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BIPOLAR INTUITIONISTIC FUZZY IMPLICATIVE IDEALS OF BCK-ALGEBRA

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Abstract. This research is focused on combining bipolar intuitionistic fuzzy set theories within the domain of BCK-algebras, leading to the establishment of a novel framework for bipolar intuitionistic fuzzy ideals of BCK-algebras. This paper introduces the concepts of a bipolar intuitionistic fuzzy implicative ideal in BCK algebra, with illustrative examples, followed by an exploration of various related properties. The conditions under which a bipolar intuitionistic fuzzy set is a bipolar intuitionistic fuzzy implicative ideal are provided. Furthermore, we introduce the notion of level subsets and disclose a theorem stating that a bipolar intuitionistic fuzzy set is a bipolar intuitionistic fuzzy implicative ideal if and only if its level subsets. The extension property of a bipolar intuitionistic fuzzy implicative ideal is presented.

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1. Introduction

In various real-world situations, information processing and decision-making often encounter situations where results or data are not well defined. This inherent lack of precision and accuracy is called uncertainty. Uncertainty arises from various factors such as incomplete information, measurement errors, inaccuracy in data collection and the inherent variability of complex systems. Facing uncertainty and dealing with it successfully is essential for making informed decisions, especially in fields like engineering, business, artificial intelligence, etc. One of the most effective approaches to solving the uncertainty problem is to use fuzzy set theory. Proposed by Lotfi A.Zadeh [28] in the mid-20th century, fuzzy set theory provides a mathematical framework to deal with uncertainty and imprecision in a

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systematic manner. The exploration of BCK/BCI-algebras started by Imai and Iseki [44], [8] in 1966, as demonstrated through the concepts of set-theoretic difference and the propositional calculi. Many researchers, including Jun ([45], [24], [21], [36]) Liu [33], Lee [31], have extensively examined the fuzzy structures of BCK/BCI-algebras, and others ([12], [11], [13], [14], [10], [15], [34]), making significant contribution to diverse branches of algebra from different points of view.

Bipolar fuzzy sets (BFS) ([47], [48]), an extension of fuzzy sets, were introduced to solve situations where both positive and negative membership degrees are relevant, important, so they cover positive and negative roles of membership. In a traditional fuzzy set,an element either fully related (membership degree = 1) or partially related (membership degree = (0,1)). On the other hand, a bipolar fuzzy set allows you to assign degrees to members in the range of [-1,1], indicating the degree to which an element positively or negatively belongs to the set. This extension provides a more comprehensive representation of uncertain and variable information, which is especially useful in real-world situations where both positive and negative aspects are important. The utilization of bipolar-valued fuzzification has been applied to investigate various concepts within BCK/BCI-algebras, such as subalgebras and ideals of BCK/BCI-algebras [29], a-ideals of BCI-algebras [25], among others, as discussed in ([19], [17], [18], [16]). A recent investigation conducted in [4] delves into bipolar-valued fuzzy BCI-implicative ideals of BCI-algebras. G. Mohiuddin et.al [38] introduces the notion of bipolar-valued fuzzy implicative ideals in BCK-algebras. Additionally, in [39] explores new types of bipolar-valued fuzzy ideals (positive implicative, closed) in BCK-algebras. Furthermore, numerous scholars have also contributed to this field in different branches of algebra and from various angles ([5], [1], [2], [3], [20], [22], [23], [26], [27], [35], [37], [40], [41], [42], [43], [46]).

Following the introduction of the concept of fuzzy sets, many studies have been undertaken to explore extensions of this idea. In 1986, K. T.Atanassov [6] introduced the concept of intuitionistic fuzzy sets, which works as an advancement of the fuzzy set theory. Ezhilmaran and Sankar [7] introduced the concept of a bipolar-valued intuitionistic fuzzy set. This type of fuzzy set provides not only the positive and negative membership levels but also includes the positive and negative non-membership levels for elements within a specified set. The presence of these literatures provides plenty of inspiration, and as far as our knowledge goes, there are no existing sources of literature concerning the bipolar intuitionistic fuzzification of ideals in BCK/BCI-algebras. This scarcity of research motivated us to embark on a theoretical investigation of the aforementioned subject. We have provided an illustration of the process through a framework diagram shown in Figure 1. Our intention is that this visual representation will enhance your understanding of the task.

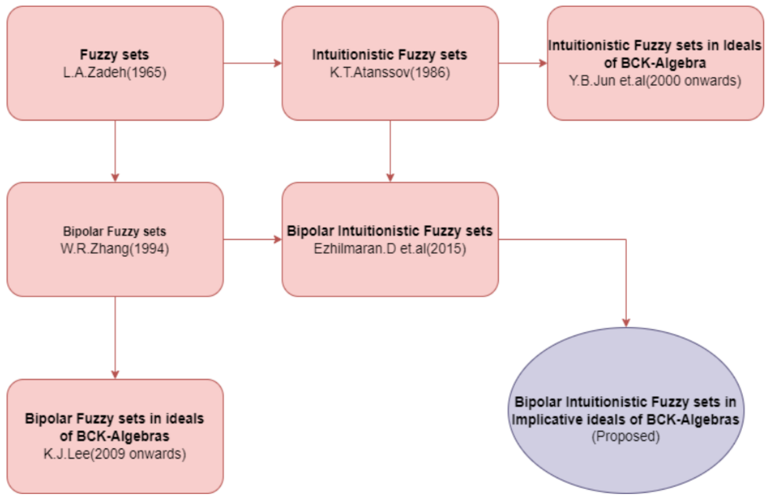

Figure.1

This paper introduces the concepts of a bipolar intuitionistic fuzzy implicative ideal in BCK-algebra, with illustrative examples. The cases where a BPIFS is a BPIFII are carefully studied.

2. Preliminaries

Definition 2.1. [9] *A BCK-algebra is an algebra of type* (2,0) *if it follows the following axioms for all* $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$

$$((\mathfrak{k}_1 * \mathfrak{k}_2) * (\mathfrak{k}_1 * \mathfrak{k}_3)) * (\mathfrak{k}_3 * \mathfrak{k}_2) = 0 \tag{1}$$

$$(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * \mathfrak{k}_2 = 0 \tag{2}$$

$$\mathfrak{k}_1 * \mathfrak{k}_1 = 0 \tag{3}$$

$$0 * \mathfrak{k}_1 = 0 \tag{4}$$

$$\mathfrak{k}_1 * \mathfrak{k}_2 = 0 \; and \; \mathfrak{k}_2 * \mathfrak{k}_1 = 0 \Rightarrow \mathfrak{k}_1 = \mathfrak{k}_2 \tag{5}$$

We can define a binary operation $\leq$ on $\mathfrak{K}$ by assuming $\mathfrak{k}_1 \leq \mathfrak{k}_2$ if and only if $\mathfrak{k}_1 * \mathfrak{k}_2 = 0$. In a BCK-algebra the following properties are holds.

$$\mathfrak{k}_1 * 0 = \mathfrak{k}_1 \tag{6}$$

$$\mathfrak{k}_1 * \mathfrak{k}_2 \leq \mathfrak{k}_1 \tag{7}$$

$$(\mathfrak{k}_1 * \mathfrak{k}_2) * \mathfrak{k}_3 = (\mathfrak{k}_1 * \mathfrak{k}_3) * \mathfrak{k}_2 \tag{8}$$

$$(\mathfrak{k}_1 * \mathfrak{k}_3) * (\mathfrak{k}_2 * \mathfrak{k}_3) \leq \mathfrak{k}_1 * \mathfrak{k}_2 \tag{9}$$

$$\mathfrak{k}_1 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) = \mathfrak{k}_1 * \mathfrak{k}_2 \tag{10}$$

$$\mathfrak{k}_1 * \mathfrak{k}_2 \leq \mathfrak{k}_3 \Rightarrow \mathfrak{k}_1 * \mathfrak{k}_3 \leq \mathfrak{k}_2 \tag{11}$$

$$\Bbbk_1 * \Bbbk_2 \leq \Bbbk_3 \Rightarrow \Bbbk_1 * \Bbbk_3 \leq \Bbbk_2 \tag{12}$$

for all $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$.

Proposition 2.1. [9] *In a BCK-algebra $\mathfrak{K}$ the following holds for all $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$*

(i) $((\Bbbk_1 * \Bbbk_3) * \Bbbk_3) * (\Bbbk_2 * \Bbbk_3) \leq (\Bbbk_1 * \Bbbk_2) * \Bbbk_3$.

(ii) $(\Bbbk_1 * \Bbbk_3) * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_3)) = (\Bbbk_1 * \Bbbk_3) * \Bbbk_3$

(iii) $(\Bbbk_1 * (\Bbbk_2 * (\Bbbk_2 * \Bbbk_1))) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)))) \leq \Bbbk_1 * \Bbbk_2$

Definition 2.2. [9] *A BCK-algebra $\mathfrak{K}$ is considered to be implicative if the following condition holds*

$$\Bbbk_1 * (\Bbbk_2 * \Bbbk_1) = \Bbbk_1 \tag{13}$$

for all $\Bbbk_1, \Bbbk_2 \in \mathfrak{K}$.

Definition 2.3. [9] *A subset $\mathfrak{I}(\neq \phi)$ of a BCK-algebra $\mathfrak{K}$ is termed as a subalgebra of $\mathfrak{K}$ if the following condition holds $\Bbbk_1 * \Bbbk_2 \in \mathfrak{I}$ for all $\Bbbk_1, \Bbbk_2 \in \mathfrak{I}$.*

Definition 2.4. [9] *A subset $\mathfrak{I}(\neq \phi)$ of a BCK-algebra $\mathfrak{K}$ is termed as an ideal of $\mathfrak{K}$ if the following condition holds*

(I 1) $0 \in \mathfrak{I}$

(I 2) $\Bbbk_1 * \Bbbk_2 \in \mathfrak{I}, \Bbbk_2 \in \mathfrak{I} \Rightarrow \Bbbk_1 \in \mathfrak{I}$ *for all* $\Bbbk_1 \in \mathfrak{K}$.

Definition 2.5. [9] *A subset $\mathfrak{I}(\neq \phi)$ of a BCK-algebra $\mathfrak{K}$ is termed as an implicative ideal of $\mathfrak{K}$ if the condition (I 1) holds and*

(II 1) $(\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3 \in \mathfrak{I}, \Bbbk_3 \in \mathfrak{I} \Rightarrow \Bbbk_1, \Bbbk_2 \in \mathfrak{I}$.

Proposition 2.2. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be ideals of $\mathfrak{K}$ with $\mathfrak{B} \subseteq \mathfrak{C}$. If $\mathfrak{B}$ is an implicative ideal, then so is $\mathfrak{C}$.*

Definition 2.6. [28] *Let $\mathfrak{K}$ be non-empty set. A fuzzy set in $\mathfrak{K}$ is a mapping $\alpha : \mathfrak{K} \to [0, 1]$.*

Definition 2.7. [28] *The complement of fuzzy set α denoted by $\overline{\alpha}$ is also a fuzzy set defined as $\overline{\alpha} = 1 - \alpha$ for all $\Bbbk_1 \in \mathfrak{K}$. Also $\overline{(\overline{\alpha})} = \alpha$.*

Definition 2.8. [32] *A bipolar fuzzy set $\mathfrak{B}$ of $\mathfrak{K}$ is defined as*

$$\mathfrak{B} = \{(\Bbbk_1, {\alpha_{\mathfrak{B}}}^p(\Bbbk_1), {\alpha_{\mathfrak{B}}}^n(\Bbbk_1)) \mid \Bbbk_1 \in \mathfrak{K}\}. \tag{14}$$

Where ${\alpha_{\mathfrak{B}}}^p : \mathfrak{K} \to [0, 1]$ and ${\alpha_{\mathfrak{B}}}^n : \mathfrak{K} \to [-1, 0]$ are mappings.The positive membership degree ${\alpha_{\mathfrak{B}}}^p$ denotes the level of satisfaction that the element $\mathfrak{K}$ to the property associated with the bipolar fuzzy set $\mathfrak{B}$, while the negative membership degree ${\alpha_{\mathfrak{B}}}^n$ denotes the level of satisfaction that the element $\mathfrak{K}$ to some implicit counter property of $\mathfrak{B}$.

We shall use the symbol $\mathfrak{B} = (\mathfrak{k}_1, \alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n)$ for a bipolar fuzzy set (see Eq.(14)).

Definition 2.9. [30] *A bipolar fuzzy set $\mathfrak{B} = (\mathfrak{k}_1, \alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n)$ in $\mathfrak{K}$ is said to be a bipolar fuzzy subalgebra if the following conditions are satisfied*

(*BPFSA-1*) $\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * \mathfrak{k}_2) \geq min\{\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_2)\}$,

(*BPFSA-2*) $\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * \mathfrak{k}_2) \leq max\{\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_2)\}$ *for all* $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$.

Definition 2.10. [30] *A bipolar fuzzy set $\mathfrak{B} = (\mathfrak{k}_1, \alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n)$ in $\mathfrak{K}$ is said to be a bipolar fuzzy ideal if the following conditions are satisfied*

(*BPFI-1*) $\alpha_{\mathfrak{B}}{}^p(0) \geq \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$, $\alpha_{\mathfrak{B}}{}^n(0) \leq \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$

(*BPFI-2*) $\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * \mathfrak{k}_2), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_2)\}$

(*BPFI-3*) $\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * \mathfrak{k}_2), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_2)\}$ *for all* $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$.

Definition 2.11. [39] *A bipolar fuzzy set $\mathfrak{B} = (\mathfrak{k}_1, \alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n)$ in $\mathfrak{K}$ is a bipolar fuzzy implicative ideal of $\mathfrak{K}$ if it satisfies*

(*BPFII-1*) $\alpha_{\mathfrak{B}}{}^p(0) \geq \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$, $\alpha_{\mathfrak{B}}{}^n(0) \leq \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$

(*BPFII-2*) $\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^p((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\}$

(*BPFII-3*) $\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^n((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\}$ *for all* $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$.

Definition 2.12. [6] *An intuitionistic fuzzy set $\mathfrak{B}$ in a non-empty set $\mathfrak{K}$ is an object having the form*

$$\mathfrak{B} = \{(\mathfrak{k}_1, \alpha(\mathfrak{k}_1), \beta(\mathfrak{k}_1)) \mid \mathfrak{k}_1 \in \mathfrak{K}\} \tag{15}$$

where $\alpha(\mathfrak{k}_1), \beta(\mathfrak{k}_1)$ *are degree of belongingness and degree of non - belongingness of* $\mathfrak{k}_1 \in \mathfrak{K}$ *respectively and* $0 \leq \alpha(\mathfrak{k}_1) + \beta(\mathfrak{k}_1) \leq 1$ *for all* $\mathfrak{k}_1 \in \mathfrak{K}$.

We shall use the symbol $\mathfrak{B} = (\mathfrak{k}_1, \alpha, \beta)$ for an intuitionistic fuzzy set (see Eq.(15)).

Definition 2.13. [7] *A bipolar intuitionistic fuzzy set $\mathfrak{B}$ in a non-empty set $\mathfrak{K}$ is an object having the form*

$$\mathfrak{B} = \{(\mathfrak{k}_1, \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1), \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1), \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1)) \mid \mathfrak{k}_1 \in \mathfrak{K}\} \tag{16}$$

where $\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) : \mathfrak{K} \to [0,1]$, $\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) : \mathfrak{K} \to [-1,0]$, $\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) : \mathfrak{K} \to [0,1]$ *and* $\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) : \mathfrak{K} \to [-1,0]$ *are such that* $0 \leq \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) + \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \leq 1$ *and* $-1 \leq \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) + \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq 0$. *Here,* $\alpha_{\mathfrak{B}}{}^p$ *represents the positive membership level, indicating the extent to which an element* $\mathfrak{K}$ *satisfies a property for a BPIFS* $\mathfrak{B}$. *On the other hand,* $\alpha_{\mathfrak{B}}{}^n$ *represents the negative membership level, indicating how well an element* $\mathfrak{K}$ *satisfies the implicit counter property associated with the bipolar intuitionistic fuzzy set. The terms* $\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$*and* $\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$ *refer to the positive non-membership level and negative non-membership level respectively. We calculate* $\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$ *and* $\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$ *as* $\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) = 1 - \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$ *and* $\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) = -1 - \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$.

Definition 2.14. *A BPIFS* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *in* $\mathfrak{K}$ *is a bipolar intuitionistic fuzzy ideal of* $\mathfrak{K}$ *if it satisfies*

(BPIFI-1) $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1)$ *and* $\beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1)$

(BPIFI-2) $\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1 * \Bbbk_2), \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_2)\}$

(BPIFI-3) $\alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1 * \Bbbk_2), \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_2)\}$

(BPIFI-4) $\beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}(\Bbbk_1 * \Bbbk_2), \beta_{\mathfrak{B}}{}^{p}(\Bbbk_2)\}$

(BPIFI-5) $\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1 * \Bbbk_2), \beta_{\mathfrak{B}}{}^{n}(\Bbbk_2)\}$ *for all* $\Bbbk_1, \Bbbk_2 \in \mathfrak{K}$.

Definition 2.15. *For any BPIFSs* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *and* $\mathfrak{C} = (\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, \delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n})$. *Then* $\mathfrak{B} \subseteq \mathfrak{C} \Rightarrow \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq \gamma_{\mathfrak{C}}{}^{p}(\Bbbk_1), \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq \gamma_{\mathfrak{C}}{}^{n}(\Bbbk_1), \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq \delta_{\mathfrak{C}}{}^{p}(\Bbbk_1)$ *and* $\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq \delta_{\mathfrak{C}}{}^{n}(\Bbbk_1)$ *for all* $\Bbbk_1 \in \mathfrak{K}$.

Theorem 2.1. *A BPIFS* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *in* $\mathfrak{K}$ *is a bipolar intuitionistic fuzzy ideal of* $\mathfrak{K}$ *if and only if it satisfies the following condition*

$\Bbbk_1 * \Bbbk_2 \leq \Bbbk_3$ *for all* $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$ *then*

$$\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_2), \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_3)\} \\ \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}(\Bbbk_2), \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_3)\} \\ \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}(\Bbbk_2), \beta_{\mathfrak{B}}{}^{p}(\Bbbk_3)\} \\ \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}(\Bbbk_2), \beta_{\mathfrak{B}}{}^{n}(\Bbbk_3)\} \end{pmatrix}$$

3. Bipolar Intuitionistic Fuzzy Implicative Ideal

Definition 3.1. *A BPIFS* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *in* $\mathfrak{K}$ *is a BPIFII of* $\mathfrak{K}$ *if it satisfies*

(BPIFII-1) $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1)$ *and* $\beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1)$

(BPIFII-2) $\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_3)\}$

(BPIFII-3) $\alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_3)\}$

(BPIFII-4) $\beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \beta_{\mathfrak{B}}{}^{p}(\Bbbk_3)\}$

(BPIFII-5) $\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \beta_{\mathfrak{B}}{}^{n}(\Bbbk_3)\}$ *for all* $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$.

Example 3.1. *Let* $\mathfrak{K} = \{0, 1, 2, 3, 4\}$ *be a set in which the binary operation "$*$" is defined as given below*

$$0 * \Bbbk_1 = 0 \; \forall \; \Bbbk_1 \in \mathfrak{K}$$

$$1 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 \in \{1, 3, 4\} \\ 1 & if \; \Bbbk_1 \in \{0, 2\} \end{cases}$$

$$2 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 \in \{2, 3, 4\} \\ 2 & if \; \Bbbk_1 \in \{0, 1\} \end{cases}$$

$$3 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 \in \{3, 4\} \\ 3 & if \; \Bbbk_1 \in \{0, 1, 2\} \end{cases}$$

$$4 * \mathfrak{k}_1 = \begin{cases} 0 & if\ \mathfrak{k}_1 = 4 \\ 1 & if\ \mathfrak{k}_1 = 3 \\ 3 & if\ \mathfrak{k}_1 = 1 \\ 4\ if\ \mathfrak{k}_1 \in \{0, 2\} \end{cases}$$

Then $\mathfrak{K}$ is a BCK-algebra.

Let us define a BPIFS $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ in $\mathfrak{K}$ as shown in Table 1.

Table 1. Bipolar Intuitionistic fuzzy set

$\mathfrak{K}$	$\alpha_{\mathfrak{B}}{}^p$	$\alpha_{\mathfrak{B}}{}^n$	$\beta_{\mathfrak{B}}{}^p$	$\beta_{\mathfrak{B}}{}^n$
0	0.69	-0.87	0.31	-0.13
1	0.69	-0.87	0.31	-0.13
2	0.69	-0.87	0.31	-0.13
3	0.32	-0.21	0.68	-0.79
4	0.32	-0.21	0.68	-0.79

By routine calculations we can show that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a BPIFII of $\mathfrak{K}$.

Theorem 3.1. *Every bipolar intuitionistic fuzzy implicative ideal of $\mathfrak{K}$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$.*

Proof. Let $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a BPIFII of $\mathfrak{K}$ and put $\mathfrak{k}_2 = 0$ in BPIFII-2,3,4,5 then

$$\begin{aligned}
\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) &\geq min\{\alpha_{\mathfrak{B}}{}^p((\mathfrak{k}_1 * (0 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\} \\
&\geq min\{\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\}, \\
\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) &\leq max\{\alpha_{\mathfrak{B}}{}^n((\mathfrak{k}_1 * (0 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\} \\
&\leq max\{\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\}, \\
\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) &\leq max\{\beta_{\mathfrak{B}}{}^p((\mathfrak{k}_1 * (0 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\} \\
&\leq max\{\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\}, \\
\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) &\geq min\{\beta_{\mathfrak{B}}{}^n((\mathfrak{k}_1 * (0 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\} \\
&\geq min\{\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\}
\end{aligned}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. This shows that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a bipolar intuitionistic fuzzy implicative of $\mathfrak{K}$. □

The following example shows that the converse of the Theorem 3.1 is not true in general.

Example 3.2. *Let* $\mathfrak{K} = \{0, 1, 2, 3\}$ *be a set in which the binary operation "* $*$ *" is defined as given below*

$$0 * \Bbbk_1 = 0 \; \forall \; \Bbbk_1 \in \mathfrak{K}$$

$$1 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 \in \{1, 2\} \\ 1 & if \; \Bbbk_1 \in \{0, 3\} \end{cases}$$

$$2 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 = 2 \\ 1 & if \; \Bbbk_1 = 1 \\ 2 & if \; \Bbbk_1 \in \{0, 3\} \end{cases}$$

$$3 * \Bbbk_1 = \begin{cases} 0 & if \; \Bbbk_1 = 3 \\ 3 & if \; \Bbbk_1 \in \{0, 1, 2\} \end{cases}$$

Then $\mathfrak{K}$ *is a BCK-algebra. Let* $\mathfrak{B} = ({\alpha_{\mathfrak{B}}}^p, {\alpha_{\mathfrak{B}}}^n, {\beta_{\mathfrak{B}}}^p, {\beta_{\mathfrak{B}}}^n)$ *be a BPIFS in* $\mathfrak{K}$ *defined as shown in Table 2.*

It is easy to check that $\mathfrak{B} = ({\alpha_{\mathfrak{B}}}^p, {\alpha_{\mathfrak{B}}}^n, {\beta_{\mathfrak{B}}}^p, {\beta_{\mathfrak{B}}}^n)$ *is a bipolar intuitionistic fuzzy ideal of* $\mathfrak{K}$ *but not a BPIFII*

Table 2. Bipolar Intuitionistic fuzzy set

$\mathfrak{K}$	${\alpha_{\mathfrak{B}}}^p$	${\alpha_{\mathfrak{B}}}^n$	${\beta_{\mathfrak{B}}}^p$	${\beta_{\mathfrak{B}}}^n$
0	0.83	-0.69	0.17	-0.31
1	0.41	-0.25	0.59	-0.75
2	0.83	-0.69	0.17	-0.31
3	0.41	-0.25	0.59	-0.75

of $\mathfrak{K}$ *because*

$$\begin{aligned} {\alpha_{\mathfrak{B}}}^p(1) = 0.41 < 0.83 &= {\alpha_{\mathfrak{B}}}^p(2) \\ &= min\{{\alpha_{\mathfrak{B}}}^p((1 * (3 * 1)) * 2), {\alpha_{\mathfrak{B}}}^p(2)\} \\ {\alpha_{\mathfrak{B}}}^n(1) = -0.25 > -0.69 &= {\alpha_{\mathfrak{B}}}^n(2) \\ &= max\{{\alpha_{\mathfrak{B}}}^n((1 * (3 * 1)) * 2), {\alpha_{\mathfrak{B}}}^n(2)\}, \\ {\beta_{\mathfrak{B}}}^p(1) = 0.59 > 0.17 &= {\beta_{\mathfrak{B}}}^p(2) \\ &= max\{{\beta_{\mathfrak{B}}}^p((1 * (3 * 1)) * 2), {\beta_{\mathfrak{B}}}^p(2)\}, \\ {\beta_{\mathfrak{B}}}^n(1) = -0.75 < -0.31 &= {\beta_{\mathfrak{B}}}^n(2) \\ &= min\{{\beta_{\mathfrak{B}}}^n((1 * (3 * 1)) * 2), {\beta_{\mathfrak{B}}}^n(2)\}. \end{aligned}$$

Theorem 3.2. *If* $\mathfrak{K}$ *is an implicative BCK-algebra, then every bipolar intuitionistic fuzzy ideal of* $\mathfrak{K}$ *is a BPIFII.*

Proof. Suppose $\mathfrak{K}$ is an implicative BCK-algebra and $\mathfrak{B} = ({\alpha_{\mathfrak{B}}}^p, {\alpha_{\mathfrak{B}}}^n, {\beta_{\mathfrak{B}}}^p, {\beta_{\mathfrak{B}}}^n)$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$. Then

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}(\text{By BPIFI-2})$$

$$\Rightarrow \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}(\text{By Eq.}(13))$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}(\text{By BPIFI-3})$$

$$\Rightarrow \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}(\text{By Eq.}(13))$$

$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}(\text{By BPIFI-4})$$

$$\Rightarrow \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}(\text{By Eq.}(13))$$

$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}(\text{By BPIFI-5})$$

$$\Rightarrow \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}(\text{By Eq.}(13))$$

It follows that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. □

Corollary 3.1. *In $\mathfrak{K}$, every bipolar intuitionistic fuzzy implicative ideal is a bipolar intuitionistic fuzzy subalgebra.*

Corollary 3.2. *Let $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ be a BPIFII of $\mathfrak{K}$. If $\mathfrak{k}_1 \leq \mathfrak{k}_2$ in $\mathfrak{K}$, then $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_2)$, $\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_2)$, $\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_2)$ and $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_2)$, that is $\alpha_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}$ are order-reversing and $\alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}$ are order-preserving.*

Theorem 3.3. *If $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII in $\mathfrak{K}$ and*

$$\mathfrak{J}(0) = \{\mathfrak{k}_1 \in \mathfrak{K} \mid \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{p}(0), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{n}(0), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{p}(0), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{n}(0)\}.$$

Then $\mathfrak{J}(0)$ is an implicative ideal of $\mathfrak{K}$.

Proof. Let $\mathfrak{k}_1, \mathfrak{k}_2, \in \mathfrak{K}$ such that $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3, \mathfrak{k}_3 \in \mathfrak{J}(0)$. Since $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII in $\mathfrak{K}$, we have

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$
$$= min\{\alpha_{\mathfrak{B}}{}^{p}(0), \alpha_{\mathfrak{B}}{}^{p}(0)\} = \alpha_{\mathfrak{B}}{}^{p}(0),$$
$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$
$$= max\{\alpha_{\mathfrak{B}}{}^{n}(0), \alpha_{\mathfrak{B}}{}^{n}(0)\} = \alpha_{\mathfrak{B}}{}^{n}(0),$$
$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$
$$= max\{\beta_{\mathfrak{B}}{}^{p}(0), \beta_{\mathfrak{B}}{}^{p}(0)\} = \beta_{\mathfrak{B}}{}^{p}(0),$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$
$$= min\{\beta_{\mathfrak{B}}{}^{n}(0), \beta_{\mathfrak{B}}{}^{n}(0)\} = \beta_{\mathfrak{B}}{}^{n}(0).$$

On the other hand, we know from (BPIFII-1) $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1)$ and $\beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1)$ for all $\Bbbk_1 \in \mathfrak{K}$. Thus $\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) = \alpha_{\mathfrak{B}}{}^{p}(0)$, $\alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) = \alpha_{\mathfrak{B}}{}^{n}(0)$, $\beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) = \beta_{\mathfrak{B}}{}^{p}(0)$ and $\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) = \beta_{\mathfrak{B}}{}^{n}(0)$. Imply $\Bbbk_1 \in \mathfrak{J}(0)$. Obviously $0 \in \mathfrak{J}(0)$. Therefore, $\mathfrak{J}(0)$ is an implicative ideal of $\mathfrak{K}$. □

Theorem 3.4. *A BPIFS $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$ if and only if the non-empty level sets*

$$U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) = \{\Bbbk_1 \in \mathfrak{K} \mid \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq p_1, \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq n_1\}$$
$$L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2) = \{\Bbbk_1 \in \mathfrak{K} \mid \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq p_2, \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq n_2\}$$

are implicative ideals of $\mathfrak{K}$, for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$.

Proof. Assume that a BPIFS $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ in $\mathfrak{K}$ is a BPIFII of $\mathfrak{K}$ and $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1), L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$ are non-empty. Let $\Bbbk_1 \in U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \cap L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$. For any $\Bbbk_1 \in \mathfrak{K}$, we have

$$\begin{aligned}
& \alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) && (\because \mathfrak{B} \text{ is a BPIFII}) \\
& \alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) \geq p_1 && (\because \Bbbk_1 \in U) \\
\Rightarrow\ & \alpha_{\mathfrak{B}}{}^{p}(0) \geq p_1 \\
& \alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) && (\because \mathfrak{B} \text{ is a BPIFII}) \\
& \alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1) \leq n_1 && (\because \Bbbk_1 \in U) \\
\Rightarrow\ & \alpha_{\mathfrak{B}}{}^{n}(0) \leq n_1 \\
& \beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) && (\because \mathfrak{B} \text{ is a BPIFII}) \\
& \beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq p_2 && (\because \Bbbk_1 \in L) \\
\Rightarrow\ & \beta_{\mathfrak{B}}{}^{p}(0) \leq p_2 \\
& \beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) && (\because \mathfrak{B} \text{ is a BPIFII}) \\
& \beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq n_2 && (\because \Bbbk_1 \in L) \\
\Rightarrow\ & \beta_{\mathfrak{B}}{}^{n}(0) \geq n_2.
\end{aligned}$$

And so $0 \in U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \cap L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$. Let $(\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3, \Bbbk_3 \in U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \cap L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$ for all $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$. Then

$$\begin{aligned}
\alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1) &\geq min\{\alpha_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_3)\} \\
&\geq min\{p_1, p_1\} = p_1,
\end{aligned}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$
$$\leq max\{n_1, n_1\} = n_1.$$

Therefore $\mathfrak{k}_1 \in U(\alpha_{\mathfrak{B}}{}^{p},\ \alpha_{\mathfrak{B}}{}^{n}, p_1,\ n_1)$.Also

$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$
$$\leq max\{p_2, p_2\} = p_2,$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$
$$\geq min\{n_2, n_2\} = n_2.$$

Therefore $\mathfrak{k}_1 \in L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$.

Hence the level sets $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1)$ and $L(\beta_{\mathfrak{B}}{}^{p},\ \beta_{\mathfrak{B}}{}^{n}, p_2,\ n_2)$ are implicative ideals of $\mathfrak{K}$, for any $p_1, p_2 \in [0,1]$ and $n_1,\ n_2 \in [-1,0]$.

Conversely, let $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFS such that the non-empty level subsets $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1)$ and $L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$ are implicative ideals of $\mathfrak{K}$, for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$. Put $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_1, \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_1, \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_2$ and $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_2$ for any $\mathfrak{k}_1 \in \mathfrak{K}$. Since $0 \in U(\alpha_{\mathfrak{B}}{}^{p},\ \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \cap L(\beta_{\mathfrak{B}}{}^{p},\ \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$ $\alpha_{\mathfrak{B}}{}^{p}(0) \geq p_1 = \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq n_1 = \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq p_2 = \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$ and $\beta_{\mathfrak{B}}{}^{n}(0) \geq n_2 = \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$.
Thus $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$ and
$\beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$ Suppose there exists $\dot{\mathfrak{k}}_1, \dot{\mathfrak{k}}_2, \dot{\mathfrak{k}}_3 \in \mathfrak{K}$ such that

$$\alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) < min\{\alpha_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\}$$
$$\alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) > max\{\alpha_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\}$$
$$\beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) > max\{\beta_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\}$$
$$\beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) < min\{\beta_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\}$$

Take $\dot{p}_1, \dot{p}_2 \in [0,1]$ and $\dot{n}_1,\ \dot{n}_2 \in [-1,0]$ as

$$\dot{p}_1 = \frac{1}{2}\left[\alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) + min\{\alpha_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\}\right]$$
$$\dot{n}_1 = \frac{1}{2}\left[\alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) + max\{\alpha_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\}\right]$$
$$\dot{p}_2 = \frac{1}{2}\left[\beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) + max\{\beta_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\}\right]$$
$$\dot{n}_2 = \frac{1}{2}\left[\beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) + min\{\beta_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\}\right]$$

Then

$$\alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) < \dot{p}_1 < min\{\alpha_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\},$$

$$\alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) > \dot{n}_1 > max\{\alpha_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \alpha_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\},$$

$$\beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_1) > \dot{p}_2 > max\{\beta_{\mathfrak{B}}{}^{p}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{p}(\dot{\mathfrak{k}}_3)\},$$

$$\beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_1) < \dot{n}_2 < min\{\beta_{\mathfrak{B}}{}^{n}((\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3), \beta_{\mathfrak{B}}{}^{n}(\dot{\mathfrak{k}}_3)\}$$

Thus $(\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3, \dot{\mathfrak{k}}_3 \in U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \dot{p}_1, \dot{n}_1)$ and $(\dot{\mathfrak{k}}_1 * (\dot{\mathfrak{k}}_2 * \dot{\mathfrak{k}}_1)) * \dot{\mathfrak{k}}_3, \dot{\mathfrak{k}}_3 \in L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, \dot{p}_2, \dot{n}_2)$ but $\dot{\mathfrak{k}}_1 \notin U(\alpha_{\mathfrak{B}}{}^{p},\ \alpha_{\mathfrak{B}}{}^{n}, \dot{p}_1,\ \dot{n}_1)$ and $\dot{\mathfrak{k}}_1 \notin L(\beta_{\mathfrak{B}}{}^{p},\ \beta_{\mathfrak{B}}{}^{n}, \dot{p}_2,\ \dot{n}_2)$ which is a contradiction with level subsets $U(\alpha_{\mathfrak{B}}{}^{p},\ \alpha_{\mathfrak{B}}{}^{n}, \dot{p}_1,\ \dot{n}_1)$ and $L(\beta_{\mathfrak{B}}{}^{p},\ \beta_{\mathfrak{B}}{}^{n}, \dot{p}_2,\ \dot{n}_2)$ are implicative ideals of $\mathfrak{K}$. Therefore

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$

$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. Hence $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. □

Theorem 3.5. *Let* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *and* $\mathfrak{C} = (\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, \delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n})$ *are Bipolar Intuitionistic fuzzy ideal of* $\mathfrak{K}$ *such that* $\mathfrak{B} \subseteq \mathfrak{C}$ *and* $\mathfrak{B}(0) = \mathfrak{C}(0)$. *If* $\mathfrak{B}$ *is a BPIFII of* $\mathfrak{K}$, *then so is* $\mathfrak{C}$.

Proof. To prove that $\mathfrak{C}$ is a bipolar intuitionistic fuzzy implicative ideal of $\mathfrak{K}$, it is sufficient to show that for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$ the non-empty level subsets $U(\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, p_1, n_1)$ and $L(\delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n}, p_2, n_2)$ are either empty or an implicative ideals of $\mathfrak{K}$. If the level subset $U(\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, p_1, n_1)$ is non-empty,then $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \neq \phi$ and $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1) \subseteq U(\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, p_1, n_1)$.
In fact, if $\mathfrak{k}_1 \in U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1)$

$$\Rightarrow \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq p_1, \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq n_1$$

$$\Rightarrow \gamma_{\mathfrak{C}}{}^{p}(\mathfrak{k}_1) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq p_1, \gamma_{\mathfrak{C}}{}^{n}(\mathfrak{k}_1) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq\ n_1$$

$$\Rightarrow \mathfrak{k}_1 \in U(\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, p_1,\ n_1).$$

Also if level subset $L(\delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n}, p_2, n_2)$ is non-empty, then
$L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2) \neq \phi$ and $L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2) \subseteq L(\delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n}, p_2, n_2)$. In fact, if $\mathfrak{k}_1 \in L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$

$$\Rightarrow \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq p_2, \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq n_2$$

$$\Rightarrow \delta_{\mathfrak{C}}{}^{p}(\mathfrak{k}_1) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq p_2, \delta_{\mathfrak{C}}{}^{n}(\mathfrak{k}_1) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq n_2$$

$$\Rightarrow \mathfrak{k}_1 \in L(\delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n}, p_2, n_2).$$

By hypothesis $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a $\mathfrak{B}$ is a bipolar intuitionistic fuzzy implicative ideal of $\mathfrak{K}$. By Theorem 3.4 the non-empty level subsets $U(\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, p_1, n_1)$ and $L(\beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n}, p_2, n_2)$ are implicative ideals of $\mathfrak{K}$, for any $p_1, p_2 \in [0,1]$ and $n_1, n_2 \in [-1,0]$. By Proposition 2.2, $U(\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, p_1,\ n_1)$

and $L(\delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n}, p_2,\ n_2)$ are an implicative ideals of $\mathfrak{K}$. It follows that again from Theorem 3.4 $\mathfrak{C} = (\gamma_{\mathfrak{C}}{}^{p}, \gamma_{\mathfrak{C}}{}^{n}, \delta_{\mathfrak{C}}{}^{p}, \delta_{\mathfrak{C}}{}^{n})$ is a bipolar intuitionistic fuzzy implicative ideal of $\mathfrak{K}$. □

Theorem 3.6. *A bipolar intuitionistic fuzzy ideal*
$\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *of* $\mathfrak{K}$ *is a BPIFII of* $\mathfrak{K}$ *if and only if it satisfies the conditions*

$$\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \end{pmatrix} \tag{17}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$.

Proof. Suppose that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. Put $\mathfrak{k}_3 = 0$ in BPIFII-2,3,4 & 5 we get

$$\begin{aligned} \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) &\geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \alpha_{\mathfrak{B}}{}^{p}(0)\} \\ &= \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) &\leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \alpha_{\mathfrak{B}}{}^{n}(0)\} \\ &= \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) &\leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \beta_{\mathfrak{B}}{}^{p}(0)\} \\ &= \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) &\geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \beta_{\mathfrak{B}}{}^{n}(0)\} \\ &= \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \end{aligned}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$.
Conversely, assume that for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$, a BPIFI $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ satisfies condition (17).
Since $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$

$$\begin{aligned} \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) &\geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ &\geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\} \\ \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) &\leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ &\leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\} \\ \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) &\leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ &\leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\} \\ \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) &\geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ &\geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\} \end{aligned}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. Therefore $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a BPIFII of $\mathfrak{K}$. □

Theorem 3.7. *A bipolar intuitionistic fuzzy sub-algebra $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ of $\mathfrak{K}$ is a BPIFII of $\mathfrak{K}$ if and only if it satisfies the conditions $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3 \leq \mathfrak{w} \Rightarrow$*

$$\begin{pmatrix} \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{w})\} \\ \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{w})\} \\ \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^p(\mathfrak{w})\} \\ \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^n(\mathfrak{w})\} \end{pmatrix} \tag{18}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{w} \in \mathfrak{K}$.

Proof. Assume that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a BPIFII of $\mathfrak{K}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{w} \in \mathfrak{K}$ be such that $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3 \leq \mathfrak{w}$. By Theorem 3.1 $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$. It follows from Theorem 3.6 and Theorem 2.1, we have

$$\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \geq \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq min\{\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{w})\}$$
$$\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq max\{\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{w})\}$$
$$\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \leq \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq max\{\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^p(\mathfrak{w})\}$$
$$\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \geq \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq min\{\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^n(\mathfrak{w})\}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3, \mathfrak{w} \in \mathfrak{K}$.

Conversely, assume that $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a bipolar intuitionistic fuzzy sub-algebra of $\mathfrak{K}$ and satisfies the condition (18) If we take $\mathfrak{k}_1 = 0$ and $\mathfrak{k}_3 = \mathfrak{w} = \mathfrak{k}_1$ in (18), then we can obtain $\alpha_{\mathfrak{B}}{}^p(0) \geq \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$,$\alpha_{\mathfrak{B}}{}^n(0) \leq \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$, $\beta_{\mathfrak{B}}{}^p(0) \leq \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1)$ and $\beta_{\mathfrak{B}}{}^n(0) \geq \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1)$ for all $\mathfrak{k}_1 \in \mathfrak{K}$. Since $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * ((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3) \leq \mathfrak{k}_3$, it follows from hypothesis

$$\alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^p((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\}$$
$$\alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^n((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\}$$
$$\beta_{\mathfrak{B}}{}^p(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^p((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^p(\mathfrak{k}_3)\}$$
$$\beta_{\mathfrak{B}}{}^n(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^n((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^n(\mathfrak{k}_3)\}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. Therefore $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a BPIFII of $\mathfrak{K}$. □

Theorem 3.8. *Let $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^p, \alpha_{\mathfrak{B}}{}^n, \beta_{\mathfrak{B}}{}^p, \beta_{\mathfrak{B}}{}^n)$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$. Then the following are equivalent:*

(i) *$\mathfrak{B}$ is a BPIFII.*

(ii) $\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \end{pmatrix}$ *for all* $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$.

(iii) $\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \\ \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \end{pmatrix}$ *for all* $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$

Proof. $(i) \Rightarrow (ii)$

Let $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. Put $\mathfrak{k}_3 = 0$ in BPIFII-2,3,4 & 5 we get

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \alpha_{\mathfrak{B}}{}^{p}(0)\} = \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \alpha_{\mathfrak{B}}{}^{n}(0)\} = \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$$

$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \beta_{\mathfrak{B}}{}^{p}(0)\} = \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$$

$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * 0), \beta_{\mathfrak{B}}{}^{n}(0)\} = \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$$

for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$. Hence the condition (ii) holds.

$(ii) \Rightarrow (iii)$

Observe that for all $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathfrak{K}$, $\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1) \leq \mathfrak{k}_1$. Using Corollary 3.2, we have $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$; $\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$; $\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$; $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$. It follows from (ii) that $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$; $\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$; $\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$ and $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1))$. Hence the condition (iii) holds.

$(iii) \Rightarrow (i)$

Since $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a bipolar intuitionistic fuzzy ideal of $\mathfrak{K}$, we have

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$

$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. Using (iii) we obtain

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

$$\beta_{\mathfrak{B}}{}^{p}(\Bbbk_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \beta_{\mathfrak{B}}{}^{p}(\Bbbk_3)\}$$

$$\beta_{\mathfrak{B}}{}^{n}(\Bbbk_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), \beta_{\mathfrak{B}}{}^{n}(\Bbbk_3)\}$$

for all $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$. Obviously $\mathfrak{B}$ satisfies $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1)$, $\alpha_{\mathfrak{B}}{}^{n}(0) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1)$, $\beta_{\mathfrak{B}}{}^{p}(0) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1)$ and $\beta_{\mathfrak{B}}{}^{n}(0) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1)$ for all $\Bbbk_1 \in \mathfrak{K}$. Therefore $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. □

Theorem 3.9. *Every bipolar intuitionistic fuzzy implicative ideal* $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ *of* $\mathfrak{K}$ *satisfies*

$$\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \beta_{\mathfrak{B}}{}^{p}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \beta_{\mathfrak{B}}{}^{n}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \end{pmatrix} \tag{19}$$

for all $\Bbbk_1, \Bbbk_2 \in \mathfrak{K}$.

Proof. In $\mathfrak{K}$, we have

$$\begin{aligned} & \Bbbk_1 * (\Bbbk_1 * \Bbbk_2) \leq \Bbbk_1 && \text{(By Eq.(7))} \\ & \Rightarrow \Bbbk_2 * \Bbbk_1 \geq \Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) && \text{(By Eq.(11))} \\ & \Rightarrow \Bbbk_2 * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2))) \leq \Bbbk_2 * (\Bbbk_2 * \Bbbk_1) && \text{(By Eq.(11))} \\ & \Rightarrow (\Bbbk_1 * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) * (\Bbbk_1 * \Bbbk_2) \leq \Bbbk_2 * (\Bbbk_2 * \Bbbk_1) && \text{(By Eq.(1))} \\ & \Rightarrow (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2))) \leq \Bbbk_2 * (\Bbbk_2 * \Bbbk_1) && \text{(By Eq.(8)))} \end{aligned}$$

It follows from Corollary 3.2

$$\begin{pmatrix} \alpha_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \alpha_{\mathfrak{B}}{}^{n}((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) \leq \alpha_{\mathfrak{B}}{}^{n}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \beta_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) \leq \beta_{\mathfrak{B}}{}^{p}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \\ \beta_{\mathfrak{B}}{}^{n}((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) \geq \beta_{\mathfrak{B}}{}^{n}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)) \end{pmatrix} \tag{20}$$

Since $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII we have

$$\begin{aligned} & \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \\ & \geq min\{\alpha_{\mathfrak{B}}{}^{p}(((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) * 0), \alpha_{\mathfrak{B}}{}^{p}(0)\} \\ & \Rightarrow \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) \geq \alpha_{\mathfrak{B}}{}^{p}((\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)) * (\Bbbk_2 * (\Bbbk_1 * (\Bbbk_1 * \Bbbk_2)))) \\ & \geq \alpha_{\mathfrak{B}}{}^{p}(\Bbbk_2 * (\Bbbk_2 * \Bbbk_1)), \end{aligned}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))$$
$$\leq max\{\alpha_{\mathfrak{B}}{}^{n}(((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)))) * 0), \alpha_{\mathfrak{B}}{}^{n}(0)\}$$
$$\Rightarrow \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) \leq \alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))))$$
$$\leq \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_2 * (\mathfrak{k}_2 * \mathfrak{k}_1)),$$
$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))$$
$$\leq max\{\beta_{\mathfrak{B}}{}^{p}(((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)))) * 0), \beta_{\mathfrak{B}}{}^{p}(0)\}$$
$$\Rightarrow \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) \leq \beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))))$$
$$\leq \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_2 * (\mathfrak{k}_2 * \mathfrak{k}_1)),$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))$$
$$\geq min\{\beta_{\mathfrak{B}}{}^{n}(((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)))) * 0), \beta_{\mathfrak{B}}{}^{n}(0)\}$$
$$\Rightarrow \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) \geq \beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2)) * (\mathfrak{k}_2 * (\mathfrak{k}_1 * (\mathfrak{k}_1 * \mathfrak{k}_2))))$$
$$\geq \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_2 * (\mathfrak{k}_2 * \mathfrak{k}_1)).$$

Hence the result is proved. □

Theorem 3.10. *Let $\mathfrak{T} \subseteq \mathfrak{K}$ and $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ be a BPIFS in $\mathfrak{K}$ defined as* $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \begin{cases} p_1 & \text{if } \mathfrak{k}_1 \in \mathfrak{T} \\ p_2 & \text{otherwise} \end{cases}$; $\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \begin{cases} n_1 & \text{if } \mathfrak{k}_1 \in \mathfrak{T} \\ n_2 & \text{otherwise} \end{cases}$; $\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \begin{cases} p_3 & \text{if } \mathfrak{k}_1 \in \mathfrak{T} \\ p_4 & \text{otherwise} \end{cases}$ *and* $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \begin{cases} n_3 & \text{if } \mathfrak{k}_1 \in \mathfrak{T} \\ n_4 & \text{otherwise} \end{cases}$ *for all $\mathfrak{k}_1 \in \mathfrak{K}$. Where $0 \leq p_2 < p_1,\ 0 \leq n_1 < n_2,\ 0 \leq p_3 < p_4, 0 \leq n_4 < n_3$ and $0 \leq \sum_{i=1}^{4} p_i \leq 1,\ -1 \leq \sum_{i=1}^{4} n_i \leq 0$. Then the following are equivalent:*

(a) *$\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$.*

(b) *$\mathfrak{T}$ is an implicative ideal of $\mathfrak{K}$.*

Proof. $(a) \Rightarrow (b)$
Suppose $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. Let $\mathfrak{k}_1 \in \mathfrak{T}$, $\alpha_{\mathfrak{B}}{}^{p}(0) \geq \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_1 \Rightarrow \alpha_{\mathfrak{B}}{}^{p}(0) \geq p_1 \Rightarrow 0 \in \mathfrak{T}$. Let $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$ such that $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3 \in \mathfrak{T}$ and $\mathfrak{k}_3 \in \mathfrak{T}$. We have $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\} \geq min\{p_1, p_1\} = p_1$ and so $\mathfrak{k}_1 \in \mathfrak{T}$. Hence $\mathfrak{T}$ is an implicative ideal of $\mathfrak{K}$.
$(b) \Rightarrow (a)$
Let $\mathfrak{k}_1 \in \mathfrak{K}$. If $\mathfrak{k}_1 \in \mathfrak{T}$, then $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_1, \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_1, \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_3$ and $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_4$. Since $0 \in \mathfrak{T}$, then $\alpha_{\mathfrak{B}}{}^{p}(0) = p_1, \alpha_{\mathfrak{B}}{}^{n}(0) = n_1, \beta_{\mathfrak{B}}{}^{p}(0) = p_3$ and $\beta_{\mathfrak{B}}{}^{n}(0) = n_3$. Therefore $\alpha_{\mathfrak{B}}{}^{p}(0) = \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1), \alpha_{\mathfrak{B}}{}^{n}(0) = \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$,
$\beta_{\mathfrak{B}}{}^{p}(0) = \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1)$ and $\beta_{\mathfrak{B}}{}^{n}(0) = \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1)$. If $\mathfrak{k}_1 \notin \mathfrak{T}$, then $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_2, \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_2, \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_4$

and ${\beta_{\mathfrak{B}}}^n(\Bbbk_1) = n_4$. Therefore

$$\begin{aligned}
{\alpha_{\mathfrak{B}}}^p(0) = p_1 > p_2 = {\alpha_{\mathfrak{B}}}^p(\Bbbk_1) \Rightarrow {\alpha_{\mathfrak{B}}}^p(0) > {\alpha_{\mathfrak{B}}}^p(\Bbbk_1)\\
{\beta_{\mathfrak{B}}}^p(0) = p_3 < p_4 = {\beta_{\mathfrak{B}}}^p(\Bbbk_1) \Rightarrow {\beta_{\mathfrak{B}}}^p(0) < {\beta_{\mathfrak{B}}}^p(\Bbbk_1)\\
{\alpha_{\mathfrak{B}}}^n(0) = n_1 < n_2 = {\alpha_{\mathfrak{B}}}^n(\Bbbk_1) \Rightarrow {\alpha_{\mathfrak{B}}}^n(0) < {\alpha_{\mathfrak{B}}}^n(\Bbbk_1)\\
{\beta_{\mathfrak{B}}}^n(0) = n_3 > n_4 = {\beta_{\mathfrak{B}}}^n(\Bbbk_1) \Rightarrow {\beta_{\mathfrak{B}}}^n(0) > {\beta_{\mathfrak{B}}}^n(\Bbbk_1).
\end{aligned}$$

Therefore ${\alpha_{\mathfrak{B}}}^p(0) \geq {\alpha_{\mathfrak{B}}}^p(\Bbbk_1)$, ${\alpha_{\mathfrak{B}}}^n(0) \leq {\alpha_{\mathfrak{B}}}^n(\Bbbk_1)$, ${\beta_{\mathfrak{B}}}^p(0) \leq {\beta_{\mathfrak{B}}}^p(\Bbbk_1)$ and ${\beta_{\mathfrak{B}}}^n(0) \geq {\beta_{\mathfrak{B}}}^n(\Bbbk_1)$ for all $\Bbbk_1 \in \mathfrak{K}$. Let $\Bbbk_1, \Bbbk_2, \Bbbk_3 \in \mathfrak{K}$. If $(\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3 \in \mathfrak{T}$ and $\Bbbk_3 \in \mathfrak{T}$, then $\Bbbk_1 \in \mathfrak{T}$ and so

$$\begin{aligned}
{\alpha_{\mathfrak{B}}}^p(\Bbbk_1) = p_1 &= min\{p_1, p_1\}\\
&= min\{{\alpha_{\mathfrak{B}}}^p((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\alpha_{\mathfrak{B}}}^p(\Bbbk_3)\},\\
{\alpha_{\mathfrak{B}}}^n(\Bbbk_1) = n_1 &= max\{n_1, n_1\}\\
&= max\{{\alpha_{\mathfrak{B}}}^n((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\alpha_{\mathfrak{B}}}^n(\Bbbk_3)\},\\
{\beta_{\mathfrak{B}}}^p(\Bbbk_1) = p_3 &= max\{p_3, p_3\}\\
&= max\{{\beta_{\mathfrak{B}}}^p((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\beta_{\mathfrak{B}}}^p(\Bbbk_3)\},\\
{\beta_{\mathfrak{B}}}^n(\Bbbk_1) = n_3 &= min\{n_3, n_3\}\\
&= min\{{\beta_{\mathfrak{B}}}^n((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\beta_{\mathfrak{B}}}^n(\Bbbk_3)\}.
\end{aligned}$$

If $(\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3 \in \mathfrak{T}$ and $\Bbbk_3 \notin \mathfrak{T}$, then $\Bbbk_1 \notin \mathfrak{T}$ and so

$$\begin{aligned}
{\alpha_{\mathfrak{B}}}^p(\Bbbk_1) = p_2 &= min\{p_1, p_2\}\\
&= min\{{\alpha_{\mathfrak{B}}}^p((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\alpha_{\mathfrak{B}}}^p(\Bbbk_3)\},\\
{\alpha_{\mathfrak{B}}}^n(\Bbbk_1) = n_2 &= max\{n_1, n_2\}\\
&= max\{{\alpha_{\mathfrak{B}}}^n((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\alpha_{\mathfrak{B}}}^n(\Bbbk_3)\},\\
{\beta_{\mathfrak{B}}}^p(\Bbbk_1) = p_4 &= max\{p_3, p_4\}\\
&= max\{{\beta_{\mathfrak{B}}}^p((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\beta_{\mathfrak{B}}}^p(\Bbbk_3)\},\\
{\beta_{\mathfrak{B}}}^n(\Bbbk_1) = n_4 &= min\{n_3, n_4\}\\
&= min\{{\beta_{\mathfrak{B}}}^n((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3), {\beta_{\mathfrak{B}}}^n(\Bbbk_3)\}.
\end{aligned}$$

If $(\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3 \notin \mathfrak{T}$ and $\Bbbk_3 \in \mathfrak{T}$, then $\Bbbk_1 \notin \mathfrak{T}$ and so

$$\begin{aligned}
{\alpha_{\mathfrak{B}}}^p(\Bbbk_1) = p_2 &= min\{p_1, p_2\}\\
&= min\{{\alpha_{\mathfrak{B}}}^p(\Bbbk_3), {\alpha_{\mathfrak{B}}}^p((\Bbbk_1 * (\Bbbk_2 * \Bbbk_1)) * \Bbbk_3)\},
\end{aligned}$$

$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_2 = max\{n_1, n_2\}$$
$$= max\{\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3)\},$$
$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_4 = max\{p_3, p_4\}$$
$$= max\{\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3)\},$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_4 = min\{n_3, n_4\}$$
$$= min\{\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3)\}.$$

If $(\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3 \notin \mathfrak{T}$ and $\mathfrak{k}_3 \notin \mathfrak{T}$, then $\mathfrak{k}_1 \notin \mathfrak{T}$ and so

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_2 = min\{p_2, p_2\}$$
$$= min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\},$$
$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_2 = max\{n_2, n_2\}$$
$$= max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\},$$
$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = p_4 = max\{p_4, p_4\}$$
$$= max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\},$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = n_4 = min\{n_4, n_4\}$$
$$= min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}.$$

Therefore

$$\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \geq min\{\alpha_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$
$$\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \leq max\{\alpha_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$
$$\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) \leq max\{\beta_{\mathfrak{B}}{}^{p}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_3)\}$$
$$\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) \geq min\{\beta_{\mathfrak{B}}{}^{n}((\mathfrak{k}_1 * (\mathfrak{k}_2 * \mathfrak{k}_1)) * \mathfrak{k}_3), \beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_3)\}$$

for all $\mathfrak{k}_1, \mathfrak{k}_2, \mathfrak{k}_3 \in \mathfrak{K}$. Hence $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$. This completes the proof. □

Corollary 3.3. *Let $\mathfrak{T} \subseteq \mathfrak{K}$ and $\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ be a BPIFS in $\mathfrak{K}$ defined as $\alpha_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \begin{cases} 1 & if\ \mathfrak{k}_1 \in \mathfrak{T} \\ 0 & otherwise \end{cases}$; $\alpha_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \begin{cases} -1 & if\ \mathfrak{k}_1 \in \mathfrak{T} \\ 0 & otherwise \end{cases}$ $\beta_{\mathfrak{B}}{}^{p}(\mathfrak{k}_1) = \begin{cases} 0 & if\ \mathfrak{k}_1 \in \mathfrak{T} \\ 1 & otherwise \end{cases}$ and $\beta_{\mathfrak{B}}{}^{n}(\mathfrak{k}_1) = \begin{cases} 0 & if\ \mathfrak{k}_1 \in \mathfrak{T} \\ -1 & otherwise \end{cases}$ for all $\mathfrak{k}_1 \in \mathfrak{K}$. Then the following are equivalent:*

(a) *$\mathfrak{B} = (\alpha_{\mathfrak{B}}{}^{p}, \alpha_{\mathfrak{B}}{}^{n}, \beta_{\mathfrak{B}}{}^{p}, \beta_{\mathfrak{B}}{}^{n})$ is a BPIFII of $\mathfrak{K}$.*

(b) *$\mathfrak{T}$ is an implicative ideal of $\mathfrak{K}$.*

4. Conclusion

This research establishes a novel framework for the integration of bipolar valued intuitionistic fuzzy set theories in the field of BCK-algebras. The introduction of bipolar intuitionistic fuzzy implicative ideals expands the theoretical landscape, offering valuable insights into algebraic structures with real-world implications. The exploration of related properties signifies the potential for further advancements at the intersection of BPIFS theories and algebraic structures. This study encourages ongoing exploration in this domain to discover new connections, properties and applications that can improve theoretical and practical understanding. In our future research on bipolar intuitionistic fuzzy structures in BCK-algebras, we may consider the following topics: (i) bipolar intuitionistic fuzzy soft ideals, (ii) $(\in, \in \vee q)-$ bipolar intuitionistic fuzzy ideals, and their relations.

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