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ON FUZZY CLOSURE FILTERS OF DECOMPOSABLE STONE ALMOST DISTRIBUTIVE LATTICES

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Abstract. The notion of fuzzy closure filters is introduced and discussed in a decomposable stone ADL. In particular, we characterize the prime fuzzy closure filters in terms of boosters. Some relationship between the lattice of fuzzy closure filters and the fuzzy ideal lattice of boosters are explored and investigated.

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1. Introduction

In [12], Swamy and Rao introduced the concept of an Almost Distributive Lattice (ADL). An ADL $(L, \wedge, \vee, 0)$ satisfies all the axioms of distributive lattice, except possibly the commutativity of the operations $\wedge$ and $\vee$. In [15], Swamy et al. introduced pseudo-complementation on almost distributive lattices and also in [16] studied Stone Almost Distributive Lattices. On the other hand, fuzzy set theory was introduced by Zadeh [17]. Next, fuzzy groups were studied by Rosenfeld [10]. Swamy and Raju introduced fuzzy ideals and congruences of lattices in [11]. In [7], Kumar studied the space of prime fuzzy ideals of a ring topologically in different way and Hadji-Abadi and Zahedi [6] extended the result of Kumar. In [13], [14] Swamy et al. studied fuzzy ideals and L-Fuzzy Filters of ADLs. [1], Abd El-Mohsen Badawy and R. El-Fawal, introduced the concept of closure filters and characterized such filters in terms of boosters. In this paper, we introduced the notion of fuzzy closure filters in a Stone ADL. Characterized fuzzy closure filters in terms of its level subsets and characteristic functions. Also,

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characterized the fuzzy closure filters in terms of boosters. Proved that the set of all fuzzy closure filters forms a complete distributive lattice. Also introduced the concepts of prime fuzzy filters and maximal fuzzy filters in a Stone ADL. Proved that the existence of prime fuzzy closure filters in a Stone ADL. Also derived that every proper fuzzy closure filter of L is the intersection of all prime fuzzy closure filters containing it. Studied some properties on the set of all prime fuzzy closure filters of a Stone ADL. We derived the properties of the set of all fuzzy closure filters of a Stone ADL topologically. In the last decades, various generalization of Boolean algebras have emerged. Along this direction, the class of $MS-$ algebras were first introduced by T.S. Blyth and J.C. Varlet [3,4] as a generalization of de Morgan algebras and Stone algebras.

2. Preliminaries

In this section, we recall certain definitions and important results, those will be required in the text of the paper.

Definition 2.1. [12] An algebra $L = (L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$
(6) $0 \wedge a = 0$
(7) $a \vee 0 = a,$ for all $a, b, c \in L$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.3. [12] *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* L

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) *If* $a \leq c, b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a.$

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.4. [12] *Let* $(L, \vee, \wedge, 0)$ *be an ADL with 0. Then the following are equivalent:*

(1) L *is a distributive lattice*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.5. [12] *Let* L *be an ADL and* $m \in L$. *Then the following are equivalent:*

(1) m *is maximal with respect to* $\leq$

(2) $m \vee a = m$, *for all* $a \in L$

(3) $m \wedge a = a$, *for all* $a \in L$

(4) $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [2,5], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.6. [12] *For any $a, y \in L$, the following are equivalent:*

(1) $(a] \subseteq (b]$

(2) $b \wedge a = a$

(3) $b \vee a = b$

(4) $[b) \subseteq [a)$.

For any $a, b \in L$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Theorem 2.7 ([9]). *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Definition 2.8. [15] Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Theorem 2.9. *Let L be an ADL and $*$ a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

(1) $0^{**} = 0$

(2) $0^* \wedge a = a$

(3) $a^{**} \wedge a = a$

(4) $a^{***} = a^*$

(5) $a \leq b \Rightarrow b^* \leq a^*$

(6) $a^* \wedge b^* = b^* \wedge a^*$

(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(8) $a^* \wedge b = (a \wedge b)^* \wedge b^*$.

Definition 2.10 ([16]). Let L be an ADL and $*$ a pseudo-complementation on L.Then L is called a Stone ADL if, for any $x \in L,\ x^* \vee x^{**} = 0^*$.

Lemma 2.11 ([16]). *Let L be a Stone ADL and $a, b \in L$. Then the following conditions hold:*

(1) $0^* \wedge a = a$ *and* $0^* \vee a = 0^*$

(2) $(a \wedge b)^* = a^* \vee b^*$.

Definition 2.12. [8] Let L be a stone ADL with maximal elements. Then for any $a \in L$, define $(a)^+ = \{x \in L \mid x \vee a^*$ is a maximal element of $L\}$. We call $(a)^+$ as booster of a.

Theorem 2.13. [8] *Let L be a stone ADL with maximal elements. Then the set $\mathcal{B}_0(L)$ of all boosters is a complete distributive lattice on its own.*

Definition 2.14. [8]

(1) For any filter F of L, define an operator β as $\beta(F) = \{(x)^+ \mid x \in F\}$.

(2) For any ideal I of $\mathcal{B}_0(L)$, define an operator $\overleftarrow{\beta}$ as $\overleftarrow{\beta}(I) = \{x \in L \mid (x)^+ \in I\}$.

We recall that for any non empty set S, the characteristic function of S,

$$\mathcal{X}_S(x) = \begin{cases} 1\ if\ x \in S \\ 0\ if\ x \notin S. \end{cases}$$

Definition 2.15. [14] Let λ be a fuzzy subset of S and let $\alpha \in [0, 1]$. Then the set

$$\lambda_\alpha = \{x \in L : \alpha \leq \lambda(x)\}$$

is called a level subset of λ.

Definition 2.16. [13] A fuzzy subset λ of an ADL L is said to be a fuzzy ideal of L, if for all $x, y \in L$,

(1) $\lambda(0) = 1$,

(2) $\lambda(x \vee y) \geq \lambda(x) \wedge \lambda(y)$,

(3) $\lambda(x \wedge y) \geq \lambda(x) \vee \lambda(y)$ for all $x, y \in L$.

In [13], Swamy and Raju observed that, a fuzzy subset λ of an ADL L is a fuzzy ideal of L if and only if $\lambda(0) = 1$ and $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$ for all $x, y \in L$.

In [14], Swamy et.al $\mu : L \to L^{'}$, where L is an ADL and $L^{'}$ is a complete lattice satisfying infinite meet distributive law. Now in our cases take $L^{'}$ as $[0, 1]$. λ is said to be a fuzzy filter of an ADL L if λ_α is a filter of L for all $\alpha \in L$.

Theorem 2.17. [14] *Let λ be a fuzzy subset of an ADL L. Then the following are equivalent to each other.*

(1) *λ is a fuzzy filter of L,*

(2) *$\lambda(m) = 1$ for all maximal elements m and $\lambda(x \wedge y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$,*

(3) *$\lambda(m) = 1$ for all maximal elements m and $\lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$ and $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$, for all $x, y \in L$.*

We define the binary operations " + " and "." on all fuzzy subsets of an ADL L as: $(\mu + \theta)(x) = sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$ and $(\mu, \theta)(x) = sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}$.

The intersection of fuzzy filters of L is a fuzzy filter. However the union of fuzzy filters may not be fuzzy filter. The least upper bound of a fuzzy filters μ and θ of L is denoted as $\mu \vee \theta = \bigcap\{\sigma \in \mathcal{FF}(L) : \mu \cup \theta \subseteq \sigma\}$. If μ and θ are fuzzy filters of L, then $\mu \cdot \theta = \mu \vee \theta$ and $\mu + \theta = \mu \cap \theta$.

Theorem 2.18. *Let λ be a fuzzy subset of L. Then λ is a fuzzy ideal of L if and only if, for any $\alpha \in [0,1]$, λ_α is an ideal of L.*

Definition 2.19.

1. A proper fuzzy ideal λ of L is called a prime fuzzy ideal if for any two fuzzy ideals η and ν of L, $\eta \cap \nu \subseteq \lambda$ implies $\eta \subseteq \lambda$ or $\nu \subseteq \lambda$.
2. A proper fuzzy filter λ of L is called a prime fuzzy filter if for any two fuzzy filters η, ν of L, $\eta \cap \nu \subseteq \lambda$ implies $\eta \subseteq \lambda$ or $\nu \subseteq \lambda$.

Theorem 2.20. *For any $\alpha \in [0,1)$, the fuzzy subset P^1_α of L given by*

$$P^1_\alpha(x) = \begin{cases} 1 \; if \; x \in P \\ \alpha \; if \; x \notin P \end{cases}$$

for all $x \in L$ is a prime fuzzy filter if and only if P is a prime filter of L.

3. Fuzzy Closure filters of Decomposable Stone ADLs

In this section, we introduce the notion of fuzzy closure filters in decomposable stone ADLs and study the properties of fuzzy closure filters.

Definition 3.1. A fuzzy subset ν of $M_0(L)$ is called a fuzzy ideal of $M_0(L)$ if $\nu((1)^+) = 1$ and $\nu((a)^+ \sqcup (b)^+) \geq \nu((a)^+) \wedge \nu((b)^+))$ and $\nu((a)^+ \cap (b)^+) \geq \nu((a)^+) \vee \nu((b)^+))$, for all $(a)^+, (b)^+ \in M_0(L)$.

Example 3.2. Let $L = \{0, 1, 2, 3\}$ be a non-empty set and $\vee, \wedge$ be binary operations and unary operations respectively which are defined by

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	1	2	3
3	0	3	3	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	2	2	2
3	3	1	2	3

Define $*$ on L, as $0^* = 1, 1^* = 2^* = 3^* = 0$. Then clearly, $(L, \wedge, \vee, *)$ is a stone ADL which is not a lattice.
For the elements $0, 2$ there exist dense element 2 such that $0^{**} \wedge 2 = 0$ and $2^{**} \wedge 2 = 2$ and
for the element 1, there exists dense element 1 such that $1^{**} \wedge 1 = 1$ and
for the element 3, there exists dense element 3 such that $3^{**} \wedge 3 = 3$.
Hence$(L, \vee, \wedge, ', 0, 1)$ is a decomposable stone ADL.
Define $\nu((1)^+) = 1,\ \nu((0)^+) = 0.5,\ \nu((2)^+) = \nu((3)^+) = 0.8$. Clearly, ν is a fuzzy ideal of $M_0(L)$.

In the following definition, we define two operators α and $\overleftarrow{\alpha}$ in L

Definition 3.3. Let L be a decomposable stone ADL.
(1) For any fuzzy filter θ of L and for any a in L, define an operator α as $\alpha(\theta)((a)^+) = \sup\{\theta(b) \mid (a)^+ = (b)^+, b \in L\}$.
(2) For any fuzzy ideal ν of $M_0(L)$ and for any a in L, define an operator $\overleftarrow{\alpha}$ as $\overleftarrow{\alpha}(\nu)(a) = \nu((a)^+)$.

Lemma 3.4. *In any decomposable stone ADL L, The following three statements hold:*

(1) For any fuzzy filter θ of L, $\alpha(\theta)$ is a fuzzy ideal of $M_0(L)$
(2) For any fuzzy ideal ν of $M_0(L)$, $\overleftarrow{\alpha}(\nu)$ is a fuzzy filter of L
(3) The maps α and $\overleftarrow{\alpha}$ are isotone.

Proof.
(1). For any fuzzy filter θ, we have $\alpha(\theta)((0^*)^+) = 1$. Let $(x)^+, (y)^+ \in M_0(L)$. Then, we have $\alpha(\theta)((x)^+) \wedge \alpha(\theta)((y)^+) = \sup\{\theta(a) \mid (a)^+ = (x)^+\} \wedge \sup\{\theta(b) \mid (b)^+ = (y)^+\} = \sup\{\theta(a) \wedge \theta(b) \mid (a)^+ = (x)^+, (b)^+ = (y)^+\} \leq \sup\{\theta(a \wedge b) \mid (a \wedge b)^+ = (x \wedge y)^+\} = \alpha(\theta)((x \wedge y)^+) = \alpha(\theta)((x)^+ \sqcup (y)^+)$ and $\alpha(\theta)((x)^+) \vee \alpha(\theta)((y)^+) = \sup\{\theta(a) \mid (a)^+ = (x)^+\} \vee \sup\{\theta(b) \mid (b)^+ = (y)^+\} = \sup\{\theta(a) \vee \theta(b) \mid (a)^+ = (x)^+, (b)^+ = (y)^+\} \leq \sup\{\theta(a \vee b) \mid (a \vee b)^+ = (x \vee y)^+\} = \alpha(\theta)((x \vee y)^+) = \alpha(\theta)((x)^+ \cap (y)^+)$. Therefore, $\alpha(\theta)$ is a fuzzy ideal of $M_0(L)$.
(2). Let ν be any fuzzy ideal of $M_0(L)$. Then $\overleftarrow{\alpha}(\nu)(0^*) = \nu((0^*)^+) = 1$. For any $x, y \in L$, $\overleftarrow{\alpha}(\nu)(x \wedge y) = \nu((x \wedge y)^+) = \nu((x)^+ \sqcup (y)^+) \geq \nu((x)^+) \wedge \nu((y)^+) = \overleftarrow{\alpha}(\nu)(x) \wedge \overleftarrow{\alpha}(\nu)(y)$ and $\overleftarrow{\alpha}(\nu)(x \vee y) = \nu((x \vee y)^+) = \nu((x)^+ \cap (y)^+) \geq \nu((x)^+) \vee \nu((y)^+) = \overleftarrow{\alpha}(\nu)(x) \vee \overleftarrow{\alpha}(\nu)(y)$.
(3). Assume that ν and θ are fuzzy filters of L with $\nu \subseteq \theta$. Now $\alpha(\nu)((a)^+) = \sup\{\nu(b) \mid (b)^+ = (a)^+\} \leq \sup\{\theta(b) \mid (b)^+ = (a)^+\} = \alpha(\theta)((a)^+)$. Therefore, α is an isotone mapping. Similarly, we deduce that $\overleftarrow{\alpha}$ is also an isotone mapping. □

Theorem 3.5. *The mapping $\nu \to \overleftarrow{\alpha}\alpha(\nu)$ is a closure operator on the lattice of fuzzy filters of L. i.e., for any fuzzy filters ν and θ of L,*
(1) $\nu \subseteq \overleftarrow{\alpha}\alpha(\nu)$
(2) $\nu \subseteq \theta \Rightarrow \overleftarrow{\alpha}\alpha(\nu) \subseteq \overleftarrow{\alpha}\alpha(\theta)$

(3) $\overleftarrow{\alpha}\alpha\{\overleftarrow{\alpha}\alpha(\nu)\} = \overleftarrow{\alpha}\alpha(\nu)$.

Proof.

(1). Clearly, we have the following equality:

$$\overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (a)^+ = (b)^+\} \geq \nu(a),$$

for all $a \in L$

(2). This part is clear.

(3). Let $a \in L$. Now, we have $\overleftarrow{\alpha}\alpha\{\overleftarrow{\alpha}\alpha(\nu)\}(a) = \alpha\{\overleftarrow{\alpha}\alpha(\nu)\}((a)^+) = \sup\{\overleftarrow{\alpha}\alpha(\nu)(b) \mid (b)^+ = (a)^+,\ b \in L\} = \sup\{\alpha(\nu)((b)^+) \mid (b)^+ = (a)^+,\ b \in L\} = \alpha(\nu)((a)^+) = \overleftarrow{\alpha}\alpha(\nu)(a)$. □

Theorem 3.6. *Let L be a decomposable stone ADL. Then α is a homomorphism of the lattice of fuzzy filters of L into the lattice of fuzzy ideals of $M_0(L)$.*

Proof. Let $\mathcal{FF}(L)$ be the set of all fuzzy filters of L and $\mathcal{FI}M_0(L)$ be the set of all fuzzy ideals in $M_0(L)$. Then, for any $\nu, \theta \in \mathcal{FF}(L)$, we have $\nu \cap \theta \subseteq \nu$ and $\nu \cap \theta \subseteq \theta$. These results imply that $\alpha(\nu \cap \theta) \subseteq \alpha(\nu)$ and $\alpha(\nu \cap \theta) \subseteq \alpha(\theta)$. The above results further imply that $\alpha(\nu \cap \theta) \subseteq \alpha(\nu) \cap \alpha(\theta)$. Now, we have $(\alpha(\nu) \cap \alpha(\theta))((a)^+) = \alpha(\nu)((a)^+) \wedge \alpha(\theta)((a)^+) = \sup\{\nu(x) \mid (x)^+ = (a)^+\} \wedge \sup\{\theta(y) \mid (y)^+ = (a)^+\} \leq \sup\{\nu(x \vee y) \mid (x \vee y)^+ = (a)^+\} \wedge \sup\{\theta(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \sup\{\nu(x \vee y) \wedge \theta(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \sup\{(\nu \cap \theta)(x \vee y) \mid (x \vee y)^+ = (a)^+\} = \alpha(\nu \cap \theta)((a)^+)$. Therefore, we deduce that $\alpha(\nu) \cap \alpha(\theta) = \alpha(\nu \cap \theta)$. Since $\nu \subseteq \nu \vee \theta$, we have $\theta \subseteq \nu \vee \theta$, we obtain that $\alpha(\nu) \subseteq \alpha(\nu \vee \theta)$ and $\alpha(\theta) \subseteq \alpha(\nu \vee \theta)$. The above results imply that $\alpha(\nu) \sqcup \alpha(\theta) \subseteq \alpha(\nu \vee \theta)$. Now, we have $(\alpha(\nu \vee \theta))((a)+) = \sup\{(\nu \vee \theta)(x) \mid (x)^+ = (a)^+\} = \sup\{\sup\{\nu(x_1) \wedge \theta(x_2) \mid x = x_1 \wedge x_2\} \mid (x)^+ = (a)^+\} \leq \sup\{\sup\{\nu(y_1) \wedge \theta(y_2) \mid (y_1)^+ = (x_1)^+, (y_2)^+ = (x_2)^+\} \mid (x_1 \wedge x_2)^+ = (a)^+\} = \sup\{\sup\{\nu(y_1) \mid (y_1)^+ = (x_1)^+\} \wedge \sup\{\theta(y_2) \mid (y_2)^+ = (x_2)^+\} \mid (x_1)^+ \sqcup (x_2)^+ = (a)^+\} = \sup\{\alpha(\nu)((x_1)^+) \wedge \alpha(\theta)((x_2)^+) \mid (x_1)^+ \sqcup (x_2)^+ = (a)^+\} = (\alpha(\nu) \sqcup \alpha(\theta))((a)^+)$. The above equalities imply that $\alpha(\nu \vee \theta) \subseteq \alpha(\nu) \sqcup \alpha(\theta)$. Hence, $\alpha(\nu \vee \theta) = \alpha(\nu) \sqcup \alpha(\theta)$. Clearly, we have shown that $\chi_{\{1\}},\ \chi_L$ are the smallest and the largest fuzzy filters of L, respectively and also we have that $\alpha(\chi_{\{1\}}),\ \alpha(\chi_L)$ are smallest and greatest fuzzy ideals of $M_0(L)$, respectively. Hence ,α is indeed a homomorphism from $\mathcal{FF}(L)$ into $\mathcal{FI}M_0(L)$. □

Corollary 3.7. *Let ν and θ be any two fuzzy filters of a decomposable stone ADL L. Then, we have $\overleftarrow{\alpha}\alpha(\nu \cap \theta) = \overleftarrow{\alpha}\alpha(\nu) \cap \overleftarrow{\alpha}\alpha(\theta)$.*

Proof. By using the above result, we obtain that $\alpha(\nu) \cap \alpha(\theta) = \alpha(\nu \cap \theta)$. Now, $\overleftarrow{\alpha}\alpha(\nu \cap \theta)(b) = \alpha(\nu \cap \theta)((b)^+) = \alpha(\nu)((b)^+) \wedge \alpha(\theta)((b)^+) = \overleftarrow{\alpha}\alpha(\nu)(b) \wedge \overleftarrow{\alpha}\alpha(\theta)(b)$. Therefore, we have $\overleftarrow{\alpha}\alpha(\nu \cap \theta) = \overleftarrow{\alpha}\alpha(\nu) \cap \overleftarrow{\alpha}\alpha(\theta)$. □

Now, we introduce the concept of fuzzy closure filters in decomposable stone ADLs.

Definition 3.8. A fuzzy filter ν of a decomposable stone ADL L is called a fuzzy closure filter if $\overleftarrow{\alpha}\alpha(\nu) = \nu$.

Example 3.9. Let $L = \{0, 1, 2, 3\}$ be a non-empty set and $\vee, \wedge,'$ be binary operations and unary operations respectively which are defined by

$'$	0	1	2	3
	1	0	0	0

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	2	2
3	0	3	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	3
3	3	1	3	3

Then $(L, \vee, \wedge,', 0, 1)$ is a decomposable MS-algebra. Define $\nu(1) = 1,\ \nu(0) = 0.5,\ \nu(2) = \nu(3) = 0.8$. Clearly, ν is a fuzzy filter of L. Clearly, we have $\overleftarrow{\alpha}\alpha\nu(x) = \nu(x)$, for all $x \in L$. Hence, ν is a fuzzy closure filter of L. Define $\theta(1) = 1,\ \theta(0) = 0,\ \theta(2) = 0.3,\ \theta(3) = 0.6$. Clearly, θ is a fuzzy filter of L. But θ is not a fuzzy closure filter of L, because $\overleftarrow{\alpha}\alpha\theta(2) \neq \theta(2)$.

Now we characterize the fuzzy closure filters in terms of its level subsets and characteristic functions.

Theorem 3.10. *Let ν be any proper fuzzy subset of L. Then ν is a fuzzy closure filter if and only if ν_t, for all $t \in [0, 1]$, is a closure filter of L.*

Proof. Let ν is a fuzzy closure filter of L. Then $(\overleftarrow{\alpha}\alpha(\nu))_t = (\nu)_t$ Now we prove every level subset of ν is a closure filter of L. It is enough to show $\overleftarrow{\alpha}\alpha(\nu_t) = \nu_t$. Clearly, we have that $\nu_t \subseteq \overleftarrow{\alpha}\alpha(\nu_t)$. Let $a \in \overleftarrow{\alpha}\alpha(\nu_t)$. That implies $(a)^+ \in \alpha(\nu_t)$. Then there exists $b \in \nu_t$ such that $(a)^+ = (b)^+$ and so, we have $\nu(b) \geq \alpha$ with $(a)^+ = (b)^+$. That implies $\alpha(\nu)((a)^+) = \sup\{\nu(b) \mid (a)^+ = (b)^+\} \geq \alpha$ and so $\overleftarrow{\alpha}\alpha(\nu)(a) \geq t$. That implies $a \in (\overleftarrow{\alpha}\alpha(\nu))_t$. Therefore, we have $\overleftarrow{\alpha}\alpha(\nu_t) \subseteq \nu_t$ and hence $\overleftarrow{\alpha}\alpha(\nu_t) = \nu_t$. Clearly, we arrive that $\nu \subseteq \overleftarrow{\alpha}\alpha(\nu)$. Let $\alpha = \overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (b)^+ = (a)^+\}$. Then for each $\epsilon > 0$, there is $x \in L,\ (x)^+ = (x)^+$ such that $\nu(a) > \alpha - \epsilon$. Since ϵ is arbitrary chosen, we have $\nu(a) \geq \alpha$ such that $(x)^+ = (a)^+$. This result implies $x \in \nu_t$. Therefore, we have $a \in \overleftarrow{\alpha}\alpha(\nu_t) = \nu_\alpha$ and hence $\nu(a) \geq \alpha = \overleftarrow{\alpha}\alpha(\nu_t)$. Thus, we conclude that $\nu\overleftarrow{\alpha}\alpha(\nu)$. □

Corollary 3.11. *Let F be any non-empty subset F of a decomposable stone ADL L. Then F is a closure filter if and only if χ_F is a fuzzy closure filter of L.*

Now we characterize the fuzzy closure filters in terms of boosters in the following result.

Theorem 3.12. *Let ν be a fuzzy filter of L. Then ν is a fuzzy closure filter if and only if for any $a, b \in L,\ (a)^+ = (b)^+$ implies $\nu(a) = \nu(b)$.*

Proof. Assume that ν is a fuzzy closure filter of L. Then we have the following equality $\nu(a) = \overleftarrow{\alpha}\alpha(\nu)(a)$, for all $a \in L$. Let $a, b \in L$ such that $(a)^+ = (b)^+$. Then, we have $\nu(a) = \overleftarrow{\alpha}\alpha(\nu)(a) = \alpha(\nu)((a)^+) = \alpha(\nu)((b)^+) = \overleftarrow{\alpha}\alpha(\nu)(b) = \nu(b)$. Conversely, assume that for any $a, b \in L$, $(a)^+ = (b)^+$ implies $\nu(a) = \nu(b)$. Now $\overleftarrow{\alpha}\alpha(\nu)(a) = \sup\{\nu(b) \mid (b)^+ = (a)^+\} = \nu(a)$. Therefore. we have $\overleftarrow{\alpha}\alpha(\nu) = \nu$. □

We now establish the following main theorem of fuzzy closure filters.

Theorem 3.13. *Let $\{\nu_i \mid i \in \Omega\}$ be any family of fuzzy closure filters of a decomposable stone ADL L. Then $\bigcap_{i\in\Omega} \nu_i$ is a fuzzy closure filter of L.*

Corollary 3.14. *Let L be a decomposable stone ADL. Then the set $\mathcal{FF}_{\mathcal{C}}(L)$ of all fuzzy closure filters of L is a complete distributive lattice with relation $\subseteq$. The sup and inf of any subfamily $\{\nu_i \mid i \in \Omega\}$ of fuzzy closure filters are $\overleftarrow{\alpha}\alpha(\bigvee \nu_i)$ and i) and $\bigcap_{i\in\Omega} \nu_i$ respectively, where $\bigvee \nu_i$ is their supremum in the lattice of fuzzy filters of L.*

Lemma 3.15. *Let ν be any fuzzy ideal of $M_0(L)$. Then $\nu = \alpha\overleftarrow{\alpha}(\nu)$.*

Proof. Let $(a)^+ \in M_0(L)$. Now $\alpha\overleftarrow{\alpha}(\nu)((a)^+) = \sup\{\overleftarrow{\alpha}(\nu)(b) \mid (b)^+ = (a)^+\} = \sup\{\nu((b)^+) \mid (b)^+ = (a)^+\} = \nu((a)^+)$. Therefore $\alpha\overleftarrow{\alpha}(\nu) = \nu$. □

Using the above Corollary 3.14 and Lemma 3.15, we are able to prove that the lattice of fuzzy closure filters of L is isomorphic to the lattice of fuzzy ideals of $M_0(L)$.

Theorem 3.16. *Let L be a decomposable stone ADL. Then there is an isomorphism of the lattice of fuzzy closure filters of L onto the lattice of fuzzy ideals of $M_0(L)$.*

Proof. Let $\mathcal{FF}_{\mathcal{C}}(L)$ be the set of all fuzzy filters of L, $\mathcal{FI}M_0(L)$ be the set of all fuzzy ideals of $M_0(L)$. Define $f : \mathcal{FF}_{\mathcal{C}}(L) \to \mathcal{FI}M_0(L)$ by $f(\nu) = \alpha(\nu)$, for any $\nu \in \mathcal{FF}_{\mathcal{C}}(L)$. It is easy to see that f is one one. Let ν be an fuzzy ideal of $M_0(L)$. Then $\overleftarrow{\alpha}(\nu)$ is a fuzzy filter of L. Now By applying the above Lemma, we deduce that $\overleftarrow{\alpha}\alpha(\overleftarrow{\alpha}(\nu)) = \overleftarrow{\alpha}(\alpha\overleftarrow{\alpha}(\nu)) = \overleftarrow{\alpha}(\nu)$. Thus $\overleftarrow{\alpha}(\nu)$ is a fuzzy closure filter of L. Hence, we derive that $f(\overleftarrow{\alpha}(\nu)) = \alpha(\overleftarrow{\alpha}(\nu)) = \nu$. This result gives that f is onto. Let ν, θ be any two fuzzy closure filters of L. Clearly, we have $f(\nu \cap \theta) = \alpha(\nu \cap \theta) = \alpha(\nu) \cap \alpha(\theta)$. Now, we further obtain $f(\overleftarrow{\alpha}\alpha(\nu \vee \theta)) = \alpha((\overleftarrow{\alpha}\alpha(\nu \vee \theta))) = \alpha(\nu \vee \theta) = \alpha(\nu) \sqcup \alpha(\theta)$. Therefore, we have shown that f is an isomorphism. □

Now, we continue to study some important properties of prime fuzzy closure filters and maximal fuzzy closure filters in decomposable stone ADLs.

Definition 3.17. A proper fuzzy closure filter ν of a decomposable stone ADL L is said to be prime if for any fuzzy filters θ and μ such that $\theta \cap \mu \subseteq \nu$, we have $\theta \subseteq \nu$ or $\mu \subseteq \nu$.

Lemma 3.18. *Let P be a proper filter of L. Then P is a prime closure filter of $L,\ t \in [0,1)$ if and only if*

$$P_t^1(a) = \begin{cases} 1 \text{ if } a \in P \\ t \text{ otherwise} \end{cases}$$

is a prime closure filter of L.

Proof. Assume that P is a proper closure filter of L and $t \in [0,1)$. It can be easily verified that P_t^1 is a proper fuzzy filter of L. Now, we prove that P_t^1 is a prime fuzzy filter of L. Let θ and λ be fuzzy filters of L such that $\theta \nsubseteq P_t^1$ and $\lambda \nsubseteq P_t^1$. Then there exist $a, b \in L$ such that $\theta(a) > P_t^1(a)$ and $\lambda(b) > P_t^1(b)$. This implies $a \notin P$ and $b \notin P$, and so we have $a \vee b \notin P$ and $P_t^1(a \vee b) = \alpha$. It follows that $\theta(x) \wedge \lambda(b) > t$. Since θ and λ are isotone mappings, we have $(\theta \cap \lambda)(a \vee b) = \theta(a \vee b) \wedge \lambda(a \vee b) \geq \theta(a) \wedge \lambda(a) > t = P_t^1(a \vee b)$. This implies $\theta \cap \lambda \nsubseteq P_t^1$. Thus, we have shown that P_t^1 is a prime fuzzy filter of L. Next, we prove that P_t^1 is a prime fuzzy closure filter of L. Since P is a prime closure filter of L and $t \in [0,1)$, for any $a, b \in L$ such that $(a)^+ = (b)^+$. If $P_t^1(a) = 1$, then $a \in P$. This implies that $b \in P$ and $P_t^1(b) = 1$. If $P_t^1(a) = t$; then $a \notin P$. This implies that $b \notin P$ and $P_t^1(b) = t$. Hence, P_t^1 is a prime fuzzy closure filter of L. Conversely, assume that P_t^1 is a prime fuzzy filter of L. If F and G are any filters of L such that $F \cap G \subseteq P$, then $(F \cap G)_t^1 = F_t^1 \cap G_t^1 \subseteq P_t^1$. This implies $F_t^1 \subseteq P_t^1$ or $G_t^1 \subseteq P_t^1$, so that $F \subseteq P$ or $G \subseteq P$. Therefore, we have shown that P is a prime filter of L. Now, suppose that P_t^1 is a prime fuzzy closure filter of L and for any $a, b \in L$ such that $(a)^+ = (b)^+$. Let $a \in P$. Then, we deduce that $1 = P_t^1(a) = P_t^1(b)$. This implies $b \in P$. Hence, P is indeed a prime closure filter of L. □

Corollary 3.19. *A proper filter P is a prime closure filter of L if and only if χ_P is a prime fuzzy closure filter of L.*

Proof. Assume that P is a prime closure filter of L. Now we prove that χ_P is a prime fuzzy filter of L. Let ν and λ be any fuzzy filters of L such that $\theta \cap \lambda \subseteq \chi_P$. Suppose $\theta \nsubseteq \chi_P$ and $\lambda \nsubseteq \chi_P$. Then there exist $a, b \in L$ such that $\lambda(a) > \chi_P(a)$ and $\theta(b) > \chi_P(b)$. This implies $a \notin P$ and $b \notin P$. Since P is a prime filter, $a \vee b \notin P$. Thus $\chi_P(a \vee b) = 0$. Now, $(\lambda \cap \theta)(a \vee b) = \lambda(a \vee b) \wedge \theta(a \vee b) \geq \lambda(a) \wedge \theta(b) > \chi_P(a) \wedge \chi_P(b) = 0 = \chi_P(a \vee b)$. This implies $\theta \cap \lambda \nsubseteq \chi_P$, which is a contradiction. Thus χ_P is a prime filter of L. Next we prove that χ_P is a prime fuzzy closure filter. Let $a, b \in L$ such that $(a)^+ = (b)^+$. If $\chi_P(a) = 1$, then $a \in P$. This implies $b \in P$. Thus $\chi_P(b) = 1$. If $\chi_P(a) = 0$, then $a \notin P$. This implies $b \notin P$. Thus $\chi_P(b) = 0$. Hence χ_P is a prime fuzzy closure filter of L. Conversely, assume that χ_P is a prime closure filter of L. Now we show that P is a prime filter of L. Let F and G be any filters of L such that $F \cap G \subseteq P$. Then $\chi_{F \cap G} \subseteq \chi_P$. That implies $\chi_F \subseteq \chi_P$ or $\chi_G \subseteq \chi_P$ and hence $F \subseteq P$ or $G \subseteq P$. Therefore P is a prime filter. We prove that P is a prime closure filter of L. Let $a, b \in L$ such that $(a)^+ = (b)^+$. Let $a \in P$. Then $\chi_P(a) = 1 = \chi_P(b)$. Thus $b \in P$. Hence P is a prime fuzzy closure filter of L. □

Theorem 3.20. *proper fuzzy filter ν of L is a prime fuzzy closure filter if and only if $Img(\nu) = \{1, t\}$, where $t \in [0, 1)$ and the set $\nu_* = \{x \in L \mid \nu(x) = 1\}$ is a prime closure filter of L.*

Proof. From the above lemma, we have the converse part. Assume that ν is a prime fuzzy closure filter. Clearly, we have $1 \in Im(\nu)$. Since ν is proper, there is $a \in L$ such that $\nu(a) < 1$. We show that $\nu(a) = \nu(b)$, for all $a, b \in L \setminus \nu_*$. Suppose $\nu(a) \neq \nu(b)$, for some $a, b \in L \setminus \nu_*$. Without loss of generality we can assume that $\nu(b) < \nu(a) < 1$. Define fuzzy subsets θ and λ as follows:

$$\theta(x) = \begin{cases} 1 \text{ if } x \in [a) \\ 0 \text{ otherwise} \end{cases}$$

and

$$\lambda(x) = \begin{cases} 1 \text{ if } x \in \nu_* \\ \nu(a) \text{ otherwise} \end{cases}$$

for all $x \in L$. Clearly, we see immediately that both θ and λ are fuzzy filters of L. Let $x \in L$. If $x \in \nu_*$, then $(\theta \cap \lambda)(x) \leq 1 = \nu(x)$. If $x \in [a) \setminus \nu_*$, then $x = a \vee x$, and we have $(\theta \cap \lambda)(x) = \theta(x) \wedge \lambda(x) = 1^{\nu}(a) = \nu(a) \leq \nu(x)$. Also if $x \notin [a)$, then $\theta(x) = 0$ and hence $(\theta \cap \lambda)(x) = 0 \leq \nu(z)$. Therefore, we get $\theta \cap \lambda \subseteq \nu$. Since $\theta(x) = 1 > \nu(x)$ and $\lambda(y) = \nu(x) > \nu(y)$, we arrive that $\lambda \nsubseteq \nu$ and $\theta \nsubseteq \lambda$, which is a contradiction. Thus $\nu(a) = \nu(b)$ for all $a, b \in L \setminus \nu_*$ and hence $Im(\nu) = \{1, t\}$ for some $t \in [0, 1)$. Let $P = \{a \in L \mid \nu(a) = 1\}$. Since ν is proper, we get that P is a proper filter of L. Let $t \neq 1$. Then

$$\nu(x) = \begin{cases} 1 \text{ if } x \in P \\ t \text{ if } x \notin P. \end{cases}$$

By the above lemma, we have shown that $P = \nu_*$. □

Definition 3.21. A proper fuzzy filter ν of a decomposable stone ADL L is said to be maximal if $Im\nu = \{1, t\}$, where $t \in [0, 1)$ and the level filter $\nu_* = \{a \in L \mid \nu(a) = 1\}$ is a maximal filter.

A proper fuzzy filter ν of a decomposable stone ADL L is said to be a maximal fuzzy closure filter of L if $Im\nu = \{1, t\}$, where $t \in [0, 1)$ and the level filter ν_* is a maximal closure filter.

Theorem 3.22. *Every maximal fuzzy filter of a decomposable stone ADL is a fuzzy closure filter.*

Proof. Let ν be a maximal fuzzy filter of L. Then ν_* is a maximal filter and $Im\nu = \{1, t\}$. That implies ν_* is maximal and $\nu_t = L$. Since every maximal filter is a closure filter of L, we get that the level subsets of L is closure filters of L. Hence ν is a fuzzy closure filter of L. □

The following corollaries follow immediately.

Corollary 3.23. *Every maximal fuzzy closure filter of L is a maximal fuzzy filter.*

Corollary 3.24. *Every maximal fuzzy closure filter of L is a prime fuzzy closure filter.*

Theorem 3.25. *Let L be a decomposable stone ADL. If ν is minimal in the class of all prime fuzzy filters containing a given fuzzy closure filter, then ν is a fuzzy closure filter.*

Proof. Let ν be a minimal in the class of all prime fuzzy filters containing a fuzzy closure filter θ of L. Since ν is a prime fuzzy filter of L, there exists a prime filter P of L such

$$\nu(z) = \begin{cases} 1 \text{ if } x \in P \\ t \text{ otherwise,} \end{cases}$$

for some $t \in [0, 1)$. Suppose that ν is not a fuzzy closure filter of L. Then there exist $a, b \in L,\ (a)^+ = (b)^+$ such that $\nu(a) \neq \nu(b)$. Without loss of generality, we may assume that $\nu(a) = 1$ and $\nu(b) = t$. Consider a fuzzy ideal ϕ of L defined by

$$\phi(x) = \begin{cases} 1 \text{ if } x \in (L \setminus P) \vee (a \vee b) \\ t \text{ otherwise.} \end{cases}$$

Then we have $\theta \cap \phi \leq t$. For if otherwise, then there exists $y \in L$ such that $\phi(y) = 1$. This implies $y \in (L \setminus P) \vee (a \vee b]$. This result again implies $y = r \vee s$ for some $r \in (L \setminus P)$ and $s \in (a \vee b]$ and hence, $y = r \vee s = r \vee (s \wedge ((a \vee b)) = (r \vee s) \wedge (s \vee a \vee b) \leq s \vee a \vee b$. Since θ is a fuzzy closure filter of $L,\ t < \theta(r \vee s) \leq \theta(r \vee a \vee b)\nu(r \vee a \vee b)$. Also, $(a)^+ = (b)^+$ implies $(r \vee a \vee b)^+ = (r \vee b)^+$. These results imply that $\theta(r \vee a \vee b) = \theta(r \vee b) \leq \nu(r \vee b) = 1$. Since ν is a prime filter, we have $\nu(r) = 1$ or $\nu(y) = 1$, which is a contradiction. Thus, we arrive that $\theta \cap \phi \leq t$. This result implies that there exists a prime fuzzy filter η such that $\eta \cap \phi \leq t$ and $\theta \subseteq \eta$. Clearly, we have $a \vee b \in (L \setminus P) \vee (a \vee b]$. This result implies $\phi(a \vee b) = 1$ and $\phi \cap \eta \leq t$. Hence, we have $\eta(a \vee b) \leq t < \nu(a \vee b) = 1$. This implies $\nu \nsubseteq \eta$. Therefore, ν is not minimal in the class of all prime fuzzy filters containing a given fuzzy closure filter, which is a contradiction. Finally, we have shown that ν is indeed a fuzzy closure filter. □

Corollary 3.26. *Let L be a decomposable stone ADL. Then prime fuzzy closure filters of L are one to one correspondence with the prime fuzzy ideals of $M_0(L)$.*

Proof. Clearly, we see that fuzzy closure filters of L are one to one correspondence with the fuzzy ideals of $M_0(L)$. Now we prove that if ν is a prime fuzzy closure filter, then $\alpha(\nu)$ is also a prime fuzzy ideal of $M_0(L)$ and vice versa. Let ν be a prime fuzzy closure filter of L. Then $\alpha(\nu)$ is a fuzzy ideal of $M_0(L)$. Let θ and ν be any ideals of $M_0(L)$. Then there exist a fuzzy closure filter of $L,\ \phi$ and ψ such that $\theta = \alpha(\phi)$ and $\nu = \alpha(\psi)$. Assume that $\alpha(\phi) \cap \alpha(\psi) \subseteq \alpha(\nu)$. Then $\alpha(\phi \cap \psi) \subseteq \alpha(\nu)$ and so $\phi \cap \psi \subseteq \nu$. Since ν is a prime closure filter of L, then $\phi \subseteq \nu$ or $\psi \subseteq \nu$. This gives $\alpha(\phi) \subseteq \alpha(\nu)$ or $\alpha(\psi) \subseteq \alpha(\nu)$. Let ν be a prime ideal of $M_0(L)$. Then there exists a fuzzy closure filter of η of L such that $\nu = \alpha(\eta)$. Let ϕ, ψ be any fuzzy filters of L such that $\phi \cap \psi \subseteq \eta$. Then $\alpha(\phi \cap \psi) = \alpha(\phi) \cap \alpha(\psi) \subseteq \alpha(\eta)$. Since $\alpha(\eta)$ is a prime ideal of L, then we have $\alpha(\phi) \subseteq \alpha(\eta)$ or $\alpha(\psi) \subseteq \alpha(\eta)$ and so $\phi \subseteq \eta$ or $\psi \subseteq \eta$. This result implies η is a prime fuzzy closure filter of L. Thus, we have shown that prime fuzzy closure filters of L are one to one correspondence with the prime fuzzy ideals of $M_0(L)$.

□

Now we turn to prove the existence of prime fuzzy closure filters in decomposable stone ADL in the following theorem.

Theorem 3.27. *Let $\alpha \in [0,1)$, ν be a fuzzy closure filter and σ be a fuzzy ideal of a decomposable stone ADL L such that$\nu \cap \sigma \leq \alpha$. Then there exists a prime fuzzy closure filter η such that $\nu \subseteq \eta$ and $\eta \cap \sigma \leq \alpha$.*

Proof. Put $\xi = \{\theta \in \mathcal{FF_C}(L) \mid \nu \subseteq \theta,\ \theta \cap \sigma \leq \alpha\}$. Clearly, $\nu \in \xi$, $\xi \neq \emptyset$ and $(\xi, \subseteq)$ is a poset. Let $Q = \{\nu_i \mid i \in \Omega\}$ be a chain in ξ. We prove that $\bigcup\limits_{i\in\Omega} \nu_i \in \xi$. Clearly $(\bigcup\limits_{i\in\Omega} \nu_i)(1) = 1$. For any $a, b \in L$, $(\bigcup\limits_{i\in\Omega} \nu_i)(a) \wedge (\bigcup\limits_{i\in\Omega} \nu_i)(b) = \sup\{\nu_i(a) \mid i \in \Omega\} \wedge \sup\{\nu_j(b) \mid j \in \Omega\} = \sup\{\nu_i(a) \wedge \nu_j(b) \mid i, j \in \Omega\} \leq \sup\{(\nu_i \cup \nu_j)(a) \wedge (\nu_i \cup \nu_j)(b) \mid i; j \in \Omega\}$. Since Q is a chain, ν_i
nu_j or $\nu_j \subseteq \nu_i$. Without loss of generality, we can assume that $\nu_j \subseteq \nu_i$ This implies $\nu_i \cup \nu_j = \nu_i$. That implies $(\bigcup\limits_{i\in\Omega} \nu_i)(a) \wedge (\bigcup\limits_{i\in\Omega} \nu_i)(b) \leq \sup\{\nu_i(a) \wedge \nu_j)(b) \mid i \in \Omega\} = \sup\{\nu_i(a \wedge b) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega} \nu_i)(a \wedge b)$. Again $(\bigcup\limits_{i\in\Omega} \nu_i)(a) = \sup\{\nu_i(a) \mid i \in \Omega\} \leq \sup\{\nu_i(a \vee b) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega} \nu_i)(a \vee b)$. Similarly, we get that $(\bigcup\limits_{i\in\Omega} \nu_i)(b) \leq (\bigcup\limits_{i\in\Omega} \nu_i)(a \vee b)$. This implies $(\bigcup\limits_{i\in\Omega} \nu_i)(a) \vee (\bigcup\limits_{i\in\Omega} \nu_i)(b) \leq (\bigcup\limits_{i\in\Omega} \nu_i)(a \vee b)$. Hence $(\bigcup\limits_{i\in\Omega} \nu_i)$ is a fuzzy filter of L. Now prove that $(\bigcup\limits_{i\in\Omega} \nu_i)$ is a fuzzy closure filter. $\overleftarrow{\alpha}\alpha(\bigcup\limits_{i\in\Omega} \nu_i)(a) = \sup\{(\bigcup\limits_{i\in\Omega} \nu_i)(x) \mid (a)^+ = (x)^+,\ x \in L\} = \sup\{\sup\{\nu_i)(x) \mid i \in \Omega\} \mid (a)^+ = (x)^+,\ x \in L\} = \sup\{\sup\{\nu_i)(x) \mid (a)^+ = (x)^+,\ x \in L\} \mid i \in \Omega\} = \sup\{\overleftarrow{\alpha}\alpha(\nu_i) \mid i \in \Omega\} = \sup\{\nu_i(a) \mid i \in \Omega\} = (\bigcup\limits_{i\in\Omega} \nu_i)(a)$. Thus $\bigcup\limits_{i\in\Omega} \nu_i$ is a fuzzy closure filter of L. Since $\nu_i \cap \sigma \leq \alpha$, for each $((\bigcup\limits_{i\in\Omega} \nu_i) \cap \sigma)(a) = (\bigcup\limits_{i\in\Omega} \nu_i)(a) \wedge \sigma)(a) = \sup\{\nu_i(a) \mid i \in \Omega\} \wedge \sigma(a) = \sup\{\nu_i(a) \wedge \sigma(a) \mid i \in \Omega\} = \sup\{(\nu_i \wedge \sigma)(a) \mid i \in \Omega\} \leq \alpha$. Thus $(\bigcup\limits_{i\in\Omega} \nu_i) \cap \sigma \leq \alpha$. Hence $\bigcup\limits_{i\in\Omega} \nu_i \in \xi$. By Zorn's Lemma, ξ has a maximal element, say δ, i.e, δ is a fuzzy closure filter of L such that $\nu \subseteq \delta$ and $\delta \cap \theta \leq \alpha$. Now we show that δ is a prime fuzzy closure filter of L. Assume that δ is not a prime fuzzy closure filter. Let $\lambda_1, \lambda_2 \in \mathcal{FF_C}(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. Suppose $\delta_1 = \overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)$. Then both δ_1, δ_2 are fuzzy closure filters of L properly containing δ. Since δ is a maximal in ξ, we get that $\delta_1, \delta_2 \notin \xi$. That implies $\delta_1 \cap \theta \lneq \alpha$ and $\delta_1 \cap \theta \lneq \alpha$. That implies there exist $a, b \in L$ such that $(\delta_1 \cap \sigma)(a) > \alpha$ and $(\delta_2 \cap \sigma)(a) > \alpha$. We have $(\delta_1 \cap \sigma)(a \vee b) \wedge (\delta_2 \cap \sigma)(a \vee b) \geq (\delta_1 \cap \sigma)(a) \wedge (\delta_2 \cap \sigma)(b) \geq \alpha$, which implies $(\delta_1 \cap \sigma)(a \vee b) \wedge (\delta_2 \cap \sigma)(a \vee b) = ((\delta_1 \cap \theta) \cap (\delta_2 \cap \sigma))(a \vee b) = ((\delta_1\ \delta_2) \cap \sigma)(a \vee b) = ((\overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta) \cap \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)) \cap \sigma)(a \vee b) = (\overleftarrow{\alpha}\alpha(\lambda_1 \cap \lambda_2) \vee \delta) \cap \sigma)(a \vee b) = (\overleftarrow{\alpha}\alpha(\delta) \cap \sigma)(a \vee b) = (\delta \cap \theta)(a \vee b) > \alpha$. That implies $(\delta \cap \sigma)(a \vee b) > \alpha$, which is a contradiction to $\delta \cap \sigma \leq \alpha$. Therefore δ is a prime fuzzy closure filter of L. □

Corollary 3.28. *Let ν be a fuzzy closure filter and σ be a fuzzy ideal of a decomposable stone ADL L such that $\nu \cap \sigma = 0$. Then there exists a prime fuzzy closure filter η such that $\nu \subseteq \eta$ and $\eta \cap \sigma = 0$.*

Corollary 3.29. *Let $t \in [0,1)$, ν be a fuzzy closure filter of a decomposable stone ADL L and $\nu(x) \leq \alpha$. Then there exists a prime fuzzy closure filter θ of L such that $\nu \subseteq \theta$ and $\theta(x) \leq t$.*

Proof. Consider $\xi = \{\theta \in \mathcal{FF}_{\mathcal{C}}(L) \mid \nu \subseteq \theta \text{ and } \theta(x) \leq t\}$. Clearly, we have that $\nu \in \xi$, $\xi \neq \emptyset$, and $(\xi, \subseteq)$ is a poset. Let $Q = \{\nu_i \mid i \in \Omega\}$ be a chain in ξ. By above theorem, $\bigcup\limits_{i\in\Omega} \nu_i$ is a fuzzy closure filter of L. Since $\nu_i \subseteq \theta$ for each $i \in \Omega$ and $\theta(a) \leq t$. $(\bigcup\limits_{i\in\Omega} \nu_i)(a) = \sup\{\nu_i(x) \mid i \in \Omega\} \leq \theta(a) \leq t$. Hence $\bigcup\limits_{i\in\Omega} \nu_i \in \xi$. By Zorn's Lemma, ξ has a maximal element say δ, i.e, δ is a fuzzy closure filter of L such that $\nu \subseteq \delta$ and $\nu(a) \leq t$. Next we show that δ is a prime fuzzy closure filter of L. Assume that δ is not a prime fuzzy closure filter. Let $\lambda_1, \lambda_2 \in \mathcal{FF}(L)$, and $\lambda_1 \cap \lambda_2 \subseteq \delta$ such that $\lambda_1 \nsubseteq \delta$ and $\lambda_2 \nsubseteq \delta$. If we put $\delta_1 = \overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta)$ and $\delta_2 = \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta)$, then both δ_1, δ_2 are fuzzy closure filters of L properly containing δ. Since δ is maximal in ξ, we get $\delta_1, \delta_2 \notin \xi$. This we show that $\delta_1(a) \lneq t$ and $\delta_2(a) \lneq t$. Thus implies $\delta_1(a) > t$ and $\delta_2(a) > t$. We get $(\delta_1(a) \wedge (\delta_2)(a) = (\delta_1 \cap \delta_2)(a) > t$, which implies $\delta_1(a) \wedge \delta_2(a) = (\overleftarrow{\alpha}\alpha(\lambda_1 \vee \delta) \cap \overleftarrow{\alpha}\alpha(\lambda_2 \vee \delta))(a) = (\overleftarrow{\alpha}\alpha((\lambda_1 \cap \lambda_2) \vee \delta))(a) = \overleftarrow{\alpha}\alpha(\delta)(a) = \delta(a) > t$. That implies $\delta(a) > t$, which is a contradiction $\delta(a) \leq t$. Thus δ is a prime fuzzy closure filter of L.

□

Corollary 3.30. *Let L be a decomposable stone ADL. Then every proper fuzzy closure filters of L is the intersection of all prime fuzzy closure filters containing it.*

Proof. Let ν be a proper fuzzy closure filter of L.
Put $\eta = \bigcap\{\theta \mid \theta$ is a prime fuzzy closure filter such that $\nu \subseteq \theta\}$. Now, we proceed to prove that $\nu = \eta$. Clearly, $\nu \subseteq \eta$. Put $t = \nu(x)$, for some $x \in L$. This implies $\nu \subseteq \nu$ and $\nu(a) \leq t$. By the above Corollary, there exists a prime fuzzy closure filter δ such that $\nu \subseteq \delta$ and $\delta(x) \leq t$. Thus,we have $\eta \subseteq \nu$. Hence, $\nu = \eta$. This result implies that every proper fuzzy closure filters of L is the intersection of all prime fuzzy closure filters containing it. □

4. Fuzzy Closure Prime Spectrum

In this section, we studied the properties of the set of all closure fuzzy filters of a decomposable stone ADL topologically.
Let L be a decomposable stone ADL and X^C denotes the set of all prime fuzzy closure filters of L. For a fuzzy subset θ of L, define $H^C(\theta) = \{\mu \in X^C : \theta \subseteq \mu\}$, and $X^C(\theta) = \{\mu \in X^C : \theta \nsubseteq \mu\}$.

Lemma 4.1. *For any fuzzy filters λ and ν of a decomposable stone ADL L, we have the following*

(1) $\lambda \subseteq \nu \Rightarrow X^C(\lambda) \subseteq X^C(\nu)$
(2) $X^C(\lambda \vee \nu) = X^C(\lambda) \cup X^C(\nu)$

(3) $X^C(\lambda \cap \nu = X^C(\lambda) \cap X^C(\nu)$.

Proof.

1. Let $\mu \in X^C(\lambda)$. Then $\lambda \nsubseteq \mu$ and so $\nu \nsubseteq \mu$. Thus $\mu \in X^C(\nu)$. Hence $X^C(\lambda) \subseteq X^C(\nu)$.
2. By condition 1, we have that $X^C(\lambda) \subseteq X^C(\lambda \vee \nu)$ and $X^C(\nu) \subseteq X^C(\lambda \vee \nu)$. That implies $X^C(\nu) \cup X^C(\lambda) \subseteq X^C(\lambda \vee \nu)$. Let $\mu \in X^C(\lambda \vee \nu)$. Then $\lambda \vee \nu \nsubseteq \mu$. Since μ is a prime fuzzy closure filter, we get that $\lambda \nsubseteq \mu$ or $\nu \nsubseteq \mu$ and hence $\mu \in X^C(\lambda)$ or $\mu \in X^C(\nu)$. Therefore $\mu \in X^C(\lambda) \cup X^C(\nu)$. Thus $X^C(\lambda \vee \nu) = X^C(\lambda) \cup X^C(\nu)$.
3. Clearly, we have that $X^C(\lambda \cap \nu) \subseteq X^C(\lambda) \cap X^C(\nu)$. Let $\mu \in X^C(\lambda) \cap X^C(\nu)$. Then $\lambda \nsubseteq \mu$ and $\nu \nsubseteq \mu$. Since μ is a prime fuzzy closure filter, we have that $\lambda \cap \mu \nsubseteq \mu$. That implies $\mu \in X^C(\lambda \cap \nu)$ and hence $X^C(\lambda) \cap X^C(\nu) \subseteq X^C(\lambda \cap \nu)$. Therefore $X^C(\lambda) \cap X^C(\nu) = X^C(\lambda \cap \nu)$. □

Lemma 4.2. *Let λ be a fuzzy subset of L. Then $X^C(\lambda) = X^C([\lambda))$*

Proof. Since $\lambda \subseteq [\lambda)$, $X^C(\lambda) \subseteq X^C([\lambda))$. Let$\mu \in X^C([\lambda))$, Then $[\lambda) \nsubseteq \mu$. That implies $\lambda \nsubseteq \mu$. Suppose $\lambda \subseteq \mu$, then $[\lambda) \subseteq \mu$, Which is not possible. Therefore $\mu \in X^C(\lambda)$ and hence $X^C(\lambda) = X^C([\lambda))$. □

Lemma 4.3. *Let $x, y \in L$, and $\alpha \in (0,1]$. Then we have the following*

(1) $\bigcup_{x \in L, \alpha \in (0,1]} X^C(x_\alpha) \subseteq X^C$
(2) $X^C(x_\alpha) \cap X^C(y_\alpha) = X^C((x \vee y)_\alpha)$
(3) $X^C(x_\alpha) \cup X^C(y_\alpha) = X^C((x \wedge y)_\alpha)$.

Proof.

1. Clearly, we have that $\bigcup_{x \in L, \alpha \in (0,1]} X^C(x_\alpha) \subset X^C$. Let $\mu \in X^C$. Then $Im\mu = \{1, r\}, r \in [0,1)$. That implies there exists an element $x \in L$ such that $\mu(x) = r$. Let us take some $\alpha \in (0,1]$ such that $\alpha > r$. That implies $\mu \in X^C(x_\alpha)$ and hence $\mu \in \bigcup_{x \in L, \alpha \in (0,1]} X^C(x_\alpha)$. Therefore $X^C \subseteq \bigcup_{x \in L, \alpha \in (0,1]} X^C(x_\alpha)$. Thus $X^C = \bigcup_{x \in L, \alpha \in (0,1]} X^C(x_\alpha)$.

2.

$$
\begin{aligned}
\text{Let } \mu \in X^C(x_\alpha) \cap X^C(y_\alpha) &\Rightarrow \mu \in X^C(x_\alpha) \text{ and } \mu \in X^C(y_\alpha) \\
&\Rightarrow x_\alpha \nsubseteq \mu \text{ and } y_\alpha \nsubseteq \mu \\
&\Rightarrow \alpha > \mu(x) \text{ and } \alpha > \mu(y) \\
&\Rightarrow \alpha > \mu(x) \vee \mu(y) = \mu(x \vee y) \\
&\Rightarrow (x \vee y)_\alpha \nsubseteq \mu \\
&\Rightarrow \mu \in X^C((x \vee y)_\alpha).
\end{aligned}
$$

Therefore $X^C(x_\alpha) \cap X^C(y_\alpha) \subseteq X^C((x \vee y)_\alpha)$.

$$\begin{aligned}
\text{Let } \mu \in X^C((x \vee y)_\alpha) \Rightarrow & (x \vee y)_\alpha \nsubseteq \mu \\
\Rightarrow & \alpha > \mu(x \vee y) = \mu(x) \vee \mu(y) \text{ as } \mu \text{ is prime} \\
\Rightarrow & \alpha > \mu(x) \text{ and } \alpha > \mu(y) \\
\Rightarrow & x_\alpha \nsubseteq \mu \text{ and } y_\alpha \nsubseteq \mu \\
\Rightarrow & \mu \in X^C(x_\alpha) \text{ and } \mu \in X^C(y_\alpha) \\
\Rightarrow & \mu \in X^C(x_\alpha) \cap X^C(y_\alpha).
\end{aligned}$$

Therefore $X^C((x \vee y)_\alpha) \subseteq X^C(x_\alpha) \cap X^C(y_\alpha)$. Hence $X^C(x_\alpha) \cap X^C(y_\alpha) = X^C((x \vee y)_\alpha)$.

3. The proof similar to 2. □

Lemma 4.4. *Let $\alpha_1, \alpha_2 \in (0, 1], \alpha = min\{\alpha_1, \alpha_2\}$ and any $x, y \in L$. Then $X^C(x_{\alpha_1}) \cap X^C(y_{\alpha_2}) = X^C((x \vee y)_\alpha)$.*

Proof. Let $\mu \in X^C(x_{\alpha_1}) \cap X^C(y_{\alpha_2})$. Then $x_{\alpha_1} \nsubseteq \mu$ and $y_{\alpha_2} \nsubseteq \mu$. That implies $\alpha_1 > \mu(x)$ and $\alpha_2 > \mu(y)$. Since μ_* is a prime filter of L and $x, y \notin \mu_*$, we have that $x \vee y \notin \mu_*$ and $\mu(x) = \mu(y) = \mu(x \vee y)$. That implies $\alpha = \alpha_1 \wedge \alpha_2 > \mu(x \vee y)$, Whence $(x \vee y)_\alpha \nsubseteq \mu$ and so $\mu \in X^C((x \vee y)_\alpha)$. Thus $X^C(x_{\alpha_2}) \cap X^C(y_{\alpha_2}) \subseteq X^C((x \vee y)_\alpha)$. Let $\mu \in X^C((x \vee y)_\alpha)$. Then $(x \vee y)_\alpha \nsubseteq \mu$. That implies $\alpha > \mu(x \vee y) = \mu(x) \vee \mu(y)$. That implies $\alpha_1 > \mu(x)$ and $\alpha_2 > \mu(y)$ and $x_{\alpha_2} \nsubseteq \mu$ and $y_{\alpha_2} \nsubseteq \mu$. Therefore $\mu \in X^C(x_{\alpha_2}) \cap X^C(y_{\alpha_2})$. Hence $X^C(x_{\alpha_2}) \cap X^C(y_{\alpha_2}) = X^C((x \vee y)_\alpha)$. □

Lemma 4.5. *The collection $\mathcal{T} = \{X^C(\theta) : \theta$ is a fuzzy filter of $L\}$ is a topology on X^C.*

Proof. Consider the fuzzy subsets λ_1, λ_2 of L defined as : $\lambda_1(x) = 0$ and $\lambda_2(x) = 1$ for all $x \in L$. Clearly $[\lambda_1)$ and λ_2 are fuzzy filters of L. $[\lambda_1) \subseteq \mu$ for all $\mu \in X^C$. Thus $X^C([\lambda_1)) = \emptyset$. Since each $\mu \in X^C$ is non-constant, $\lambda_2 \nsubseteq \mu$ for all $\mu \in X^C$. Thus $X^C(\lambda_2) = X^C$. This implies $\emptyset, X^C \in \mathcal{T}$. Also for any fuzzy filters λ_1 and λ_2 of L, by Lemma-4.1 we have $X^C(\lambda_1) \cap X^C(\lambda_2) = X^C(\lambda_1 \cap \lambda_2)$. This show that $\mathcal{T}$ is closed under finite intersections. Next, let $\{\lambda_i, i \in \Omega\}$ be any family of fuzzy filters of L. Now we prove that $\bigcup\limits_{i\in\Omega} X^C(\lambda_i) = X^C([\bigcup\limits_{i\in\Omega} \lambda_i))$. Let $\mu \in X^C([\bigcup\limits_{i\in\Omega} \lambda_i))$, then $[\bigcup\limits_{i\in\Omega} \lambda_i) \nsubseteq \mu$, which implies that $\lambda_i \nsubseteq \mu$ for some $i \in \Omega$. Otherwise if $\lambda_i \subseteq \mu$ for each $i \in \Omega$, it will be true that $[\bigcup\limits_{i\in\Omega} \lambda_i) \subseteq \mu$. Thus $\mu \in \bigcup\limits_{i\in\Omega} X^C(\lambda_i)$ Whence $X^C([\bigcup\limits_{i\in\Omega} \lambda_i)) \subseteq \bigcup\limits_{i\in\Omega} X^C(\lambda_i)$. Clearly $\bigcup\limits_{i\in\Omega} X^C(\lambda_i) \subseteq X^C([\bigcup\limits_{i\in\Omega} \lambda_i))$. Hence $\bigcup\limits_{i\in\Omega} X^C(\lambda_i) = X^C([\bigcup\limits_{i\in\Omega} \lambda_i))$. Therefore, $\mathcal{T}$ is closed under arbitrary unions and hence, it is topology on X^C. □

Theorem 4.6. *Let $\mathfrak{B} = \{X^C(x_\alpha) : x \in L, \alpha \in (0, 1]\}$. Then $\mathfrak{B}$ forms a base for some topology on $\mathcal{T}$.*

Proof. By conditions 1 and 2 from Lemma-4.3, it follows that $\mathfrak{B}$ forms a base for some topology on X^C. □

Theorem 4.7. *The space X^C is a T_0-space.*

Proof. Let $\mu, \theta \in X^C$ such that $\mu \neq \theta$. Then either $\mu \nsubseteq \theta$ or $\theta \nsubseteq \mu$. Without loss of generality, we can assume that $\mu \nsubseteq \theta$. Then $\theta \in X^C(\mu)$ and $\mu \notin X^C(\mu)$. Thus X^C is a T_0-space. □

Theorem 4.8. *For any fuzzy filter μ of L, $X^C(\mu) = X^C(\overleftarrow{\alpha}\alpha(\mu))$.*

Proof. Clearly we have that for any fuzzy filter μ of L. $\mu \subseteq \overleftarrow{\alpha}\alpha(\mu)$. Then $X^C(\mu) \subseteq X^C(\overleftarrow{\alpha}\alpha(\mu))$. Conversely, let $\theta \in X^C(\overleftarrow{\alpha}\alpha(\mu))$. Then $\overleftarrow{\alpha}\alpha(\mu) \nsubseteq \theta$. Suppose $\theta \notin X^C(\mu)$, then $\mu \subseteq \theta$. This implies $\overleftarrow{\alpha}\alpha(\mu) \subseteq \theta^e = \theta$, which is not possible. Thus $\theta \in X^C(\mu)$ and so $X^C(\overleftarrow{\alpha}\alpha(\mu)) \subseteq X^C(\mu)$. Hence $X^C(\mu) = X^C(\overleftarrow{\alpha}\alpha(\mu))$. □

Theorem 4.9. *For any fuzzy filter μ of L, $X^C(\mu) = \bigcup\limits_{x_\alpha \in \mu} X^C(x_\alpha)$.*

Theorem 4.10. *The lattice $\mathcal{FF}_C(L)$ is isomorphic with the lattice of all open sets X^C.*

Proof. The lattice of all open sets in X^C is $(\mathcal{T}, \cap, \cup)$. Define the mapping $f : \mathcal{FF}_C(L) \to \mathcal{T}$ by $f(\mu) = X^C(\mu)$ for all $\mu \in \mathcal{FF}_C(L)$. Let $\mu, \theta \in \mathcal{FF}_C(L)$. Then $f(\mu \sqcup \theta) = f((\mu \vee \theta)^e) = X^C(\mu \vee \theta) = X^C(\mu) \cup X^C(\theta) = f(\mu) \cup f(\theta)$, and $f(\mu \cap \theta) = X^C(\mu \cap \theta) = X^C(\mu) \cap X^C(\theta) = f(\mu) \cap f(\theta)$. That implies f is homomorphism. Since $X^C(\mu) = X^C(\overleftarrow{\alpha}\alpha(\mu))$ and $\overleftarrow{\alpha}\alpha(\mu) \in \mathcal{FF}_C(L), \forall X^C(\mu) \in \mathcal{T}$, there exists $\overleftarrow{\alpha}\alpha(\mu) \in \mathcal{FF}_C(L)$ such that $f(\overleftarrow{\alpha}\alpha(\mu)) = X^C(\mu)$. Hence f is onto. Next we prove that f is one to one. Let $f(\mu) = f(\theta)$. Suppose that $\mu \neq \theta$, then there exists $x \in L$ such that either $\mu(x) < \theta(x)$ or $\theta(x) < \mu(x)$. Without loss of generality, we can assume that $\mu(x) < \theta(x)$. Put $\theta(x) = \alpha$, then by Corollary-3.29, we can find a prime fuzzy closure filter δ of L such that $\mu \subseteq \delta$ and $\delta(x) < \alpha$. This implies $\delta \notin X^C(\mu)$ and $\theta \nsubseteq \delta$. This show that $\delta \notin X^C(\mu)$ and $\delta \in X^C(\theta)$. This is a contradiction $f(\mu) = f(\theta)$. Thus $\mu = \theta$. Hence f is an isomorphism. □

For any fuzzy subset θ of L, $X^C(\theta) = \{\mu \in X^C : \mu \nsubseteq \theta\}$ is open set of X^C and $H^C(\theta) = X^C \setminus X^C(\theta)$ is a closed set of X^C. Also every closed set in X^C is the form of $H^C(\theta)$ for all fuzzy subset of L. Then we have the following:

Theorem 4.11. *The closure of any $A \subseteq X^C$ is given by $\overline{A} = H^C(\bigcap\limits_{\mu \in A} \mu)$.*

Proof. Let $A \subseteq X^C$ and $\gamma \in A$. Then $\bigcap \mu \in A\mu \subseteq \gamma$. Thus $\gamma \in H^C(\gamma) \subseteq H^C(\bigcap\limits_{\mu \in A} \mu)$. Therefore, $H^C(\bigcap\limits_{\mu \in A} \mu)$ is a closed set containing A. Let $\mathcal{C}$ be any closed set containing A in X^C. Then $\mathcal{C} = H^C(\theta)$ for some fuzzy subset of θ of L. Since $A \subseteq C = H^C(\theta)$, we have $\theta \subseteq \mu$ for all $\mu \in A$. Hence $\theta \subseteq \bigcap\limits_{\mu \in A} \mu$.

Therefore, $H^C(\bigcap_{\mu \in A} \mu) \subseteq H^C(\theta) = C$. Hence $H^C(\bigcap_{\mu \in A} \mu)$ is the smallest closed set containing A. Therefore, $A = H^C(\bigcap_{\mu \in A} \mu)$. □

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