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FUZZY FRACTIONAL BOUNDARY VALUE PROBLEMS WITH HILFER FRACTIONAL DERIVATIVES

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Abstract. This work is concerned with the existence and Ulam-Hyers (UH) stability results of nonlocal fuzzy fractional boundary value problems (FFBVPs) with Hilfer fractional derivative. By using Banach fixed point theorem, we prove the existence and uniqueness of the solution of our problem. And by applying the generalized Gronwall inequality, we explore the UH stability result.

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1. Introduction

Fractional calculus has recently gained widespread acceptance and significance as a result of its numerous application in many fields. Fractional derivatives were discovered by Leibnitz in 1695, and since then, a growing number of scientists have dedicated themselves to study of fractional calculus. The most often used fractional calculus definitions are Riemann-Liouiville (RL) definition and Caputo definition [1–4]. When Hilfer [5] studied fractional time development in physical phenomena, he gave a generalization of both RL and Caputo derivative, and many authors call it the Hilfer fractional derivative [6].

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Fuzzy analysis and fuzzy differential equations (FDEs) have recently been proposed as solutions to the uncertainty imposed by insufficient information in various mathematical models that prodict real-world situations [7–9].

The notion of fuzzy RL differentiability based on Hukuhara differentiability was introduced in [10, 11], and the authors investigated the existence of solutions for specific fuzzy integral equations using the Hausdroff measure of non compactness. Bed et al [12] developed and studied new generalized differentiability notions for fuzzy-valued functions based on Hukuhara differentiability.

On the other hand, stability analysis is a fundamental aspect of mathematical analysis that is a crucial in a variety of engineering and science fields. Arhrrabi et al. [29–31] studied the existence and stability of solutions for a coupled system of fuzzy fractional pantograph stochastic differential equations, averaging principle for fuzzy stochastic differential equations and existence and uniqueness results of fuzzy fractional stochastic differential equations with impulsive. The Ulam's stability may be considered as a special type of data veliance that began with Ulam [13]. Rassias expanded on the notion of UH stability in [14]. Following that, several authors used various methodologies to explore distinct UH stability problems for various types of FFDEs, see [15–20].

In 2018, Hoa et al [21] investigated the FFDEs with $\alpha \in (0,1)$:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{a^+}x(t) = f(t, x(t)), \\ x(a) = x_0, \end{cases}$$

where the function f and initial condition x_0 are fuzzy.

In 2021, Arhrrabi et al [22] discussed the following fuzzy fractional boundary value problem:

$$\begin{cases} \mathcal{D}^{\alpha}x(t) = f(t, \mathcal{D}^{\alpha-1}x(t), x(t)), \\ x(0) = \tilde{0}, \qquad x(a) = A \in E^1, \end{cases}$$

where $f : [0,a] \times E^1 \times E^1 \longrightarrow E^1$ is a continuous mapping and $\alpha \in (1,2)$. They proved existence and uniqueness of the fuzzy solution by using Banach fixed point theorem.

In [23], the authors considered the following FDEs with Hilfer-Katugampola fractional derivative and nonlocal conditions:

$$\begin{cases} {}^{\rho}\mathcal{D}^{\alpha,\beta}_{a^+}x(t) = f(t, x(t)), \\ {}^{\rho}\mathcal{I}^{1-\gamma}_{a^+}x(t) = x_0 = \sum_{i=1}^{m} C_i x(t_i), \gamma = \alpha + \beta(1-\alpha), \end{cases}$$

where $x \in \mathbb{R}$, $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\rho > 0$. They proved existence and uniqueness of fuzzy solution. Rashid et al [24] studied the same problem but this time they used the Hilfer generalized proportional fractional derivative.

Inspired by the above discussion, in this manuscript, we will explore the existence, uniqueness and stability results of the following FFBVPs with fuzzy Hilfer fractional derivatives and nonlocal integral boundary condition:

$$\begin{cases} {}^{H}\mathcal{D}^{\alpha,\beta}_{a^+}x(t) = f(t,x(t)), \qquad t \in [a,b], \\ x(a) = 0, \qquad x(b) = \sum_{i=1}^{m} \delta_i \mathcal{I}^{\varphi_i} x(\xi_i), \end{cases} \tag{1}$$

where $f : [a,b] \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is a fuzzy function, $\varphi_i > 0$, $\delta_i \in \mathbb{R}$, $\xi_i \in [a,b]$, $1 < \alpha < 2$ and $0 \leq \beta \leq 1$. ${}^{H}\mathcal{D}^{\alpha,\beta}_{a^+}$ is the Hilfer fractional derivative of order α and parameter β, and $\mathcal{I}^{\varphi_i}$ is the RL fractional integral of order φ_i, $i = 1,..,m$.

The goal of this work is to create an innovative class of FFBVPs involving fuzzy Hilfer type fractional derivative, as well as to establish existence and stability results for their solutions. However, to the best of our knowledge, there is no works on FFBVPs with fuzzy Hilfer fractional derivative in the literature.

The remainder of this work is organized as follows: In Section 2 we will go through the fundamental definitions of fuzzy fractional and differential equations followed by several necessary Lemmas and assumptions. The existence and uniqueness results for FFBVPs are investigated in Section 3. Following that, in Section 4, the UHs and GUHs results is studied. Finally, in Section 5 an example is given to show the pratical utility of the analytical results.

2. Preliminaries

In this section, we introduce some definitions and notations of fuzzy fractional calculus. Let $\mathcal{F}_{\mathbb{R}^n}$ denote the set of fuzzy subsets of the real axis, if $\Lambda : \mathbb{R}^n \longrightarrow [0,1]$, satisfying the following properties:

(i) Λ is normal, that is, there exists $z_0 \in \mathbb{R}^n$ such that $\Lambda(z_0) = 1$,

(ii) Λ is fuzzy convex, that is, for $0 \leq \lambda \leq 1$

$$\Lambda\big(\lambda z_1 + (1-\lambda)z_2\big) \geq min\left\{\Lambda(z_1), \Lambda(z_2)\right\}, \ for\ any\ z_1, z_2 \in \mathbb{R}^n,$$

(iii) Λ is upper semicontinous on $\mathbb{R}^n$,

(iv) $[\Lambda]^0 = cl\{z \in \mathbb{R}^n : \Lambda(z) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^n, |.|)$.

Then $\mathcal{F}_{\mathbb{R}^n}$ is called the space of fuzzy number. For $q \in (0,1]$, we denote $[\Lambda]^q = \{z \in \mathbb{R}^n | \Lambda(z) \geq q\}$ and $[\Lambda]^0 = \{z \in \mathbb{R}^n | \Lambda(z) > 0\}$. From the conditions (i) to (iv), it follows that the $q - level$

set of Λ, $[\Lambda]^q$, is a nonempty compact interval, for all $q \in [0,1]$ and any $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$.
The notation $[\Lambda]^q = [\underline{\Lambda}(q), \overline{\Lambda}(q)]$, denotes explicitly the $q - level$ set of Λ, for $q \in [0,1]$. We refer to $\underline{\Lambda}$ and $\overline{\Lambda}$ as the lower and upper branches of Λ, respectively. For $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$, we define the lengh of the $q - level$ set of Λ as $len([\Lambda]^q) = \overline{\Lambda}(q) - \underline{\Lambda}(q)$. For addition and scalar multiplication in fuzzy set space $\mathcal{F}_{\mathbb{R}^n}$, we have $[\Lambda_1 + \Lambda_2]^q = [\Lambda_1]^q + [\Lambda_2]^q$, $[\lambda\Lambda]^q = \lambda[\Lambda]^q$.
Let $\Lambda_1, \Lambda_2 \in \mathcal{F}_{\mathbb{R}^n}$, if there exists $\Lambda_3 \in \mathcal{F}_{\mathbb{R}^n}$ such that $\Lambda_1 = \Lambda_2 + \Lambda_3$, then Λ_3 is called the Hukuhara difference of Λ_1 and Λ_2 and it is noted by $\Lambda_1 \ominus \Lambda_2$.

Definition 2.1. [24] The generalized Hukuhara difference (gH-difference) of two fuzzy numbers $\Lambda_1, \Lambda_2 \in \mathcal{F}_{\mathbb{R}^n}$ is defined as follows:

$$\Lambda_1 \ominus_{gH} \Lambda_2 = \Lambda_3 \Leftrightarrow (i)\ \Lambda_1 = \Lambda_2 + \Lambda_3 \text{ Or } (ii)\ \Lambda_2 = \Lambda_1 + (-1)\Lambda_3. \tag{2}$$

Remark 2.2. [7] Based on the definition of the diameter of the $q - level$ set of $\Lambda \in \mathcal{F}_{\mathbb{R}^n}$, from (2), we have:
$*$ The condition of the existence of $\Lambda_1 \ominus_{gH} \Lambda_2$ in the case (i) is $len([\Lambda_1]^q) \geq len([\Lambda_2]^q)$.
$*$ The condition of the existence of $\Lambda_1 \ominus_{gH} \Lambda_2$ in the case (ii) is $len([\Lambda_2]^q) \geq len([\Lambda_1]^q)$.

Definition 2.3. [24] A function $\Lambda : [a,b] \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ is called increasing (decreasing) on $[a,b]$ if for every $q \in [0,1]$, the function $t \longmapsto len[\Lambda(t)]^q$ is increasing (decreasing) on $[a,b]$, then we say that Λ is monotone on $[a,b]$.

The Hausdorff distance between fuzzy numbers is given by

$$\begin{aligned}\mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) &= \sup_{0 \leq q \leq 1} \left\{ |\underline{\Lambda}_1(q) - \underline{\Lambda}_2(q)|, |\overline{\Lambda}_1(q) - \overline{\Lambda}_2(q)| \right\}, \\ &= \sup_{0 \leq q \leq 1} \mathcal{D}_H\big([\Lambda_1]^q, [\Lambda_2]^q\big).\end{aligned}$$

The metric space $(\mathcal{F}_{\mathbb{R}^n}, \mathcal{D}_\infty)$ is complet metric space and the following properties of the metric $\mathcal{D}_\infty$ are valid.

$$\mathcal{D}_\infty\big(\Lambda_1 + \Lambda_3, \Lambda_2 + \Lambda_3\big) = \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\lambda\Lambda_1, \lambda\Lambda_2\big) = |\lambda| \mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big),$$

$$\mathcal{D}_\infty\big(\Lambda_1, \Lambda_2\big) \leq \mathcal{D}_\infty\big(\Lambda_1, \Lambda_3\big) + \mathcal{D}_\infty\big(\Lambda_3, \Lambda_2\big),$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{F}_{\mathbb{R}^n}$ and $\lambda \in \mathbb{R}^n$. Let $C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ denote the set of all continuous fuzzy functions and $AC\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ the set of all absolutely continuous fuzzy functions on $[a,b]$ with value in $\mathcal{F}_{\mathbb{R}^n}$ and $AC^1\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ the set of all absolutely continuously differentiable fuzzy functions on the interval $[a,b]$ with value in $\mathcal{F}_{\mathbb{R}^n}$. For $\gamma \in (0,1)$, let $C_\gamma\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ the space of continuous functions defined by

$$C_\gamma := \{x \in (a,b] \longrightarrow \mathcal{F}_{\mathbb{R}^n} : \ (t-u)^{1-\gamma}x(t) \in C[a,b]\}.$$

Denote by $L\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ the set of all fuzzy functions $x:[a,b] \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ such that $t \mapsto \mathcal{D}_\infty[x(t),\hat{0}]$ belong to $L^1[a,b]$.

Definition 2.4. [25] The RL fractional integral of order $\alpha > 0$ of a continuous function is defined by

$$\mathcal{I}^\alpha_{a+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}x(u)du.$$

Definition 2.5. [25] The RL fractional derivative of order $\alpha > 0$ of a continuous function x is given by

$$\begin{aligned} {}^{RL}\mathcal{D}^\alpha_{a+}x(t) :&= D^n\mathcal{I}^{n-\alpha}_{a+}x(t), \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-u)^{n-\alpha-1}x(u)du, \end{aligned}$$

where $n = [\alpha]+1$.

For $x \in L\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$, we define the RL fractional integral of order α of the fuzzy function x:

$$x_\alpha(t) := \mathcal{I}^\alpha_{a+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}x(u)du, \ t \geq a.$$

Since $[x(t)]^q = \big[\underline{x}(t,q),\overline{x}(t,q)\big]$, we can define the fuzzy RL fractional integral of fuzzy function x based on lower and upper functions

$$[\mathcal{I}^\alpha_{a+}x(t)]^q = \big[\mathcal{I}^\alpha_{a+}\underline{x}(t,q),\mathcal{I}^\alpha_{a+}\overline{x}(t,q)\big], \qquad t \geq a.$$

Where

$$\mathcal{I}^\alpha_{a+}\underline{x}(t,q) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}\underline{x}(u,q)du,$$

and

$$\mathcal{I}^\alpha_{a+}\overline{x}(t,q) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}\overline{x}(u,q)du;$$

It follows that the operator $x_\alpha(t)$ is linear and bounded from $C\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ to $C\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ and we have

$$K \leq \frac{\|x\|}{\Gamma(\alpha)}\int_a^t (t-u)^{\alpha-1}du = \frac{\|x\|}{\Gamma(\alpha+1)}(t-u)^\alpha,$$

where $\|x\| = \sup_{a\leq t\leq b}\mathcal{D}_\infty\big(x(t),\hat{0}\big)$.

Definition 2.6. [23] The fuzzy Hilfer fractional derivative of order α and parameter β of a function $x \in C_{1-\gamma}[a,b]$ is defined by

$$^{H}\mathcal{D}_{a^+}^{\alpha,\beta} x(t) = \mathcal{I}_{a^+}^{\beta(n-\alpha)} \left(\frac{d}{dt}\right)^n \mathcal{I}_{a^+}^{(1-\beta)(n-\alpha)} x(t),$$

if the gH-derivative $x'_{1-\alpha}(t)$ exists, where $n-1<\alpha<n$, and $0 \leq \beta \leq 1$.

The composition of the fuzzy fractional integration operator $\mathcal{I}_{a^+}^{\alpha}$ with the fuzzy fractional differentiation operator $^{RL}\mathcal{D}_{a^+}^{\alpha}$ is given by the following result.

Lemma 2.7. [25,28] *Let $\alpha > 0$, $n = [\alpha]+1$ and let $f \in L^1(a,b)$, $f_{n-\alpha}(x) = \mathcal{I}_{a^+}^{n-\alpha} f(x) \in AC^n[a,b]$. Then*

$$\mathcal{I}_{a^+}^{\alpha}(^{RL}\mathcal{D}_{a^+}^{\alpha} f)(t) = f(t) - \sum_{i=1}^{n} \frac{f_{n-\alpha}^{(n-i)}(a)}{\Gamma(\alpha-i+1)}(t-a)^{\alpha-1}.$$

In particular, if $1<\alpha<2$, then

$$\mathcal{I}_{a^+}^{\alpha}\left(^{RL}\mathcal{D}_{a^+}^{\alpha} f\right)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} - \frac{f_{2-\alpha}(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}.$$

Lemma 2.8. *Let $1 < \alpha \leq 2$ and $0 \leq \gamma \leq 1$. If $x \in AC\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ and $\mathcal{I}_{a^+}^{1-\alpha} x \in C_{\gamma}^1[a,b]$, $\mathcal{I}_{a^+}^{2-\alpha} x \in C_{\gamma}^2[a,b]$, then we have*

$$\mathcal{I}_{a^+}^{\alpha}\left(^{RL}\mathcal{D}_{a^+}^{\alpha} x\right)(t) = x(t) \ominus \left(\frac{\mathcal{I}_{a^+}^{1-\alpha} x(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} + \frac{\mathcal{I}_{a^+}^{2-\alpha} x(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right),$$

if x is increasing, and we have

$$\mathcal{I}_{a^+}^{\alpha}\left(^{RL}\mathcal{D}_{a^+}^{\alpha} x\right)(t) = -\left(\frac{\mathcal{I}_{a^+}^{1-\alpha} x(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} + \frac{\mathcal{I}_{a^+}^{2-\alpha} x(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\right) \ominus \big(-x(t)\big),$$

if x is decreasing and provided that the mentioned Hukuhara differences exist.

Proof. By using Lemma 2.7, we have by direct computation for case of x is increasing:

$$\begin{aligned}\mathcal{I}_{a^+}^{\alpha}\left(^{RL}\mathcal{D}_{a^+}^{\alpha} x\right)(t,q) &= \Big[\mathcal{I}_{a^+}^{\alpha}\big(^{RL}\mathcal{D}_{a^+}^{\alpha} \underline{x}\big)(t,q), \mathcal{I}_{a^+}^{\alpha}\big(^{RL}\mathcal{D}_{a^+}^{\alpha} \overline{x}\big)(t,q)\Big],\\ &= \Bigg[\underline{x}(t,q) - \frac{\underline{x}_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} - \frac{\underline{x}_{2-\alpha}(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2},\\ &\qquad \overline{x}(t,q) - \frac{\overline{x}_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} - \frac{\overline{x}_{2-\alpha}(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\Bigg].\end{aligned}$$

And for case of x is decreasing, we have:

$$\begin{aligned}\mathcal{I}_{a^+}^{\alpha}\left({}^{RL}\mathcal{D}_{a^+}^{\alpha}x\right)(t,q) =&\Big[\mathcal{I}_{a^+}^{\alpha}\left({}^{RL}\mathcal{D}_{a^+}^{\alpha}\overline{x}\right)(t,q),\mathcal{I}_{a^+}^{\alpha}\left({}^{RL}\mathcal{D}_{a^+}^{\alpha}\underline{x}\right)(t,q)\Big],\\ =&\Bigg[\overline{x}(t,q)-\frac{\overline{x}_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1}-\frac{\overline{x}_{2-\alpha}(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2},\\ &\underline{x}(t,q)-\frac{\underline{x}_{1-\alpha}(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1}-\frac{\underline{x}_{2-\alpha}(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}\Bigg],\end{aligned}$$

for all $q\in[0,1]$ which complete the proofs. □

Lemma 2.9. [23] *If $x\in AC\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ is a monotone fuzzy function, we have*

(i) $\left[{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}x(t)\right]^{\alpha}=\left[{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}\underline{x}(t,\alpha),{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}\overline{x}(t,\alpha)\right]$, *if x is increasing.*

(ii) $\left[{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}x(t)\right]^{\alpha}=\left[{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}\overline{x}(t,\alpha),{}^{H}\mathcal{D}_{a^+}^{\alpha,\beta}\underline{x}(t,\alpha)\right]$, *if x is decreasing.*

Now, we define the Henry-Gronwall inequality [26], which can be used in the proof of our result.

Lemma 2.10. *Let $f,g:[0,T)\longrightarrow\mathbb{R}^+$ be continuous functions. If g is nondescreasing and there exists constants $K\geq 0$ and $\alpha>0$ as*

$$f(u)\leq g(u)+K\int_0^u(u-v)^{\alpha-1}f(v)dv,\qquad u\in[0,T),$$

then

$$f(u)\leq g(u)+\int_0^u\left[\sum_{m=1}^{\infty}\frac{(K\Gamma(\alpha))^m}{\Gamma(m\alpha)}(u-v)^{m\alpha-1}g(v)\right]dv,\quad u\in[0,T).$$

If $g(u)=b$ is constant on $[0,T)$, the previous inequality is transformed into

$$f(u)\leq bE_{\alpha}\left(K\Gamma(\alpha)u^{\alpha}\right),\quad u\in[0,T),$$

where E_α is given by

$$E_{\alpha}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\alpha+1)}.$$

Remark 2.11. [26] For all $u\in[0,T)$, $\exists N_K^*>0$ doesn't depend to b such that

$$f(u)\leq N_K^*b.$$

3. Existence and uniqueness results

In this section, the existence and uniqueness of solution to problem (1) are investigated by using a Banach fixed point theorem. The following lemma shows the equivalence between a fuzzy fractional differential equation and a fuzzy fractional integral equation.

Lemma 3.1. *Let $\gamma = \alpha + \beta(2-\alpha)$ where $1 < \alpha < 2$ and $0 \leq \beta \leq 1$. Let $f : (a,b] \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ be a fuzzy function such that $t \longmapsto f(t,x(t)) \in C_\gamma\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ for all $x \in \mathcal{F}_{\mathbb{R}^n}$. Then a monotone fuzzy function $x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ is a solution of problem (1) if and only if x satisfies the fractional integral equation*

$$x(t) = \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)} \left(\frac{1}{\Gamma(\alpha)} \int_a^b (b-s)^{\alpha-1} f(s,x(s))ds \ominus \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)} \int_a^{\xi_i} (\xi_i - s)^{\alpha+\varphi_i-1} f(s,x(s))ds \right) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s,x(s))ds, \quad (3)$$

where $A = \sum_{i=1}^m \frac{\delta_i(\xi_i - a)^{\gamma+\varphi_i-1}}{\Gamma(\alpha+\varphi_i)} \ominus \frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)} \neq 0$, $\varphi_i > 0$, $\xi_i \in [a,b]$, $a > 0$ and $\delta_i \in \mathbb{R}$, $i = 1,..,m$ and provided that the mentioned Hukuhara differences exists.

Proof. Let $x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ be a monotone fuzzy solution of (1). The first equation of (1) can be written as

$$\mathcal{I}_{a^+}^{\beta(2-\alpha)} \left(\frac{d}{dt}\right)^2 \mathcal{I}_{a^+}^{(1-\beta)(2-\alpha)} x(t) = f(t,x(t)),$$

since $f(t,x(t)) \in C_\gamma\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ $\forall x \in \mathcal{F}_{\mathbb{R}^n}$, we applying the RL fractional integral of order α to both sides of last equation, we get

$$\mathcal{I}_{a^+}^{\alpha} \mathcal{I}_{a^+}^{\beta(2-\alpha)} \left(\frac{d}{dt}\right)^2 \mathcal{I}_{a^+}^{(1-\beta)(2-\alpha)} x(t) = \mathcal{I}_{a^+}^{\alpha} f(t,x(t)),$$

or, we have $2 - \gamma = (1-\beta)(2-\alpha)$, then

$$\begin{aligned} \mathcal{I}_{a^+}^{\alpha} \mathcal{I}_{a^+}^{\beta(2-\alpha)} \left(\frac{d}{dt}\right)^2 \mathcal{I}_{a^+}^{(1-\beta)(2-\alpha)} x(t) &= \mathcal{I}_{a^+}^{\gamma} \left(\frac{d}{dt}\right)^2 \mathcal{I}_{a^+}^{2-\gamma} x(t), \\ &= \mathcal{I}_{a^+}^{\gamma} \left({}^{RL}\mathcal{D}_{a^+}^{\gamma} x(t) \right). \end{aligned}$$

Then

$$\mathcal{I}_{a^+}^{\gamma} \left({}^{RL}\mathcal{D}_{a^+}^{\gamma} x(t) \right) = \mathcal{I}_{a^+}^{\alpha} f(t,x(t)).$$

By using Lemma 2.8 and setting $c_0 = \mathcal{I}_{a^+}^{1-\alpha} x(a)$, $c_1 = \mathcal{I}_{a^+}^{2-\alpha} x(a)$, we get

$$x(t) = \frac{c_0}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{c_1}{\Gamma(\gamma-1)}(t-a)^{\gamma-2} + \mathcal{I}_{a^+}^{\alpha} f(t,x(t)).$$

Since $\lim_{t\longrightarrow a}(t-a)^{\gamma-2}=\infty$, in the view of boundary conditions $x(a)=0$, we must have $c_1=0$, then we get

$$x(t)=\frac{c_0}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\mathcal{I}^{\alpha}_{a^+}f(t,x(t)). \tag{4}$$

Then, to determine the constant c_0 we use the second boundary condition. We have

$$\sum_{i=1}^{m}\delta_i\mathcal{I}^{\varphi_i}_{a^+}x(\xi_i)=c_0\sum_{i=1}^{m}\frac{\delta_i(\xi_i-a)^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)}+\sum_{i=1}^{m}\delta_i\mathcal{I}^{\alpha+\varphi_i}_{a^+}f(\xi_i,x(\xi_i)),$$

or we have $x(b)=\sum_{i=1}^{m}\delta_i\mathcal{I}^{\varphi_i}_{a^+}x(\xi_i)$, then

$$c_0\left(\sum_{i=1}^{m}\frac{\delta_i(\xi_i-a)^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)}\ominus\frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)}\right)=\mathcal{I}^{\alpha}_{a^+}f(b,x(b))\ominus\sum_{i=1}^{m}\delta_i\mathcal{I}^{\alpha+\varphi_i}_{a^+}f(\xi_i,x(\xi_i)),$$

thus, we find

$$c_0=\frac{1}{A}\left(\mathcal{I}^{\alpha}_{a^+}f(b,x(b))\ominus\sum_{i=1}^{m}\delta_i\mathcal{I}^{\alpha+\varphi_i}_{a^+}f(\xi_i,x(\xi_i))\right).$$

Substituting the value of c_0 into (4), we obtain fractional integral equation (3). The converse follows by direct computation. □

Theorem 3.2. *Assume that the following conditions holds:*

$(C1)$ *There exist $N_f>0$ such that*

$$\mathcal{D}_\infty\big(f(t,\hat{0}),\hat{0}\big)\le N_f,\ \forall t\in[a,b].$$

$(C2)$ *For every $t\in[a,b]$ and $x,y\in\mathcal{F}_{\mathbb{R}^n}$, there exist $L>0$ such that*

$$\mathcal{D}_\infty\big(f(t,x),f(t,y)\big)\le L\mathcal{D}_\infty(x,y),\ \forall t\in[a,b],\ x,y\in\mathcal{F}_{\mathbb{R}^n}.$$

If $LP<1$, then the boundary value problem (1) has a unique solution on $[a,b]$, where

$$P:=\frac{(b-a)^{\alpha+\gamma-1}}{A\Gamma(\gamma)\Gamma(\alpha+1)}-\frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)}\sum_{i=1}^{m}\frac{\delta_i(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)},$$

is a positive constant.

Proof. We switch the boundary value problem (1) into a fixed point problem. For this, we consider the operator $\mathcal{R}:C\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)\longrightarrow C\big([a,b],\mathcal{F}_{\mathbb{R}^n}\big)$ as

$$(\mathcal{R}x)(t)=\frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\left(\frac{1}{\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}f(s,x(s))ds\ominus\frac{\sum_{i=1}^{m}\delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1}\right.$$
$$\left.f(s,x(s))ds\right)+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,x(s))ds. \tag{5}$$

For each positive number r, we define

$$\mathbb{B}_r := \left\{x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big) \mid \mathcal{D}_\infty(x, \hat{0}) \le r\right\}.$$

Step1: We prove that $\mathcal{R}(\mathbb{B}_r) \subseteq \mathbb{B}_r$. We choose $r \ge \frac{N_f P}{1-LP}$. For any $x \in \mathbb{B}_r$, we have

$$\begin{aligned}
\mathcal{D}_\infty\big((\mathcal{R}x)(t), \hat{0}\big) &= \mathcal{D}_\infty\Bigg(\frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\left[\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,x(s))ds \ominus \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i - s)^{\alpha+\varphi_i-1} f(s,x(s))ds\right] \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s))ds, \hat{0}\Bigg), \\
&\le \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\Bigg[\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mathcal{D}_\infty\big(f(s,x(s)),\hat{0}\big)ds - \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1} \\
&\mathcal{D}_\infty\big(f(s,x(s)),\hat{0}\big)ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{D}_\infty\big(f(s,x(s)),\hat{0}\big)ds, \\
&\le \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\Bigg[\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\Big\{\mathcal{D}_\infty\big(f(s,x(s)), f(s,\hat{0})\big) + \mathcal{D}_\infty\big(f(s,\hat{0}),\hat{0}\big)\Big\}ds - \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)} \\
&\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1}\Big\{\mathcal{D}_\infty\big(f(s,x(s), f(s,\hat{0})\big) + \mathcal{D}_\infty\big(f(s,\hat{0}),\hat{0}\big)ds\Big\}\Bigg] \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\{\mathcal{D}_\infty\big(f(s,x(s)), f(s,\hat{0})\big) + \mathcal{D}_\infty\big(f(s,\hat{0}),\hat{0}\big)\Big\}ds, \\
&\le \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\Bigg[\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\Big\{L\mathcal{D}_\infty\big(x(s),\hat{0}\big) + N_f\Big\}ds - \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)} \\
&\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1}\Big\{L\mathcal{D}_\infty\big(x(s),\hat{0}\big) + N_f\Big\}ds\Bigg] + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\{L\mathcal{D}_\infty\big(x(s),\hat{0}\big) + N_f\Big\}ds, \\
&\le (Lr + N_f)\left(\frac{(b-a)^{\gamma-1+\alpha}}{A\Gamma(\gamma)\Gamma(\alpha+1)} - \frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)}\sum_{i=1}^m \frac{\delta_i(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right), \\
&:= (Lr + N_f)P \le r,
\end{aligned}$$

which implies that $\mathcal{R}(\mathbb{B}_r) \subseteq \mathbb{B}_r$.
Step2: We show that $\mathcal{R}$ is a contraction operator. For $x, y \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ and $t \in [a,b]$, we have

$$\begin{aligned}
\mathcal{D}_\infty\big((\mathcal{R}x)(t), (\mathcal{R}y)(t)\big) &\le \frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)}\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\mathcal{D}_\infty\Big(f(s,x(s)), f(s,y(s))\Big)ds - \frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)} \\
&\frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1}\mathcal{D}_\infty\Big(f(s,x(s)), f(s,y(s))\Big)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{D}_\infty\Big(f(s,x(s)), f(s,y(s))\Big)ds, \\
&\le L\left(\frac{(b-a)^{\gamma-1+\alpha}}{A\Gamma(\gamma)\Gamma(\alpha+1)} - \frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)}\sum_{i=1}^m \delta_i\frac{(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)\mathcal{D}_\infty(x,y), \\
&:= LP\mathcal{D}_\infty(x,y),
\end{aligned}$$

which implies that $\mathcal{D}_\infty\big(\mathcal{R}x, \mathcal{R}y\big) \le LP\mathcal{D}_\infty(x,y)$. So, since $LP < 1$, then $\mathcal{R}$ is a contraction operator. Therefore, by using Banach's contraction mapping principle, we conclude that $\mathcal{R}$ has a fixed point which is the unique solution of (1). □

4. Stability result

In this section, we study Ulam's type stability for FFBVPs (1). First we recall the definitions of those types of Ulam stability.

Definition 4.1. [27] The first equation of problem (1) is said to be UH stable if there is a constant $\omega > 0$ such that for each $\epsilon > 0$ and each solution $x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ of the inequality

$$\mathcal{D}_\infty\bigg(\quad {}^H\mathcal{D}^{\alpha,\beta}_{a^+} x(t), f(t, x(t))\bigg) \leq \epsilon, \qquad t \in [a,b], \tag{6}$$

there is a solution $v \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ of the first eqation of (1) such that

$$\mathcal{D}_\infty\big(x(t), v(t)\big) \leq \omega\epsilon, \qquad t \in [a,b].$$

Definition 4.2. [27] The first equation of problem (1) is said to be generalized UH (GUH) stable if there exists $\varphi \in C^1\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$, $\varphi(0) = 0$ such that for each solution $x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ of (6), there exists a solution $v \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ of the first equation of (1) such that

$$\mathcal{D}_\infty\big(x(t), v(t)\big) \leq \varphi(\epsilon), \qquad t \in [a,b].$$

Remark 4.3. A function $x \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ is a solution of (6) if and only if $\exists \phi \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ such that

(i) $\mathcal{D}_\infty\big(\phi(t), \hat{0}\big) \leq \epsilon$, $t \in [a,b]$.

(ii) ${}^H\mathcal{D}^{\alpha,\beta}_{a^+} x(t) = f(t, x(t)) + \phi(t)$, $t \in [a,b]$.

Now, we prove the UH and GUH stability of the problem (1).

Theorem 4.4. *Assume that the condition $(C2)$ is satisfied. Then, the problem (1) will be UH stable and consequently GUH stable.*

Proof. Let x be the solution of Eq. (6) and v be the solution of the proposed problem (1). Then, $\exists \phi \in C\big([a,b], \mathcal{F}_{\mathbb{R}^n}\big)$ such that

$${}^H\mathcal{D}^{\alpha,\beta}_{a^+} x(t) = f(t, x(t)) + \phi(t), \qquad t \in [a,b], \tag{7}$$

and $\mathcal{D}_\infty\big(\phi(t), \hat{0}\big) \leq \epsilon$, $t \in [a,b]$. It follow from lemma 3.1, that the solution of (7) can be expressed as follows

$$x(t) = \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\Bigg(\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s, x(s))ds \ominus \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i} (\xi_i - s)^{\alpha+\varphi_i-1}$$
$$f(s, x(s))ds\Bigg) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s, x(s))ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\phi(s)ds. \tag{8}$$

We set

$$\mathbf{B}_x := \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,x(s))ds \ominus \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i - s)^{\alpha+\varphi_i-1} f(s,x(s))ds,$$

and

$$\mathbf{B}_v := \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s,v(s))ds \ominus \frac{\sum_{i=1}^m \delta_i}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i - s)^{\alpha+\varphi_i-1} f(s,v(s))ds.$$

Then

$$\begin{aligned}\mathcal{D}_\infty(x(t),v(t)) = \mathcal{D}_\infty\Bigg(&\frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\mathbf{B}_x + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,x(s))ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\phi(s)ds, \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\mathbf{B}_v \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s,v(s))ds\Bigg), \\ &\leq \frac{(t-a)^{\gamma-1}}{A\Gamma(\gamma)}\mathcal{D}_\infty(\mathbf{B}_x,\mathbf{B}_v) + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\mathcal{D}_\infty(f(s,x(s)),f(s,v(s)))}{(t-s)^{1-\alpha}}ds + \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\mathcal{D}_\infty(\phi(t),\hat{0})}{(t-s)^{1-\alpha}}ds,\end{aligned}$$

since $x(b) = v(b)$, we can see that $\mathcal{D}_\infty\big(\mathbf{B}_x,\mathbf{B}_v\big) = 0$, therefore, by using the condition $(C2)$, lemma 2.10, remark 2.11 and remark 4.3, we get

$$\begin{aligned}\mathcal{D}_\infty\big(x(t),v(t)\big) &\leq \frac{L}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{D}_\infty\big(x(s),v(s)\big)ds + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}, \\ &\leq \int_a^t\left[\sum_{k=1}^\infty \frac{(L)^k}{\Gamma(k\alpha)}(t-s)^{k\alpha-1}\right]ds + \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)},\end{aligned}$$

since $\frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}$ is a constant on $[a,b]$, the previous inequality is transformed into

$$\mathcal{D}_\infty\big(x(t),v(t)\big) \leq \frac{\epsilon(b-a)^\alpha}{\Gamma(\alpha+1)}E_\alpha\big(L(b-a)^\alpha\big),$$

by setting

$$\omega := \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}E_\alpha\big(L(b-a)^\alpha\big),$$

we get

$$\mathcal{D}_\infty\big(x(t),v(t)\big) \leq \omega\epsilon.$$

Hence, the problem (1) is UH stable. Therefore, if we put $\varphi(\epsilon) = \omega\epsilon$, we have $\varphi(0) = 0$ and $\mathcal{D}_\infty\big(x(t),v(t)\big) \leq \varphi(\epsilon)$. Then, the problem (1) is GUH stable.

□

5. Example

In this section, we consider an example to apply our results in the study of existence and stability results.

Consider the following FFBVPs with Hilfer fractional derivative

$$\begin{cases} {}^{H}\mathcal{D}^{\frac{5}{4},\frac{3}{7}}x(t) = \frac{\cos(t)x(t)}{(e^{t^2-1}+6)(1+x(t))}, & t \in [1,8], \\ x(1) = 0, \qquad x(8) = \frac{7}{3}\mathcal{I}^{\frac{4}{3}}x(\frac{3}{2}) + \frac{10}{3}\mathcal{I}^{\frac{7}{3}}x(\frac{5}{4}) + \frac{11}{3}\mathcal{I}^{\frac{8}{3}}x(5), \end{cases} \tag{9}$$

Consider the function $f : [1,8] \times \mathcal{F}_{\mathbb{R}^n} \longrightarrow \mathcal{F}_{\mathbb{R}^n}$ given by

$$f(t,x) = \frac{\cos(t)x(t)}{(e^{t^2-1}+6)(1+x(t))}.$$

For $x, y \in \mathcal{F}_{\mathbb{R}^n}$ and $t \in [1,8]$, we have

$$\mathcal{D}_\infty\big(f(t,x), f(t,y)\big) \leq \frac{1}{7}\mathcal{D}_\infty\big(x,y\big).$$

Also, by using the given data, we have

$$P = \frac{(b-a)^{\alpha+\gamma-1}}{A\Gamma(\gamma)\Gamma(\alpha+1)} - \frac{(b-a)^{\gamma-1}}{A\Gamma(\gamma)}\sum_{i=1}^{3}\frac{\delta_i(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \approx 6.7382,$$

thus $LP = \frac{6.7382}{7} = 0.9626 < 1$. Then, by theorem 3.2, the boundary value problem (9) has unique solution on $[1,8]$. Futher, by theorem 4.4, we conclude that the problem (9) is UH stable and therefore GUH stable with

$$\omega = \frac{(8-1)^{\frac{5}{4}}}{\Gamma(\frac{5}{4}+1)}E_{\frac{5}{4}}\big(\frac{1}{7}(8-1)^{\frac{5}{4}}\big) > 0.$$

6. Conclusion

In this work, we have studied the existence, uniqueness and UH stability of nonlocal FFBVPs with fuzzy Hilfer fractional derivative. Our survery based on the Banach fixed point theorem and generalized Gronwall inequality.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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