Asia Pac. J. Math. 2023 **10**:42

MAXIMAL INVERSE SUBSEMIGROUP AND MAXIMAL SUBGROUP OF $Hyp_G(n)$

SARAWUT PHUAPONG[1], AMPIKA BOONMEE[2,*]

[1]Department of Mathematics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna, Chiang Mai, Thailand

[2]Faculty of Science at Siracha, Kasetsart University Sriracha Campus, Sriracha Chonburi, Thailand

[*]Corresponding author: ampika.boo@ku.th

Received Sep. 18, 2023

Abstract. A generalized hypersubstisution of type τ is a mapping which maps from the set of all any operation symbols of type τ to the set of all terms. The set of all generalized hypersubstisutions of type τ with a binary operation defined on this set forms a monoid. The monoid of all generalized hypersubstisutions of type $\tau = (n)$ denote by $Hyp_G(n)$. In semigroup theory, a regular element is a special element in semigroup. The principle special study of a regular element is a completely regular element and inverse of element with a great diversity of their various generalization. In this paper, we use the concept of regular element in the moind $Hyp_G(n)$ to study inverse of an element in this monoid. We characterize the set of all elements in the minoid $Hyp_G(n)$ which has a unique inverse and we show that this set is a maximal inverse subsemigroup of the monoid $Hyp_G(n)$. Furthermore, we have maximal inverse subsemigroup and maximal subgroup of $Hyp_G(n)$ are identical.

2020 Mathematics Subject Classification. 20B30, 20M05, 20M17.

Key words and phrases. generalized hypersubstitution; regular element; inverse of an element; inverse semigroup.

1. Introduction and Preliminaries

We first recall from [6] that an element a in a semigroup S is *regular* if there exists an element b in S with $a = aba$. A semigroup S is *regular semigroup* if all its elements are regular. An element b in S such that $a = aba$ and $b = bab$ is an inverse of a. Notice that an element with an inverse is necessarily regular. Less obviously, every regular element has an inverse. An element a may well have more than one inverse. Denote $V(a)$ is the set of all an inverse of a, then $|V(a)| \geq 1$. An *inverse semigroup* is a semigroup which every element has unique inverse, i.e. a regular semigroup in which every element has a unique inverse.

DOI: 10.28924/APJM/10-42

A semigroup S is a *monoid* if a binary operation is defined on S has an *identity*, i.e. there exist unique element e in S sucth that $ae = a = ea$ for all a in S. For each a monoid S with identity e, an element u in S is called *unit* if there exist u^{-1} in S such that $uu^{-1} = e = u^{-1}u$. Then $uu^{-1}u = u$ and $u^{-1}uu^{-1} = u^{-1}$, i.e. u^{-1} is an inverse (in semigroup) of u. The set of all unit elements of S denoted by $U(S)$.

In this paper, we study an inverse element in the monoid of all generalized hypersubstisution of type τ. Henceforth, we recall the concept of the monoid of all generalized hypersubstisution of type τ.

Let $X := \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and $X_n := \{x_1, x_2, \ldots, x_n\}$ which $n \in \mathbb{N}$ be an n-element alphabet of variables. Let $\{f_i \mid i \in I\}$ be a set of n_i-ary operation symbols indexed by the set I. The squecence $\tau = (n_i)_{i \in I}$ which $n_i \in \mathbb{N}$ is a type with operation symbols f_i. An n-ary term of type τ is defined inductively as follows:

(i) The variables $x_1, x_2, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms of type τ then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term.

Denote $W_\tau(X_n)$ is the set of all n-ary terms of type τ. $W_\tau(X_n)$ is the smallest set which contains $x_1, x_2, \ldots, x_n$ and is closed under finite application of (ii). It is clear that every n-ary term is also an m-ary for all $m \geq n$. Let $W_\tau(X) = \cup_{n=1}^{\infty} W_\tau(X_n)$. Recent trends in the study of terms can be found in [5,7,11,13].

The concept of a generalized hypersubstisution of type τ was first defined by Leeratanavalee and Denecke [10]. A generalized hypersubstisution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to the set of all terms of type τ which does not necessarily preserve the arity. The set of all generalized hypersubstisutions of type τ denoted by $Hyp_G(\tau)$. Leeratanavalee and Denecke use the concept of a generalized superposition of term and the concept of the extension of generalized hypersubstitution to define a binary operation on $Hyp_G(\tau)$ and show that $Hyp_G(\tau)$ with this binary operation forms the monoid. Firstly we will recall the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ which is defined by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then
$S^m(t, t_1, t_2, \ldots, t_m) = S^m(x_j, t_1, t_2, \ldots, t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then
$S^m(t, t_1, t_2, \ldots, t_m) = S^m(x_j, t_1, t_2, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, \ldots, s_{n_i})$, then
$S^m(t, t_1, t_2, \ldots, t_m) := f_i(S^m(s_1, t_1, t_2, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, t_2, \ldots, t_m))$.

Each generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined as follows:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, t_2, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \hat{\sigma}[t_2], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$ where $\circ$ denotes the usual composition of mappings. Then $Hyp_G(\tau)$ forms a monoind under the operation $\circ_G$ where the identity σ_{id} is a generalized hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, ..., x_{n_i})$. See [4,8,9] for other developments of generalized hypersubstitutions.

2. Maximal Inverse Subsemigroup and Maximal Subgroup of $Hyp_G(n)$

In this paper, we study inverse of an element in the moind of all generalized hypersubstitution of type $\tau = (n)$. We fix the type $\tau = (n)$ be a type with an n-ary operation symbol f and let $t \in W_{(n)}(X)$. We denote

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t) :=$ the set of all variables occurring in the term t.

In 2010, Puninagool and Leeratanavalee [12] characterized all regular elements of the monoid generalized hypersustitutions of type $\tau = (n)$. Next, Boonmee and Leeratanavalee [1] used the concept of regular elements to classify the partition of the set of all regular elements of the monoid generalized hypersustitutions of type $\tau = (n)$ by the set R_1, R_2 and R_3.

Let $\sigma_t \in Hyp_G(n)$, denote

$R_1 := \{\sigma_{x_i} \mid x_i \in X\}$;

$R_2 := \{\sigma_t \mid var(t) \cap X_n = \emptyset\}$;

$R_3 := \{\sigma_t \mid t = f(t_1, t_2, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, t_{i_2} = x_{j_2}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, i_2, \ldots, i_m$ and for distinct $j_1, j_2 \ldots, j_m \in \{1, 2, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}\}$.

Then $R_1 \cup R_2 \cup R_3$ is the set of all regular elements of the monoid $Hyp_G(n)$. We know that every regular element has an inverse, so every element in $R_1 \cup R_2 \cup R_3$ has an inverse. If $\sigma_t \in R_1 \cup R_2 \cup R_3$ then σ_t may well have more than one inverse.

For each $\sigma_{x_i} \in R_1$, σ_{x_j} is an inverse of σ_{x_i} such that

$$\sigma_{x_j} \circ_G \sigma_{x_i} \circ_G \sigma_{x_j} = \sigma_{x_j} \quad \text{and} \quad \sigma_{x_i} \circ_G \sigma_{x_j} \circ_G \sigma_{x_i} = \sigma_{x_i}$$

for all $\sigma_{x_j} \in R_1$. Similary, for each $\sigma_t \in R_2$ then σ_s is an inverse of σ_t such that

$$\sigma_s \circ_G \sigma_t \circ_G \sigma_s = \sigma_s \quad \text{and} \quad \sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$$

for all $\sigma_s \in R_2$. We see that, every element in $R_1 \cup R_2$ has more than one inverse.

For each $\sigma_t \in R_3$, $|V(\sigma_t)| \geq 1$. In the main results of this paper, we will characterize inverse of an element in R_3. Then we use the characteristics of inverse of an element in R_3 to characterize the set of all elements in the minoid $Hyp_G(n)$ which has a unique inverse. Finally, we show that the set of all elements in the minoid $Hyp_G(n)$ which has a unique inverse is a maximal inverse subsemigroup of the minoid $Hyp_G(n)$. Morever, we have this set is a maximal subgroup of the minoid $Hyp_G(n)$.

First of all, we recall some notation that need to be referenced in this paper [2].

Let $t \in W_{(n)}(X)$. A subterm of t is defined inductively by the following

(i) Every variable $x \in var(t)$ is a subterm of t.

(ii) If $t = f(t_1, t_2, \ldots, t_n)$, then $t_1, t_2, \ldots, t_n$ and t itself are subterms of t.

We denote the set of all subterms of t by $sub(t)$.

Example 2.1. Let $t \in W_{(3)}(X) \setminus X$ where $t = f(x_2, f(x_4, f(x_4, x_1, x_3), x_5), x_6)$. Then

$$sub(t) = \{x_1, x_2, x_3, x_4, x_5, x_6, f(x_4, x_1, x_3), f(x_4, f(x_4, x_1, x_3), x_5), t\}.$$

Let $t \in W_{(n)}(X) \setminus X$ where $t = f(t_1, t_2, \ldots, t_n)$ for some $t_1, t_2, \ldots, t_n \in W_{(n)}(X)$ and let $\pi_{i_l} : W_{(n)}(X)\setminus X \to W_{(n)}(X)$ with $\pi_{i_l}(t) = \pi_{i_l}(f(t_1, t_2, \ldots, t_n)) = t_{i_l}$. Maps π_{i_l} are defined for $i_l = 1, 2, \ldots, n$. Let $s \in sub(t)$ where $s \neq t$ and let $s^{(j)}$ be a subterm s occurring in the j^{th} order of t (from the left). If $s^{(j)} = \pi_{i_m} \circ \cdots \circ \pi_{i_1}(t)$ for some $m \in \mathbb{N}$, then the sequence of $s^{(j)}$ in t denote by $seq^t(s^{(j)})$ and the depth of $s^{(j)}$ in t denote by $depth^t(s^{(j)})$ such that

$$seq^t(s^{(j)}) = (i_1, i_2, \ldots, i_m) \quad \text{and} \quad depth^t(s^{(j)}) = m.$$

The set of all a sequences of s in term t denote by $seq^t(s)$, then

$$seq^t(s) = \{seq^t(s^{(j)}) \mid j \in \mathbb{N}\}.$$

Example 2.2. Let $t \in W_{(5)}(X) \setminus X$ where $t = f(x_1, s, f(x_2, f(s, x_4, x_6, s, x_3), s, s, x_5), x_1, x_7)$ for some $s \in W_{(5)}(X)$. Then

$$t = f(x_1, s^{(1)}, f(x_2, f(s^{(2)}, x_4, x_6, s^{(3)}, x_3), s^{(4)}, s^{(5)}, x_5), x_1, x_7)$$

and then

$$\begin{aligned}
seq^t(s^{(1)}) &= (2), & depth^t(s^{(1)}) &= 1,\\
seq^t(s^{(2)}) &= (3,2,1), & depth^t(s^{(2)}) &= 3,\\
seq^t(s^{(3)}) &= (3,2,4), & depth^t(s^{(3)}) &= 3,\\
seq^t(s^{(4)}) &= (3,3), & depth^t(s^{(4)}) &= 2,\\
seq^t(s^{(5)}) &= (3,4), & depth^t(s^{(5)}) &= 2
\end{aligned}$$

and $seq^t(s) = \{(2), (3,2,1), (3,2,4), (3,3), (3,4)\}$.

In this paper, we introduce the following definition.

Definition 2.3. Let $t \in W_{(n)}(X) \setminus X$ and let $m \in \mathbb{N}$. The set of all distinct a variable $x_i \in var(t) \cap X_n$ which $depth^t(x_i^{(j)}) = m$ for some $j \in \mathbb{N}$ denote by $var(t)^{d(m)}_{X_n}$, then

$$var(t)^{d(m)}_{X_n} = \{x_i \in var(t) \cap X_n \mid depth^t(x_i^{(j)}) = m \text{ for some } j \in \mathbb{N}\}.$$

Defined the set $codn(t)^{d(m)}_{X_n}$ by

$codn(t)^{d(m)}_{X_n} = \{i_m \in \{1,2,\dots,n\} \mid seq^t(x_i^{(j)}) = (i_1,i_2,\dots,i_m)$ where $x_i \in var(t) \cap X_n$ which $depth^t(x_i^{(j)}) = m$ for some $j \in \mathbb{N}\}$.

Example 2.4. Let $t \in W_{(4)}(X) \setminus X$ where $t = f(x_2, f(x_1,x_3,x_5,f(x_2,x_1,x_6,x_8)), x_4, f(x_1,x_3,x_7,x_3))$, then

$$var(t)^{d(1)}_{X_n} = \{x_2,x_4\}, \qquad codn(t)^{d(1)}_{X_n} = \{1,3\},$$
$$var(t)^{d(2)}_{X_n} = \{x_1,x_3\}, \qquad codn(t)^{d(2)}_{X_n} = \{1,2,4\},$$
$$var(t)^{d(3)}_{X_n} = \{x_1,x_2\}, \qquad cond(t)^{d(3)}_{X_n} = \{1,2\}.$$

For $m \geq 5$, then $var(t)^{d(m)}_{X_n} = \emptyset$ and $codn(t)^{d(m)}_{X_n} = \emptyset$.

Lemma 2.5. *Let $t = f(t_1,t_2,\dots,t_n)$ where $var(t) \cap X_n = \{x_{i_1}, x_{i_2}, \dots, x_{i_m}\}$ for some $i_1,i_2,\dots,i_m \in \{1,2,\dots,n\}$ and let $s = f(s_1,s_2,\dots,s_n)$ where $s_{i_l} = x_{i_l}$ for all $i_l \in \{i_1,i_2,\dots,i_m\}$ then $\sigma_t \circ_G \sigma_s = \sigma_t$.*

Proof. Assume that the condition holds and denote

$$(\sigma_t \circ_G \sigma_s)(f) = f(u_1,...,u_n)$$

where $u_i = S^n(t_i, \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2], ..., \hat{\sigma}_t[t_n])$ for all $i \in \{1,2,...,n\}$. We will prove that $\sigma_t \circ_G \sigma_s = \sigma_t$ by showing that $u_i = t_i$ for all $i \in \{1,2,\dots,n\}$. Let $t_i \in \{t_1,t_2,\dots,t_n\}$. If $var(t_i) \cap X_n = \emptyset$ then $u_i = t_i$. If $t_i = x_{i_j}$ for some $i_j \in \{i_1,i_2,\dots,i_m\}$ then

$$u_i = S^n(x_{i_j}, \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2], ..., \hat{\sigma}_t[s_n]) = \hat{\sigma}_t[s_{i_j}] = \hat{\sigma}_t[x_{i_j}] = x_{i_j} = t_i.$$

For $t_i \in W_{(n)}(X) \setminus X$ and $var(t_i) \cap X_n \neq \emptyset$ where $t_i = f(w_1,w_2,\dots,w_n)$. Then

$$u_i = S^n(f(w_1,w_2,\dots,w_n), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2], ..., \hat{\sigma}_t[s_n]).$$

So $u_i \in W_{(n)}(X) \setminus X$ and $var(u_i) \cap X_n \neq \emptyset$. Let $u_i = f(u'_1,u'_2,\dots,u'_n)$ where $u'_i = S^n(w_i, \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2], ..., \hat{\sigma}_t[t_n])$ for all $i \in \{1,2,...,n\}$. The proof in this case is simmilar to the previous case, then $u'_k = w_k$ for all $k \in \{1,2,\dots,n\}$. Therefore $u_i = t_i$ for all $i \in \{1,2,\dots,n\}$, i.e. $\sigma_t \circ_G \sigma_s = \sigma_t$. □

By the definition of set R_3 and the definition of $var^{d(m)}_{X_n}$, we can rewrite the set R_3 as follows:

$$R_3 = \{\sigma_t \mid t \in W_{(n)}(X) \setminus X \text{ where } var(t)^{d(1)}_{X_n} \neq \emptyset \text{ and } var(t) \cap X_n = var(t)^{d(1)}_{X_n}\}.$$

Theorem 2.6. *Let $\sigma_t \in R_3$ where $t = f(t_1,t_2,\dots,t_n)$ and $var(t)^{d(1)}_{X_n} = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$ and let $s = f(s_1,s_2,\dots,s_n)$. Then $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ if and only if $s_{j_l} = x_{\pi(j_l)}$ where π is a bijective map from $\{j_1,j_2,\dots,j_m\}$ into $\{i_1,i_2,\dots,i_m\}$ for some $i_1,i_2,\dots,i_m \in codn(t)^{d(1)}_{X_n}$ such that $t_{i_l} = x_{\pi^{-1}(i_l)}$ for all $i_l \in \{i_1,i_2,\dots,i_m\}$.*

Proof. Let

$$u = \sigma_s \circ_G \sigma_t(f) = f(u_1, u_2, \ldots, u_n)$$

where $u_i = S^n(s_i, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], ..., \hat{\sigma}_s[t_n])$ and let

$$w = \sigma_t \circ_G \sigma_u(f) = f(w_1, w_2, \ldots, w_n)$$

where $w_i = S^n(t_i, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2], ..., \hat{\sigma}_t[u_n])$.

($\Rightarrow$) Assume that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. We prove the result by contradiction. Suppose that, there exists $j_l \in \{j_1, j_2, \ldots, j_m\}$ such that $s_{j_l} \notin \{x_i | i \in codn(t)^{d(1)}_{X_n}\}$. Since $x_{j_l} \in var(t)^{d(1)}_{X_n}$, there exist $i_l \in codn(t)^{d(1)}_{X_n}$ such that $t_{i_l} = x_{j_l}$. Then

$$\begin{aligned} w_{i_l} &= S^n(t_{i_l}, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2], ..., \hat{\sigma}_t[u_n]) \\ &= S^n(x_{j_l}, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2], ..., \hat{\sigma}_t[u_n]) \\ &= \hat{\sigma}_t[u_{j_l}] \end{aligned}$$

and $u_{j_l} = S^n(s_{j_l}, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], ..., \hat{\sigma}_s[t_n])$. If $s_{j_l} \in X_n$ then $s_{j_l} = x_k$ for some $k \in \{1, 2, \cdots, n\} \backslash codn(t)^{d(1)}_{X_n}$. So

$$u_{j_l} = S^n(x_k, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], ..., \hat{\sigma}_s[t_n]) = \hat{\sigma}_s[t_k].$$

Since $k \notin codn(t)^{d(1)}_{X_n}$, so $t_k \in W_{(n)}(X) \setminus X_n$ and so $u_{j_l} \in W_{(n)}(X) \setminus X_n$. If $s_{j_l} \notin X_n$ then $u_{j_l} \in W_{(n)}(X) \setminus X_n$. Therefore $w_{i_l} = \hat{\sigma}_t[u_{j_l}] \in W_{(n)}(X) \setminus X_n$. So $t_{i_l} = x_{j_l} \neq w_{i_l}$. We have therefore reached a contradiction, because we assumed that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$.

($\Leftarrow$) Assume that the condition holds. Let $j_l \in \{j_1, j_2, \ldots, j_m\}$. Then there exist $i_l \in \{i_1, i_2, \ldots, i_m\}$ such that $\pi(j_l) = i_l$ and $\pi^{-1}(i_l) = j_l$. So $s_{j_l} = x_{\pi(j_l)} = x_{i_j}$ and $t_{i_l} = x_{\pi^{-1}(i_l)} = x_{j_l}$. Consider

$$\begin{aligned} u_{j_l} &= S^n(s_{j_l}, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], ..., \hat{\sigma}_s[t_n]) \\ &= S^n(x_{i_l}, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2], ..., \hat{\sigma}_s[t_n]) \\ &= \hat{\sigma}_s[t_{i_l}] \\ &= x_{j_l}. \end{aligned}$$

It follows that $u_{j_l} = x_{j_l}$ for all $j_l \in \{j_1, j_2, \ldots, j_m\}$. By Lemma 2.5, $\sigma_t \circ_G \sigma_u = \sigma_t$. Therefore $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. $\square$

By the characteristics of σ_s in the previous theorem we see that $var(s)^{d(1)}_{X_n} = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}$. If $var(s) \cap X_n = var(s)^{d(1)}_{X_n}$ then $\sigma_s \in R_3$ and $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. By the previous theorem, we have the characteristics of inverse of an element in the set R_3 are as follows:

Theorem 2.7. *Let $\sigma_t \in R_3$ where $t = f(t_1, t_2, \ldots, t_n)$ and $var(t)^{d(1)}_{X_n} = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$. Then $V(\sigma_t) = \{\sigma_s \in R_3 \mid s = f(s_1, s_2, \ldots, s_n)$ where $s_{j_l} = x_{\pi(j_l)}$ and $\pi : \{j_1, j_2, \ldots, j_m\} \to \{i_1, i_2, \ldots, i_m\}$ is a bijective map, for some $i_1, i_2, \ldots, i_m \in codn(t)^{d(1)}_{X_n}$ such that $t_{i_l} = x_{\pi^{-1}(i_l)}$ for all $i_l \in \{i_1, i_2, \ldots, i_m\}\}$.*

Proof. Let $H = \{\sigma_s \in R_3 \mid s = f(s_1, s_2, \dots, s_n)$ where $s_{j_l} = x_{\pi(j_l)}$ and $\pi : \{j_1, j_2, \dots, j_m\} \to \{i_1, i_2, \dots, i_m\}$ is a bijective map, for some $i_1, i_2, \dots, i_m \in codn(t)^{d(1)}_{X_n}$ such that $t_{i_l} = x_{\pi^{-1}(i_l)}$ for all $i_l \in \{i_1, i_2, \dots, i_m\}\}$. We will show that $V(\sigma_t) = H$. Let $\sigma_u \in V(\sigma_t)$ then $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$ and $\sigma_u \circ_G \sigma_t \circ_G \sigma_u = \sigma_u$. So σ_u is regular where $var(u) \cap X_n \neq \emptyset$, i.e. $\sigma_u \in R_3$. By theorem 2.6, we have $\sigma_u \in H$ so is $V(\sigma_t) \subseteq H$. Let $\sigma_u \in H$ where $u = f(u_1, u_2, \dots, u_n)$ then $u_{j_l} = x_{\pi(j_l)}$ and $\pi : \{j_1, j_2, \dots, j_m\} \to \{i_1, i_2, \dots, i_m\}$ is a bijective map, for some $i_1, i_2, \dots, i_m \in codn(t)^{d(1)}_{X_n}$ such that $t_{i_l} = x_{\pi^{-1}(i_l)}$ for all $i_l \in \{i_1, i_2, \dots, i_m\}$. By Theorem 2.6, we have $\sigma_t \circ_G \sigma_u \circ_G \sigma_t = \sigma_t$ and $var(u)^{d(1)}_{X_n} = \{x_{i_1}, x_{i_2}, \dots, x_{i_m}\}$ and $j_1, j_2, \dots, j_m \in codn(u)^{d(1)}_{X_n}$ such that $\pi^{-1} : \{i_1, i_2, \dots, i_m\} \to \{j_1, j_2, \dots, j_m\}$ is a bijective map and $u_{j_l} = x_{(\pi^{-1})^{-1}(j_l)}$. By Theorem 2.6, we have $\sigma_u \circ_G \sigma_t \circ_G \sigma_u = \sigma_u$. Hence $\sigma_u \in V(\sigma_t)$ so is $H \subseteq V(\sigma_t)$. Therefore $V(\sigma_t) = H$. □

Lemma 2.8. *Let $\sigma_t \in R_3$. If $var(t)^{d(1)}_{X_n} \neq X_n$ then $|V(\sigma_t)| > 1$.*

Proof. Let $var(t)^{d(1)}_{X_n} = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \subset X_n$. Then $\{j_1, j_2, \dots, j_m\} \subset \{1, 2, \dots, n\}$ and then $\{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_m\} \neq \emptyset$. By Theorem 2.7, there exist $\sigma_s \in V(\sigma_t)$ where $\sigma_s \in R_3$ and $j_1, j_2, \dots, j_m \in codn(s)^{d(1)}_{X_n}$. If $s = f(s_1, s_2, \dots, s_n)$ then σ_s is an inverse of σ_t for all $s_i \in W_{(n)}(X)$ where $i \in \{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_m\}$ and $var(s_i) \cap X_n \subseteq var(s) \cap X_n$. Hence $|V(\sigma_t)| > 1$. □

Example 2.9. Let $t \in R_3$ where

$$t = f(x_2, f(x_1, x_4, x_7, x_7, x_7), x_4, f(x_4, x_7, x_7, x_8, x_1), x_1).$$

Then $var(t)^{d(1)}_{X_5} = \{x_1, x_2, x_4\}$ and $codn(t)^{d(1)}_{X_n} = \{1, 3, 5\}$. By Theorem 2.7, there exist $\sigma_s \in V(\sigma_t)$ where $var(s)^{d(1)}_{X_5} = \{x_1, x_3, x_5\}$ and $1, 2, 4 \in codn(t)^{d(1)}_{X_n}$ such that $s = f(x_5, x_1, s_3, x_3, s_5)$ where $s_3, s_5 \in W_{(n)}(X)$ and $var(s_i) \cap X_5 \subseteq \{x_1, x_3, x_5\}$ for all $i \in \{3, 5\}$.

If $s_3 = x_1$ and $s_5 = f(x_3, x_6, x_6, x_7, x_7)$ then

$$s = f(x_5, x_1, x_1, x_3, f(x_3, x_6, x_6, x_7, x_7))$$

such that $\sigma_s \in V(\sigma_t)$.

If $s_3 = x_8$ and $s_5 = x_7$ then $s = f(x_5, x_1, x_8, x_3, x_7)$ such that $\sigma_s \in V(\sigma_t)$.

We see that $|V(\sigma_t)| > 1$.

Theorem 2.10. *Let $\sigma_t \in R_3$. $|V(\sigma_t)| = 1$ if and only if $var(t)^{d(1)}_{X_n} = X_n$.*

Proof. Let $\sigma_t \in R_3$ then $\emptyset \neq var(t)^{d(1)}_{X_n} \subseteq X_n$ and $|V(\sigma_t)| \geq 1$.
($\Rightarrow$) By contrapositive of Lemma 2.8, if $|V(\sigma_t)| = 1$ then $var(t)^{d(1)}_{X_n} = X_n$.
($\Leftarrow$) Assume that $var(t)^{d(1)}_{X_n} = X_n$. Then $codn(t)^{d(1)}_{X_n} = \{1, 2, \dots, n\}$. So $t = f(x_{\pi_t(1)}, x_{\pi_t(2)}, \dots, x_{\pi_t(n)})$ where π_t is a bijective map on $\{1, 2, \dots, n\}$. Hence there exist unique a bijective map on $i \in \{1, 2, \dots, n\}$, say π_s where $\pi_s(i) = j$ such that $\pi_t(j) = i$. Then there exist unique $\sigma_s \in Hyp_G(n)$ where $s = f(s_1, s_2, \dots, s_n) = f(x_{\pi_s(1)}, x_{\pi_s(2)}, \dots, x_{\pi_s(n)})$ which σ_s is an inverse of σ_t. Therefore $|V(\sigma_t)| = 1$. □

Theorem 2.11. [3] *An element $\sigma_t \in U(Hyp_G(n))$ if and only if $t = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ where $\pi \in S_n$ and S_n is a set of all permutation of $\{1, 2, \ldots, n\}$.*

Let $Inv(Hyp_G(n))$ is the set of all elements in the minoid $Hyp_G(n)$ which has a unique inverse. Then

$$Inv(Hyp_G(n)) = \{\sigma_t \in Hyp_G(n) \mid t \in W_{(n)}X \setminus X \text{ and } var(t)^{d(1)}_{X_n} = X_n\}.$$

We see that $Inv(Hyp_G(n)) = U(Hyp_G(n))$. Since $U(Hyp_G(n))$ is a subsemigroup of $Hyp_G(n)$, so $Inv(Hyp_G(n))$ is an inverse subsemigroup of $Hyp_G(n)$. Hence $Inv(Hyp_G(n))$ is maximal inverse subsemigroup of $Hyp_G(n)$, because $Inv(Hyp_G(n))$ contains all elements in the minoid $Hyp_G(n)$ which has a unique inverse. It is clear that $Inv(Hyp_G(n)) = U(Hyp_G(n))$ is a maximal subgroup of $Hyp_G(n)$.

Acknowledgments

The research was supported by Faculty of Science at Siracha, Kasetsart University Sriracha Campus, Sriracha Chonburi and Rajamangala University of Technology Lanna, Chiang Mai, Thailand. The authors are thankful to the referees for their valuable comments.

References

[1] A. Boonmee, S. Leeratanavalee, All completely regular elements in $Hyp_G(n)$, Discuss. Math. - Gen. Algebra Appl. 33 (2013), 211–219. https://doi.org/10.7151/dmgaa.1203.

[2] A. Boonmee, S. Leeratanavalee, All intra-regular generalized hypersubstitutions of type (2), Acta Univ. Sapient. Math. 11 (2019), 29–39. https://doi.org/10.2478/ausm-2019-0003.

[3] A. Boonmee, S. Leeratanavalee, Factorisable monoid of generalized hypersubstitutions of type $\tau = (2)$, Thai J. Math. 13 (2015), 213–225.

[4] N. Chansuriya, All maximal idempotent submonoids of generalized cohypersubstitutions of type $\tau = (2)$, Discuss. Math. Gen. Algebra Appl. 41 (2021), 45–54.

[5] K. Denecke, H. Hounnon, Partial Menger algebras of terms, Asian-European J. Math. 14 (2020), 2150092. https://doi.org/10.1142/s1793557121500923.

[6] J.M. Howie, Fundamentals of semigroup theory, Academic Press, London, 1995.

[7] P. Kitpratyakul, B. Pibaljommee, On substructures of semigroups of inductive terms, AIMS Math. 7 (2022), 9835–9845. https://doi.org/10.3934/math.2022548.

[8] T. Kumduang, S. Leeratanavalee, Semigroups of terms, tree Languages, Menger algebra of n-ary functions and their embedding theorems, Symmetry. 13 (2021), 558. https://doi.org/10.3390/sym13040558.

[9] P. Kunama, S. Leeratanavalee, Green's relations on submonoids of generalized hypersubstitutions of type (n), Discuss. Math. Gen. Algebra Appl. 41 (2021), 239–248.

[10] S. Leeratanavalee, K. Denecke, Generalized hypersubstitutions and strongly solid varieties, in: General Algebra and Applications, Proceeding of the 59 th Workshop on General Algebra, 15 th Conference for Young Algebraists Potsdam 2000, Shaker Verlag, (2000), 135-145.

[11] S. Phuapong, T. Kumduang, Menger algebras of terms induced by transformations with restricted range, Quasigroup Related Syst. 29 (2021), 255–268.

[12] W. Punainagool, S. Leeratanavalee, The monoid of generalized hypersubstitutions of type $\tau = (n)$, Discuss. Math. Gen. Algebra Appl. 30 (2010), 173–191.

[13] K. Wattanatripop, T. Changphas, The Menger algebra of terms induced by order-decreasing transformations, Commun. Algebra. 49 (2021), 3114–3123. https://doi.org/10.1080/00927872.2021.1888385.