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NUMERICAL IMPLEMENTATION OF HIGHER ORDER EXPLICIT METHOD FOR PHYSICAL MODELS VIA A HYBRID INTERPOLATING FUNCTION

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Abstract. This work introduces and analyzes a novel higher order explicit method for solving physical models in real-life situations. This new method is derived via a hybrid interpolating function which is the combination of Chebyshev polynomials of first kind and exponential function. Its properties, including consistency, local truncation error, stability, order of accuracy, and convergence, are thoroughly examined and studied. To evaluate its effectiveness, the proposed method is applied to four numerical examples derived from real-world scenarios. Furthermore, this study compares the results obtained from the new numerical method with those of the well-known PJS method [28], in terms of the exact solution. The study concludes that the method provides accurate solutions and can be considered as one of the suitable approaches for solving first-order initial value problems (IVPs).

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1. Introduction

The study of differential equations plays a crucial role in various scientific and engineering disciplines as it involves modeling natural processes such as growth and decay, heat transfer, population dynamics,

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and object motion [1]. Differential equations provide mathematical expressions that describe the connection between the current value of a quantity and its rate of change. By analyzing how quantities evolve over time, these equations enable us to make predictions about future events [2]. Differential equations can be categorized into two main types: ordinary and partial. Ordinary differential equations are concerned with functions that involve a single variable, while partial differential equations describe the behavior of functions that depend on multiple variables [3]. To solve these equations accurately, various methods are employed in the study of differential equations, including analytical, numerical, and qualitative approaches. A fundamental aspect of computational mathematics involves studying numerical techniques for resolving ordinary differential equations (ODEs). When ODEs cannot be solved analytically, numerical approaches are used to approximate them, enabling simulations and predictions of real-world processes [4]. Furthermore, numerical solutions allow for predicting the dynamic behavior of complex systems, which is not achievable through analytical solutions [5]. The efficiency of numerical methods in solving ODEs depends on factors such as accuracy, stability, and convergence. Traditional numerical methods like Euler's method, Runge-Kutta methods, and multistep methods have limitations concerning these factors. Consequently, recent efforts have focused on developing and analyzing new and effective numerical methods for solving ODEs. Different numerical techniques can differ in terms of convergence, accuracy order, local errors, stability, and computational complexity. Research in the field of numerical methods for solving ordinary differential equations (ODEs) is actively focusing on enhancing the accuracy, stability, and efficiency of these methods. Many algorithms have been proposed in scholarly works, taking into account the specific characteristics and form of the differential equations that need to be solved. Numerous examples of these algorithms can be found in the literature, such as [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], among others. Fadugba and Idowu [27] applied a new numerical method with third-order accuracy to solve initial value problems (IVPs) in ODEs, analyzing its properties in the process. In this study we will consider a new numerical method via the transcendental function of exponential type with an initial value problem,

$$y' = f(x, y);\ y(x_0) = y_0,\ x \in [a, b],\ -\infty < y < \infty \tag{1}$$

A trigonometric function-based numerical integration formula is applicable if the answer to (1) is known to be periodic or oscillate with a known frequency [15]. On the other hand, a numerical approach will be far more useful if the solution of (1) contains singularities.

2. Methodology

This section presents the derivation of a new numerical method and its properties.

2.1. **Derivation of newly proposed numerical method.** In this paper, we derive a new numerical method via the combination of Chebyshev polynomials of first kind and exponential function of the form

$$F(x) = \sum_{i=0}^{3} b_i H_i(x) + b_4 e^{-2x} \tag{2}$$

for the solution of (1). By assumption, the theoretical solution $y(x)$ to equation (1) can be represented locally in the interval $[x_n, x_n + 1],\ n \geq 0$ by interpolating polynomial (1). From (2), we have

$$F(x) = b_0 H_0(x) + b_1 H_1(x) + b_2 H_2(x) + b_3 H_3(x) + b_4 e^{-2x} \tag{3}$$

where $b_0......b_4$ are constants and $H_0(x),\ H_1(x),\ H_2(x),$ and $H_3(x)$ are the first, second, third and fourth Chebyshev polynomials of the first kind. Therefore,

$$H_0(x) = 1 \tag{4}$$

$$H_1(x) = x \tag{5}$$

$$H_2(x) = 2x^2 - 1 \tag{6}$$

$$H_3(x) = 4x^3 - 3x \tag{7}$$

Using (3)-(7)

$$F(x) = a_0 + a_1 x + a_2(2x^2 - 1) + a_3(4x^3 - 3x) + a_4 e^{-2x} \tag{8}$$

Assuming y_n is the numerical estimate to the theoretical solution $y(x)$ and $F_n = F(x_n, y_n)$, we define mesh points as

$$x_{n+1} - x_n = h,\ n = 0,\ 1,\ 2,\ 3, ...$$

To obtain undetermined coefficients, the following constants are imposed on the interpolating polynomial (8). Interpolating function must coincide with theoretical solution at $x = x_n$ and $x = x_{n+1}$

$$F(x_n) = a_0 + a_1 x_n + a_2(2x_n^2 - 1) + a_3(4x_n^3 - 3x_n) + a_4 e^{-2x_n} \tag{9}$$

and

$$F(x_n + 1) = a_0 + a_1 x_n + 1 + a_2(2x_n + 1^2 - 1) + a_3(4x_n + 1^3 - 3x_n + 1) + a_4 e^{-2x_n + 1} \tag{10}$$

The derivatives $F'(x),\ F''(x),\ F'''(x), F''''$ coincide with f_n, f_n', f_n'', f_n'''
respectively.

$$F^(x_n) = f_n \tag{11}$$

$$F''(x_n) = f_n' \tag{12}$$

$$F'''(x_n) = f_n'' \tag{13}$$

$$F^{iv}(x_n) = f_n''' \tag{14}$$

Therefore

$$F'(x) = a_1 + 4a_2x_n + a_3(12x_n^2 - 3) - 2a_4e_n^{-2x} = f_n \tag{15}$$

$$F''(x) = 4a_2 + a_3(24x_n) + 4a_4e^{-2x_n} = f_n' \tag{16}$$

$$F'''(x) = 24a_3 - 8a_4 - 4e^{-2x_n} = f_n'' \tag{17}$$

$$F^{iv}(x) = 16a_4e^{-2x_n} = f_n''' \tag{18}$$

From (18),

$$a_4 = \frac{f_n'''}{16e^{2x_n}} \tag{19}$$

Substituting (19) into (17), we obtain

$$a_3 = \frac{2f_n'' + f_n'''}{48} \tag{20}$$

Now substituting (19), (20) into (16), we obtain

$$a_2 = \frac{4f_n' - 4nhf_n'' - 2nhf_n''' - f_n''}{16} \tag{21}$$

In order to obtain a_1, we substitute (19), (20) and (21) into (15). then one obtains

$$a_1 = f_n - nhf_n' + \left(\frac{(nh)^2}{2} + \frac{1}{8}\right) f_n'' + \left(\frac{(nh)^2}{4} + \frac{nh}{4} + \frac{3}{16}\right) f_n''' \tag{22}$$

The undetermined coefficients a_1 a_2 a_3 and a_4 are given by (19), (20), (21) and (22). By definition, the mesh points x_n and x_{n+1} are given by

$$x_n = x_0 + nh \tag{23}$$

$$x_{n+1} = x_0 + (n+1)h \tag{24}$$

Let $x_0 = 0$ from (23) and (24), we obtain

$$x_n = nh, x_n + 1 = (n+1)h$$

Thus,

$$x_{n+1} - x_n = (n+1)h - (nh) = h \tag{25}$$

$$x_{n+1}^2 - x_n^2 = \left[(n+1)h^2\right] - (nh)^2 = h^2(2n+1) \tag{26}$$

$$x_{n+1}^3 - x_n^3 = \left[(n+1)h^3\right] - (nh)^3 = h^3(3n^2 + 3n + 1) \tag{27}$$

Subtracting (9) from (10), we obtain

$$F(x_n + 1) - F(x_n) = a_0 + a_1x_n + 1 + a_2(2x_n^2 + 1 - 1) + a_3(4x_n^3 + 1 - 3x_n + 1$$
$$+ a_4e^{(-2x_n+1)}) - \left[a_0 + a_1x_n + a_2(2x_n^2 - 1) + a_3(4x_n^3 - 3x_n) + a_4e^{-(2x_n)}\right]$$

$$F(n+1) - F(x_n) = a_1(x_n + 1 - x_n) + a_2(2x_n^2 + 1 - 2x_n^2)$$
$$+ a_3\left[4x_n + 1 - 4x_n^3 - (3x_n + 1 + 3x_n)\right] + a_4(e^{(-2x_n+1)} - e^{(-2x_n)}) \tag{28}$$

Substituting (25), (26) and (27) into (28), we have

$$a_1(h) + 2a_2h^2(2_n + 1) + a_3\left[4h^3(3n^2 + 3n + 1) - 3h\right] + a_4\left[e^{-(2n+1)h} - e^{(-2nh)}\right] \tag{29}$$

From (19), (20), (21) and (22) with $x_n = nh$, we have

$$a_4 = \frac{f_n'''}{16e^{(-2nh)}} \tag{30}$$

$$a_3 = \frac{2f_n'' + f_n'''}{48} \tag{31}$$

$$a_2 = \frac{4f_n' - 4nhf_n'' - 2nhf_n''' - f_n'''}{16} \tag{32}$$

$$a_1 = f_n - nhf_n' + \left[\frac{(nh)^2}{2} + \frac{1}{8}\right] f_n'' + \left[\frac{(nh)^2}{4} + \frac{(nh)}{4} + \frac{3}{16}\right] f_n''' \tag{33}$$

Then, substituting (30), (31), (32) and (33) into (29), we have

$$f_n - nhf_n' + \left[\frac{(nh)^2}{2} + \frac{1}{8}\right] f_n'' = \left[\frac{(nh)^2}{4} + \frac{(nh)}{4} + \frac{3}{16}\right] f_n'''(h) +$$
$$\left(\frac{4f_n' - 4nhf_n'' - 2nhf_n''' - f_n'''}{16}\right) 2h^2(2_n + 1) + \frac{2f_n'' + f_n'''}{48}\left[4h^3(3n^2 + 3n + 1) - 3h\right]$$
$$+ \frac{f_n'''}{16e^{(-2nh)}}\left[e^{-(2n+1)h} - e^{(-2nh)}\right] = F(x_{n+1}) - F(x_n) \tag{34}$$

Simplifying (34) yields

$$F(x_{n+1}) - F(x_n) = hf_n + h^2\left[\frac{f_n'}{2} - \frac{f_n'''}{8}\right] + h^3\left[\frac{f_n''}{6} + \frac{f_n'''}{12}\right] + \left[\frac{h}{8} + \frac{e^{-2h}}{16} - \frac{1}{16}\right] f_n''' \tag{35}$$

Let

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n \tag{36}$$

Therefore,

$$y_{n+1} = y_n + hf_n + h^2\left[\frac{f_n'}{2} - \frac{f_n'''}{8}\right] + h^3\left[\frac{f_n''}{6} + \frac{f_n'''}{12}\right] + \left[\frac{h}{8} + \frac{e^{-2h}}{16} - \frac{1}{16}\right] f_n''' \tag{37}$$

Thus, (37) is the newly derived numerical method from the combination of both exponential function and Chebyshev polynomial of the first kind

3. Analysis of the Newly Derived Numerical Method

3.1. Convergence property of the derived method. The convergence property of the newly derived method is presented in the result below.

Theorem 1. *Let y_n^* be defined as a point in the interior of the interval whose end points are y_n and $\bar{y}_n$. Applying the mean value theorem, then the derived scheme (37) is convergent and the increment function is Lipschitzian.*

Proof. The newly derived scheme is simplified as

$$y_{n+1} - y_n = h(f_n + D_1 f_n' + D_2 f_n'' + D_3 f_n''') \tag{38}$$

with

$$D_1 = \frac{h}{2}, D_2 = \frac{h^2}{3!}, D_3 = \frac{h^3}{4!} \tag{39}$$

Comparing equation (39) with the general one-step method, we have the increment function defined as

$$\Phi(x_n, y_; h) = f(x_n, y_n) + D_1 f'(x_n, y_n) + D_2 f''(x_n, y_n) + D_3 f'''(x_n, y_n) \tag{40}$$

Similarly,

$$\phi(x_n, \bar{y_n}) = f(x_n, \bar{y_n}) + D_1 f'(x_n, \bar{y_n}) + D_2 f''(x_n, \bar{y_n}) + D_3 f'''(x_n, \bar{y_n})$$

Therefore,

$$\begin{aligned} \Phi(x_n, y_; h) - \phi(x_n, \bar{y_n}) &= [f(x_n, y_n) - f(x_n, \bar{y_n})] + D_1[f'(x_n, y_n) - f'(x_n, y_n)] \\ &+ D_2[f''(x_n, y_n) - f''(x_n, y_n)] + D_3[f'''(x_n, y_n) - f'''(x_n, y_n)] \end{aligned} \tag{41}$$

Where

$$\begin{aligned} f(x_n, y_n) - f(x_n, \bar{y_n}) &= \sup_{(x_n, y_n) \in D} \frac{\partial f(x_n, y{*}_n)}{\partial y}(y_n - \bar{y_n}) = P(y_n - \bar{y_n}) \\ f'(x_n, y_n) - f'(x_n, \bar{y_n}) &= \sup_{(x_n, y_n) \in D} \frac{\partial f^1(x_n, y{*}_n)}{\partial y}(y_n, \bar{y_n}) = Q(y_n - \bar{y_n}) \\ f''(x_n, y_n) - f''(x_n, \bar{y_n}) &= \sup_{(x_n, y_n) \in D} \frac{\partial f^2(x_n, y{*}_n)}{\partial y}(y_n, \bar{y_n}) = R(y_n - \bar{y_n}) \\ f'''(x_n, y_n) - f'''(x_n, \bar{y_n}) &= \sup_{(x_n, y_n) \in D} \frac{\partial f^3(x_n, y{*}_n)}{\partial y}(y_n, \bar{y_n}) = S(y_n - \bar{y_n}) \end{aligned} \tag{42}$$

Thus, substituting (42) into (41) and taking modulus, yields

$$|\Phi(x_n, y_; h) - \phi(x_n, \bar{y_n})| = |(P + D_1 Q + D_2 R + D_3 S)(y_n - \bar{y_n})|$$

$$|\Phi(x_n, y_; h) - \phi(x_n, \bar{y_n})| \leq |(P + D_1 Q + D_2 R + D_3 S)||(y_n - \bar{y_n})|$$

Hence,

$$|\Phi(x_n, y_; h) - \phi(x_n, \bar{y_n})| \leq L|y_n - \bar{y_n}|, L = |(P + D_1 Q + D_2 R + D_3 S)| \tag{43}$$

This completes the proof. □

3.2. **Local truncation error (LTE) of the derived method.** To find the order of the scheme derived, the derived numerical scheme is subtracted from Taylor's series expansion for $y(x)$ in power of h which is illustrated as follows:

$$y(x_n+h) = y(x_n) + \frac{hy'(x_n)}{1!} + \frac{hy''(x_n)}{2!} + \frac{hy'''(x_n)}{3!} + \frac{hy^{iv}(x_n)}{4!} + O(h^5) \tag{44}$$

By (11), (12), (13), (14), we obtain

$$F'(x_n) = y'(x_n) = f_n \tag{45}$$

$$F''(x_n) = y''(x_n) = f'_n \tag{46}$$

$$F'''(x_n) = y'''(x_n) = f''_n \tag{47}$$

$$F^{iv}(x_n) = y^{iv}(x_n) = f'''_n \tag{48}$$

Using (45) - (48) and (37), then we obtain

$$\begin{aligned} &LTE = y(x_n+h) - y_{n+1} \\ &= \left[y(x_n) + \frac{hy'(x_n)}{1!} + \frac{hy''(x_n)}{2!} + \frac{hy'''(x_n)}{3!} + \frac{hy^{iv}(x_n)}{4!} + O(h^5)\right] \\ &- \left[y_n + hf_n + h^2\left(\frac{f'_n}{3} - \frac{f'''_n}{8}\right) + h^3\left(\frac{f''_n}{6} + \frac{f'''_n}{12}\right) + \left(\frac{h}{8} + \frac{1}{16}f'''_n(e^{-2h}-1)\right)\right] \end{aligned} \tag{49}$$

Replacing ϵ^{-2h} by Maclaurin's series and implifying further, we obtain our local truncation error with the leading term containing h^5.

$$LTE = O(h^5)$$

. This shows that our newly derived scheme is of order four.

3.3. **Consistency property of the method.** A numerical method is said to be consistent if $\phi(x_n, y_n; h) = f_n$. From the derived scheme, we have

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}f'_n - \frac{h^2}{8}f'''_n + \frac{h^3}{6}f''_n + \frac{h}{12}f'''_n + \left[\frac{h}{8} + \frac{1}{16}(e^{-2h}-1)\right]f'''_n \tag{50}$$

. Thus,

$$\frac{y_n+1-y_n}{h} = f_n + h\left(\frac{f'_n}{2} - \frac{f'''_n}{8}\right) + h^2\left(f''_n + \frac{f'''_n}{12}\right) + \frac{4h^2}{32}f'''_n + \ldots. \tag{51}$$

From the RHS, the increment function is obtained as:

$$\phi(x_n, y_n; h) = f_n + h\left(\frac{f''_n}{2} - \frac{f'''_n}{8}\right) + h^2\left(f''_n + \frac{f'''_n}{12}\right) + \frac{4h^2}{32}f'''_n \tag{52}$$

Therefore, as $h \to 0$, (52) becomes

$$\phi(x_n, y_n; 0) = f_n \tag{53}$$

Equation (53) confirms the consistency of the derived method.

3.4. **Stability of the derived Method.** Consider the test problem

$$y' = -\lambda y \qquad y(0) = 1 \tag{54}$$

In which its theoretical solution is given as

$$y(x) = e^{-\lambda x}, \qquad \lambda > 0$$

where λ is in general a complex constant. The exact solution of (54) at point $x = x_{n+1}$ is

$$y(x_{n+1}) = e^{-\lambda(x_n+h)} \tag{55}$$

From the derived method (37)

$$\begin{aligned} y_{n+1} = y_n + h[-\lambda e^{-\lambda x_n}] + h^2\left[\frac{\lambda^2 e^{-\lambda x_n}}{2} - \frac{\lambda^4 e^{-\lambda x_n}}{8}\right] \\ + h^3\left[\frac{-\lambda e^{\lambda x_n}}{6} + \frac{\lambda^4 e^{-\lambda x_n}}{12}\right] + \left[\frac{h}{8} - \frac{(e^{-2h}) - 1}{16}\right]\lambda^4 e^{-\lambda x_n} \end{aligned} \tag{56}$$

Let

$$e^{-\lambda x_n} = y_n \tag{57}$$

Then (56) becomes

$$y_{n+1} = y_n\left[1 - \lambda h + \frac{h^2\lambda^2}{2} + \frac{h^2\lambda^4}{8} - \frac{h^3\lambda^3}{6} + \frac{h^3\lambda^4}{12} + \left(\frac{h}{8} - \frac{e^{-2h} - 1}{16}\lambda^4\right)\right] \tag{58}$$

Setting

$$B = 1 - \lambda h + \frac{h^2\lambda^2}{2} + \frac{h^2\lambda^4}{8} - \frac{h^3\lambda^3}{6} + \frac{h^3\lambda^4}{12} + \left(\frac{h}{8} - \frac{e^{(-2h)} - 1}{16}\lambda^4\right) \tag{59}$$

Therefore, (58) yields

$$y_{n+1} = By_n \tag{60}$$

Comparing (55) and (59), show the factor D is merely an approximation for the factor $e^{-\lambda h}$ in the exact solution. The factor D error growth can be control by $||D|| \leq 1$, in order for the error not to be magnify. Therefore, the stability of the derived method requires that

$$|B| = \left|1 - \lambda h + \frac{h^2\lambda^2}{2} + \frac{h^2\lambda^4}{8} - \frac{h^3\lambda^3}{6} + \frac{h^3\lambda^4}{12} + \left(\frac{h}{8} - \frac{e^{(-2h)} - 1}{16}\lambda^4\right)\right| \leq 1 \tag{61}$$

Also, let $\zeta = h\lambda$, the stability function of the derived method is given by

$$\begin{aligned} B &= 1 - \zeta + \frac{\zeta^2}{2} + \frac{\zeta^2\lambda^2}{8} - \frac{\zeta^3}{6} + \frac{\zeta^3\lambda}{12} + \left(\frac{h}{8} - \frac{e^{-2h} - 1}{16}\right)(\lambda^4) \\ &= 1 - \zeta + \frac{\zeta^2}{2} - \frac{\zeta^3}{6} + \frac{\zeta^4}{24} - \ldots \end{aligned} \tag{62}$$

This shows that the derived method is absolutely stable. Alternatively, the stability property of the derived method is given by the following result [18].

Theorem 2. *Let $y_n = y(x_n)$ and $M_n = M(x_n)$ be two different numerical solutions of differential equation $y\prime = f(x, y)$ with initial conditions as $y(x_0) = \alpha$ and $M(x_0) = \bar{\alpha}$, respectively. If the numerical estimates are generated by interpolation scheme derived, we have*

$$y_{n+1} = y_n + h\phi(x_n, y_n; h)$$

$$M_{n+1} = M_n + h\phi(x_n, m_n; h)$$

The condition that $|y_{n+1} - m_{n+1}| \leq W|\alpha - \bar{\alpha}|$ is the necessary and sufficient condition that our derived scheme is stable and convergent

Proof. From the derived method, we have that

$$y_{n+1} = y_n + f(x_n, y_n) + D_1 f'(x_n, y_n) + D_2 f''(x_n, y_n) + D_3 f'''(x_n, y_n)$$

Also, define

$$m_{n+1} = m_n + f(x_n, m_n) + D_1 f'(x_n, m_n) + D_2 f''(x_n, m_n) + D_3 f'''(x_n, m_n)$$

Therefore,

$$\begin{aligned} y_{n+1} - m_{n+1} = y_n - m_n + [f(x_n, y_n) - f(x_n, m_n)] + D_1 \left[f'(x_n, y_n) - f'(x_n, m_n)\right] \\ + D_2 \left[f''(x_n, y_n) - f''(x_n, m_n)\right] + D_3 \left[f'''(x_n, y_n) - f'''(x_n, m_n)\right] \end{aligned} \tag{63}$$

Apply the mean value theorem with an assumption that y_n^* is a point in the interval whose end points are y_n and m_n. we have

$$y_{n+1} - m_{n+1} = (1 + P + D_1 Q + D_2 R + D_3 S)(y_n - m_n) \tag{64}$$

Taking absolute value of (64), yields

$$|y_{n+1} - m_n + 1| \leq |1 + P + D_1 Q + D_2 R + D_3 S||y_n - m_n| \tag{65}$$

Let

$$|1 + P + D_1 Q + D_2 R + D_3 S| = W, y_n(x_0) = \alpha, m_n(x_0) = \bar{\alpha} \tag{66}$$

Hence,

$$|y_{n+1} - m_{n+1}| \leqslant W|\alpha - \bar{\alpha}| \tag{67}$$

This establishes the proof. □

4. Numerical Examples

Example 1. Consider an initial value problem:

$$\frac{dy}{dx} = y, \quad 0 \leq x \leq 1, \tag{68}$$

whose exact solution is

$$y(x) = e^x. \tag{69}$$

Table 1. Table of results for comparison

h	XN	NNM	PJS	ES
0.1	1.0000	2.718279744135	2.718277728147	2.718281828459
0.05	1.0000	2.718281692656	2.718281559098	2.718281828459
0.025	1.0000	2.718281819793	2.718281811199	2.718281828459
0.0125	1.0000	2.718281827912	2.718281827367	2.718281828459

Table 2. Table of results for comparison

h	XN	AE(NNM)	AE(PJS)
0.1	1.0000	0.000002084324	0.000004100312
0.05	1.0000	0.000000135803	0.000000269361
0.025	1.0000	0.000000008666	0.000000017260
0.0125	1.0000	0.000000000547	0.000000001092

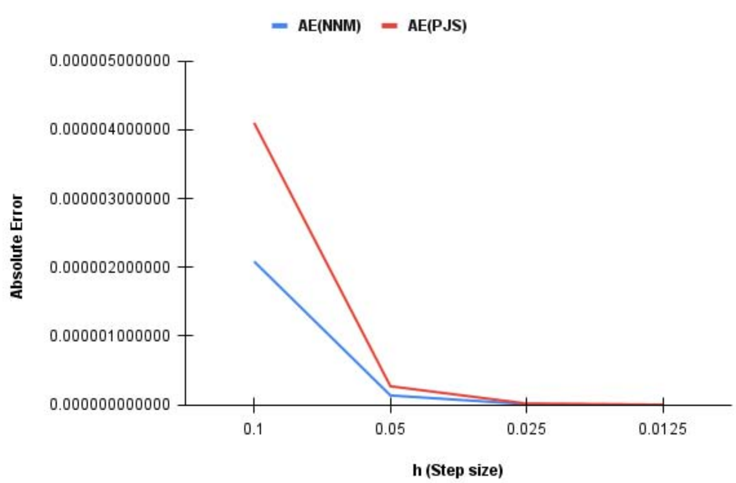

Figure 1. The comparative absolute errors analyses using Table 2

Example 2. Suppose an investment fund is growing continuously at a rate of 5% per year. The initial investment made was $GH¢100$. What is the value of the investment after 7 years?

Using the continuous growth model, the value of the investment after 7 years can be modeled by the first-order differential equation:

$$\frac{dV(t)}{dt} = rV(t) \ , \quad V(0) = 100 \ , \tag{70}$$

where $V(t)$ represents the value of the investment at time t and $r = 0.05$ is the rate. Solving this differential equation using separation of variables, the exact solution is

$$V(t) = 100 \, e^{0.05t} \ . \tag{71}$$

Table 3. Table of results for comparison

h	XN	NNM	PJS	ES
0.1	7.0000	141.906754859068	141.906754854005	141.906754859326
0.05	7.0000	141.906754859310	141.906754858990	141.906754859326
0.025	7.0000	141.906754859324	141.906754859305	141.906754859326
0.0125	7.0000	141.906754859326	141.906754859324	141.906754859326

Table 4. Table of results for comparison

h	XN	AE(NNM)	AE(PJS)
0.1	7.0000	0.000000000258	0.000000005321
0.05	7.0000	0.000000000016	0.000000000336
0.025	7.0000	0.000000000001	0.000000000021
0.0125	7.0000	0.000000000000	0.000000000002

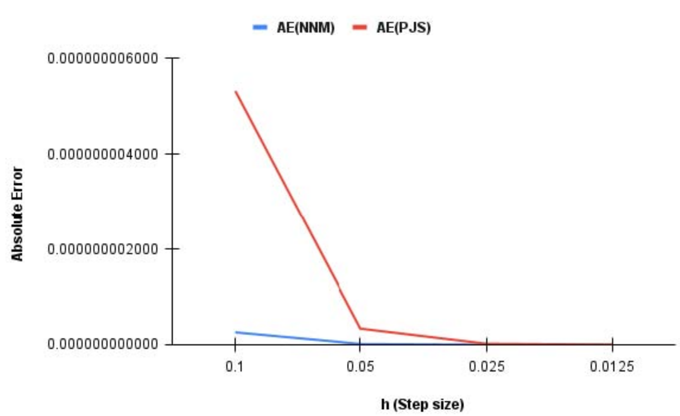

Figure 2. The comparative absolute errors analyses using Table 4

Example 3. Suppose there is a group of bacteria reproducing at a rate of 0.8 per hour per individual. If there are no limitations to the growth of the colony, how many bacteria will there be after one hour?

The growth of the colony can be represented by a first-order differential equation of the form

$$\frac{dN(t)}{dt} = rN(t) \;, \quad N(0) = 1000 \;, \tag{72}$$

where $N(t)$ represents the number of bacteria at time t, r is the rate of growth per individual per hour. The exact solution is obtained as

$$N(t) = 1000\, e^{0.8t} \;. \tag{73}$$

Table 5. Table of results for comparison

h	XN	NNM	PJS	ES
0.1	1.0000	2225.540359928990	2225.539670205250	2225.540928492460
0.05	1.0000	2225.540891754530	2225.540846515410	2225.540928492460
0.025	1.0000	2225.540926157780	2225.540923261220	2225.540928492460
0.0125	1.0000	2225.540928345320	2225.540928162100	2225.540928492460

Table 6. Table of results for comparison

h	XN	AE(NNM)	AE(PJS)
0.1	1.0000	0.000568563474	0.001258287212
0.05	1.0000	0.000036737933	0.000081977051
0.025	1.0000	0.000002334681	0.000005231242
0.0125	1.0000	0.000000147141	0.000000330365

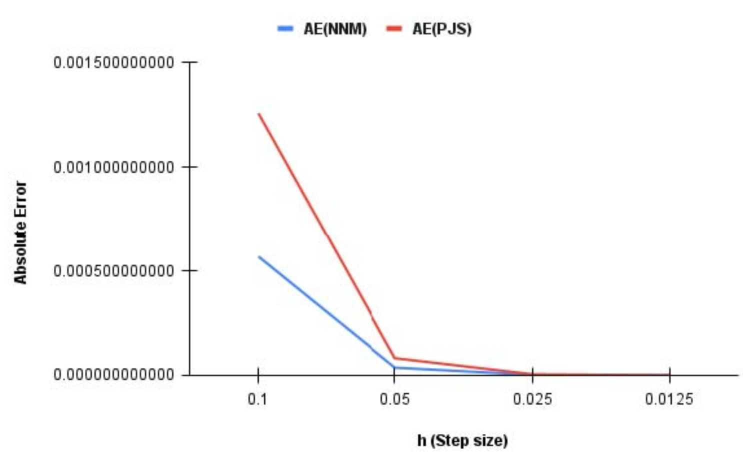

Figure 3. The comparative absolute errors analyses using Table 6

Example 4. Suppose we have a sample of radioactive material with an initial mass of 500 grams. The half-life of the material is 10 days. How much radioactive material will be left after 30 days have elapsed? Radioactive decay can be modeled by the exponential decay equation:

$$\frac{dN(t)}{dt} = -\lambda N(t)\ , \quad N(0) = 500\ , \tag{74}$$

where $N(t)$ is the amount of radioactive material remaining at time t and λ is the decay constant. The half-life is 10 days, so the decay constant is

$$\lambda = \frac{\ln(2)}{10} \approx 0.0693\ .$$

Solving this differential equation using separation of variables, the exact solution is

$$N(t) = 500\, e^{-0.0693t}\ . \tag{75}$$

Table 7. Table of results for comparison

h	XN	NNM	PJS	ES
0.1	30.0000	62.527602450870	62.527602415162	62.527602448357
0.05	30.0000	62.527602448514	62.527602446271	62.527602448357
0.025	30.0000	62.527602448367	62.527602448227	62.527602448357
0.0125	30.0000	62.527602448358	62.527602448348	62.527602448357

Table 8. Table of results for comparison

h	XN	AE(NNM)	AE(PJS)
0.1	30.0000	0.000000002513	0.000000033195
0.05	30.0000	0.000000000157	0.000000002086
0.025	30.0000	0.000000000010	0.000000000130
0.0125	30.0000	0.000000000001	0.000000000009

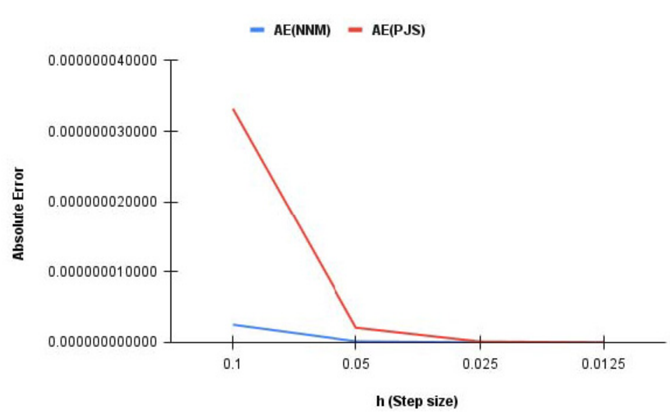

Figure 4. The comparative absolute errors analyses using Table 8

5. Discussion of Results and Concluding Remarks

The results presented in Tables 1, 3, 5 and 7 and Tables 2, 4, 6 and 8 indicate that the proposed numerical method is more accurate than the PJS method [28] in terms of accuracy and convergence, respectively. Moreover, an analysis of the computational progress with different step sizes, as depicted in Figures 1, 2, 3, and 4, reveals that the proposed method incurs smaller errors compared to the PJS method. Also, this finding is further supported by Figures 1 to 4, which demonstrates that the proposed method closely aligns with the exact solution as the step size decreases. It can be concluded that the proposed method indeed exhibits fourth-order accuracy. Therefore, the proposed numerical method represents an improvement over the PJS method [28].

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