -2r$, hence $\lim_{t \to +\infty}(z(t; \phi(\theta))) = 0$. By $rankF = m$ we deduce that $\lim_{t \to +\infty}(x(t; (\theta))) = 0$. □

4. Algorithm

The results presented in Section 3 rely on the existence of matrices H and G. It is evident that the existence of these matrices is dependent on the matrices A_0, A_1, B, and F. To establish this dependency, we introduce the following lemmas:

Lemma 4.1 (Hmamed et al [17]). *There exists a matrix* $H \in \mathbb{R}^{m \times m}$ $(G \in \mathbb{R}^{m \times m})$ *solution of*

$$\begin{cases} F(A_0 + BF) = HF \\ FA_1 = GF \end{cases} \tag{35}$$

where $F \in \mathbb{R}^{m \times n}$ *and* $rankF = m$, $m \leq n$ *if and only if*

$$rank \begin{bmatrix} F(A_0 + BF) \\ F \end{bmatrix} = m \; (rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m) \tag{36}$$

Corollary 4.2 (Porter [25]). *If condition (36) is satisfied, then the solution of (35) is given by*

$$H = [F_1((A_0)_{11} + F_2((A_0)_{21} + BF_1)]F_1^{-1} \tag{37}$$

and

$$G = [F_1(A_1)_{11} + F_2(A_1)_{21}]F_1^{-1} \tag{38}$$

with $F = \begin{bmatrix} F_1 & F_2 \end{bmatrix}$, $F_1 \in \mathbb{R}^{m \times m}$, $F_2 \in \mathbb{R}^{m \times n-m}$, $rankF_1 = m$,
$B \in \mathbb{R}^{n \times m}, \; A_0 = \begin{bmatrix} (A_0)_{11} & (A_0)_{12} \\ (A_0)_{21} & (A_0)_{22} \end{bmatrix}$, $A_1 = \begin{bmatrix} (A_1)_{11} & (A_1)_{12} \\ (A_1)_{21} & (A_1)_{22} \end{bmatrix}$,

$(A_0)_{11}, (A_1)_{11} \in \mathbb{R}^{m\times m}$, $(A_0)_{12}, (A_1)_{21} \in \mathbb{R}^{m\times n-m}$,
$(A_0)_{21}, (A_1)_{21} \in \mathbb{R}^{n-m\times m}$ *and* $(A_0)_{22}, (A_1)_{22} \in \mathbb{R}^{n-m\times n-m}$.

The search for such a matrix F the solution of the LCRP problem can be done according to the following algorithm:

Algorithm 1: (Independent of delay case)

Step1: Given a set $\Sigma = \{\lambda_1, ..., \lambda_n\}$ with $\Re(\lambda_i) < 0,\ i = 1, ..., n$. Compute F which satisfies $\sigma(A_0+BF) = \Sigma$ (see [28]).

Step2: If $rankF = m$, go to Step3, else go to Step1 and change set Σ.

Step3: If $rank \begin{bmatrix} F(A_0+BF) \\ F \end{bmatrix} = m$ and $rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m$, else go to Step1 and change set Σ.

Step4: Compute H and G .

Step5: Compute $\overline{H}$ and $\mid G \mid$.

Step6: If $\overline{H}$ and $\mid G \mid$ satisfying the condition $c3$ of theorem 3.4, else go to Step1 and change set Σ.

Algorithm 2: (Dependent of delay case)

Step1: Given a set $\Sigma = \{\lambda_1, ..., \lambda_n\}$ with $\Re(\lambda_i) < 0,\ i = 1, ..., n$. Compute F which satisfies $\sigma(A_0+BF) = \Sigma$ (see [28]).

Step2: If $rankF = m$, go to Step3, else go to Step1 and change set Σ.

Step3: If $rank \begin{bmatrix} F(A_0+BF) \\ F \end{bmatrix} = m$ and $rank \begin{bmatrix} FA_1 \\ F \end{bmatrix} = m$, else go to Step1 and change set Σ.

Step4: Compute H and G .

Step5: Compute M, V and W by equations $M = H + G$, $V = GH$ and $W = G^2$.

Step6: Compute $\overline{M}$, $\mid V \mid$ and $\mid W \mid$.

Step7: If $\overline{M}$, $\mid V \mid$ and $\mid W \mid$ satisfying the condition $c3$ of theorem 3.6, else go to Step1 and change set Σ.

Example

Consider the linear discrete-time system with time delay described by the following

$$\dot{x}(t) = A_0x(t) + A_1x(t-r) + Bu(t) \tag{39}$$

where $A_0 = \begin{pmatrix} 1 & -0.125 & 0 \\ 0 & 0 & -0.5 \\ 0 & 0 & 0.1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0.05 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.25 & 0.5 \\ 0 & 0 & 0.5 \end{pmatrix}$

The control vector $u \in \mathbb{R}^2$ is subject to constraints

$$-1 \leq u_1 \leq 1,\ -2 \leq u_2 \leq 2 \text{ and } u_3 = 0$$

Note that A_0 is unstable. The eigenvalues of A_0 are $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = 0.1$.

Let

$$F = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

in our case we choose $H = \begin{pmatrix} -0.5 & 0 & 0 \\ 0 & -0.1875 & 0.1875 \\ 0 & 1.15 & -1.5 \end{pmatrix}$ and $G = \begin{pmatrix} 05 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

We can verify that H and G satisfy the hypothesis of Theorem 2.2

Then $u_1(k) = -3x_1$, $u_2(k) = \frac{1}{4}x_2$ and $u_3(k) = x_2 - 2x_3$ stabilizes the system on

$$\begin{aligned} D(F,q,q) &= \{x \in \mathbb{R}^3 \mid -1 \leq -3x_1 \leq 1 \;;\; -2 \leq \tfrac{1}{4}x_2 \leq 2 \;;\; x_2 - 2x_3 = 0\} \\ &= \{x \in \mathbb{R}^3 \mid -\tfrac{1}{3} \leq x_1 \leq \tfrac{1}{3} \;;\; -8 \leq x_2 \leq 8 \;;\; x_3 = \tfrac{1}{2}x_2\} \end{aligned}$$

We will draw the solution of system (39), for the initial condition

$\varphi_1 = [\frac{1}{3}, 8, 4]^T$in $D(F,p,q)$ with a delay $r = 2$, we notice that the trajectories of our system converge asymptotically to the equilibrium $x_e = 0$. The same results are obtained for arbitrarily initial conditions $\varphi \in D(F,p,q)$.

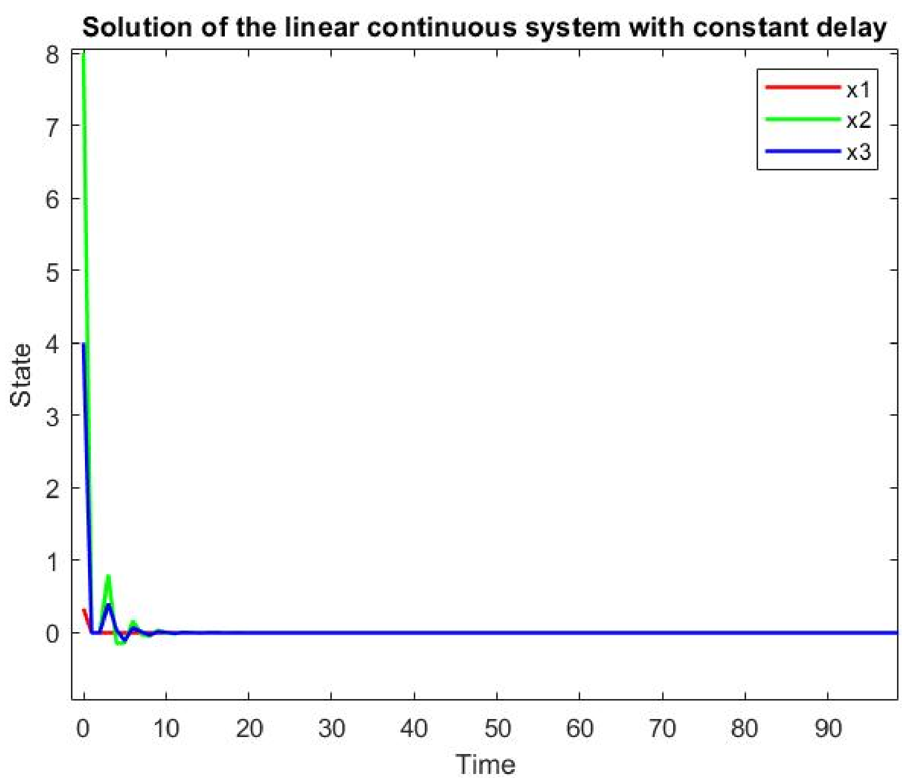

FIGURE 1. The asymptotic stability of the closed-loop system for initial state $\varphi_1 = [\frac{1}{3}, 8, 4]^T$ with $r = 2$.

5. Conclusion

his paper focuses on the study of symmetrical constrained regulation in continuous-time delay dynamical systems, specifically when the equilibrium point lies on the boundary of the constraint domain. The investigation considers two cases: delay-independent and delay-dependent.
In each case, the properties of positive invariance are utilized to provide sufficient conditions for a state feedback control law, represented as $u = Fx$, to be a solution to the linear constrained regulation problem.

Furthermore, the paper concludes by presenting an example that demonstrates the practical application of the obtained results.

References

[1] A. Belhouari, E. Tissir, A. Hmamed, Stability of interval matrix polynomial in continuous and discrete cases, Syst. Control Lett. 18 (1992), 183–189. https://doi.org/10.1016/0167-6911(92)90004-c.

[2] H. Bensalah, L. Baron, Positive invariance of constrained linear continuous-time delay system with delay dependence, in: Proceedings of the International Conference of Control, Dynamic Systems, and Robotics Ottawa, Ontario, Canada, 179, (2015).

[3] A. Benzaouia, The regulator problem for linear discrete-time systems with nonsymmetrical constrained control, in:Proceedings of the 30th IEEE Conference on Decision and Control, IEEE, Brighton, UK, 1991: pp. 1742–1743. https://doi.org/10.1109/CDC.1991.261705.

[4] A. Benzaouia, Further results on the saturated controller design for linear continuous-time systems, in: Proceedings of the 10th Mediterranean Conference on Control and Automation-MED2002, Lisbon, Portugal, July 9-12, 2002.

[5] A. Benzaouia, C. Burgat, Regulator problem for linear discrete-time systems with non-symmetrical constrained control, Int. J. Control. 48 (1988), 2441–2451. https://doi.org/10.1080/00207178808906339.

[6] A. Benzaouia, S. El Faiz, The regulator problem for linear systems with constrained control: an LMI approach, IMA J. Math. Control Inform. 23 (2006), 335–345. https://doi.org/10.1093/imamci/dni062.

[7] A. Benzaouia, A. Hmamed, Regulator problem for linear continuous-time systems with nonsymmetrical constrained control, IEEE Trans. Automat. Control. 38 (1993), 1556–1560. https://doi.org/10.1109/9.241576.

[8] A. Benzaouia, A. Hmamed, F. Tadeo, Stabilisation of controlled positive delayed continuous-time systems, Int. J. Syst. Sci. 41 (2010), 1473–1479. https://doi.org/10.1080/00207720903353641.

[9] G. Bitsoris, Positively invariant polyhedral sets of discrete-time linear systems, Int. J. Control. 47 (1988), 1713–1726. https://doi.org/10.1080/00207178808906131.

[10] G. Bitsoris, S. Olaru, Further results on the linear constrained regulation problem, in: 21st Mediterranean Conference on Control and Automation, IEEE, Platanias, Chania - Crete, Greece, 2013: pp. 824–830. https://doi.org/10.1109/MED.2013.6608818.

[11] G. Bitsoris, S. Olaru, M. Vassilaki, On the linear constrained regulation problem for continuous-time systems, IFAC Proc. Volumes. 47 (2014), 4004–4009. https://doi.org/10.3182/20140824-6-za-1003.02558.

[12] M. Dambrine, J.P. Richard, P. Borne, Feedback control of time-delay systems with bounded control and state, Math. Probl. Eng. 1 (1995), 77–87.

[13] P.O. Gutman, P. Hagander, A new design of constrained controllers for linear systems, IEEE Trans. Automat. Control. 30 (1985), 22–33. https://doi.org/10.1109/tac.1985.1103785.

[14] J.K. Hale, Theory of functional differential equations, Spring-Verlag, New York, (1977).

[15] J.K. Hale, S.M.V. Lunel, Introduction to Function Differntial Equations, Springer-Verlag, New York, (1993).

[16] A. Hmamed, Constrained regulation of linear discrete-time systems with time delay: Delay-dependent and delay-independent conditions, Int. J. Syst. Sci. 31 (2000), 529–536. https://doi.org/10.1080/002077200291109.

[17] A. Hmamed, A. Benzaouia, H. Bensalah, Regulator problem for linear continuous-time delay systems with nonsymmetrical constrained control, IEEE Trans. Automat. Control. 40 (1995), 1615–1619. https://doi.org/10.1109/9.412630.

[18] A. Hmamed, M.A. Rami, A. Benzaouia, F. Tadeo, Stabilization under constrained states and controls of positive systems with time delays, Eur. J. Control. 2 (2012), 182–190.

[19] A. Hmamed, A. Benzaouia, M.A. Rami, F. Tadeo, Memoryless control to drive states of delayed continuous-time systems within the nonnegative orthant, IFAC Proc. Volumes. 41 (2008), 3934–3939. https://doi.org/10.3182/20080706-5-KR-1001.00662.

[20] A. Hmamed, E. Tissir, Further results on the stability of discrete-time matrix polynomials, Int. J. Syst. Sci. 29 (1998), 819–821. https://doi.org/10.1080/00207729808929574.

[21] J.C. Hennet, S. Tarbouriech, Stability and stabilization of delay differential systems, Automatica. 33 (1997), 347–354. https://doi.org/10.1016/s0005-1098(96)00185-9.

[22] N. Stankovic, S. Olaru, S.I. Niculescu, Further remarks on asymptotic stability and set invariance for linear delay-difference equations, Automatica. 50 (2014), 2191–2195. https://doi.org/10.1016/j.automatica.2014.05.019.

[23] P. Lancaster, M. Tismenetsky, The theory of matrices, 2nd ed., Academic Press, New York, (1985).

[24] M.T. Laraba, S. Olaru, S.I. Niculescu, F. Blanchini, G. Giordano, D. Casagrande, S. Miani, Guide on set invariance for delay difference equations, Ann. Rev. Control. 41 (2016), 13–23. https://doi.org/10.1016/j.arcontrol.2016.04.020.

[25] B. Porter, Eigenvalue assignment in linear multivariable systems by output feedback, Int. J. Control. 25 (1977), 483–490. https://doi.org/10.1080/00207177708922246.

[26] S.B. Stojanovic, D. Lj. Debeljkovic, Delay dependent stability of linear time-delay systems, Theor. Appl. Mech. 40 (2012), 223–245.

[27] M. Vassilaki, G. Bitsoris, Constrained regulation of linear continuous-time dynamical systems, Syst. Control Lett. 13 (1989), 247–252. https://doi.org/10.1016/0167-6911(89)90071-6.

[28] W. Wonham, On pole assignment in multi-input controllable linear systems, IEEE Trans. Automat. Control. 12 (1967), 660–665. https://doi.org/10.1109/tac.1967.1098739.