Asia Pac. J. Math. 2023 **10**:6

MILD NONNEGATIVE SOLUTIONS FOR FRACTIONAL ITERATIVE DIFFERENTIAL EQUATIONS

ABDERRAHIM GUERFI[1], ABDELOUAHEB ARDJOUNI[2,*]

[1]Applied Mathematics Lab, Department of Mathematics, University of Annaba, Algeria

[2]Department of Mathematics and Informatics, University of Souk Ahras, Algeria

[*]Corresponding author: abd_ardjouni@yahoo.fr

Received Jan. 7, 2023

Abstract. This research work deals with the existence and uniqueness of mild nonnegative solutions of a fractional iterative differential equation by first inverting it as a fixed point problem. After that, we create a suitable mapping and then use the Schauder fixed point theorem and the contraction mapping principle to demonstrate our new findings. We conclude by providing an example to illustrate our findings.

2020 Mathematics Subject Classification. 34A08; 34A12; 45G05; 47H09; 47H10.

Key words and phrases. fractional iterative differential equations; fixed point theorem; existence; uniqueness.

1. Introduction

Fractional and iterative differential equations have many applications in the automatic control problems, the physical problems, the mechanical problems, the biological problems, the disease transmission problems, the problems of long-range planning in economics, and in many other areas of science and technology. Many authors have focused their attention, in particular, on issues related to the existence, uniqueness, and stability of solutions for iterative and fractional differential equations, see [1]- [13], [15]- [18] and the references therein.

In [7], Bouakkaz discussed the existence of positive periodic solutions for the iterative functional differential equation

$$\upsilon'(\tau) = -a(\tau)\upsilon(\tau) + p(\tau)\upsilon^{m}(\tau - r(\tau)) f\left(\tau, \upsilon^{[1]}(\tau), ..., \upsilon^{[n]}(\tau)\right).$$

DOI: 10.28924/APJM/10-6

By applying the Schauder fixed point theorem, the existence of positive periodic solutions has been obtained.

The following fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha}\upsilon(\tau) = f(\tau, \upsilon(\tau)) + {}^{C}D^{\alpha-1}g(\tau, \upsilon(\tau)), \ 0 < \tau \leq \tau_0, \\ \upsilon(0) = \upsilon_1 > 0, \ \upsilon'(0) = \upsilon_2 > 0, \end{cases}$$

has been discussed in [8], where $1 < \alpha \leq 2$. By using the Schauder and Banach fixed point theorems and the method of the upper and lower solutions, the authors obtained positivity results.

In [10], Guerfi and Ardjouni studied the existence and uniqueness of mild solutions for the fractional iterative differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha}\upsilon(\tau) = f\left(\tau, \upsilon^{[1]}(\tau), ..., \upsilon^{[n]}(\tau)\right), \ \tau \in [0, \tau_0], \\ \upsilon(0) = \upsilon'(0) = 0, \end{cases}$$

where $1 < \alpha \leq 2$. By applying the Schauder and Banach fixed point theorems, the existence and uniqueness of solutions have been established.

Motivated and inspired by the aforementioned works, in this manuscript we concentrate on the existence and uniqueness of mild nonnegative solutions for the nonlinear fractional iterative differential equation

$$(1) \qquad \begin{cases} {}^{C}D_{0+}^{\alpha}\upsilon(\tau) = p(\tau)\,\upsilon^{m}(\tau)\, f\left(\tau, \upsilon^{[1]}(\tau), ..., \upsilon^{[n]}(\tau)\right), \ \tau \in [0, \tau_0], \\ \upsilon(0) = \upsilon'(0) = 0, \end{cases}$$

where $\tau_0 > 0$, ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative of order $1 < \alpha \leq 2$, $\upsilon^{[i]}(\tau)$, $i = 1, ..., n$ are the iterates of $\upsilon(\tau)$, $p \in C(\mathbb{R}, \mathbb{R}^{+})$ and $m \geq 0$. The function $f : \mathbb{R} \times \mathbb{R}^{n} \to \mathbb{R}^{+}$ is continuous. In order to obtain our desired results in this manuscript, we convert (1) into a fixed point problem. Next, we demonstrate the existence of mild nonnegative solutions by using the Schauder fixed point theorem, as well as the uniqueness of the mild nonnegative solution by applying the contraction mapping principle.

The following is how this manuscript is structured. Section 2 presents various definitions and lemmas relevant to fractional calculus as well as some preliminary findings that will be useful in later sections. The Schauder fixed point theorem is also stated. The main findings on the existence and uniqueness of mild nonnegative solutions for the problem (1) are presented in Section 3 along with an example to help clarify our findings.

2. Preliminaries

We denote by $C\left([0,\tau_0],\mathbb{R}\right)$ the Banach space of all continuous functions defined on $[0,\tau_0]$, equipped with the norm

$$\|\upsilon\| = \sup_{\tau\in[0,\tau_0]} |\upsilon(\tau)|.$$

For $0<\rho\leq\tau_0$ and $\lambda>0$, let

$$C(\rho,\lambda) = \{\upsilon\in C([0,\tau_0],\mathbb{R}) : 0\leq\upsilon(\tau)\leq\rho,\ \forall\tau\in[0,\tau_0]$$
$$\text{and } |\upsilon(\tau_2)-\upsilon(\tau_1)|\leq\lambda|\tau_2-\tau_1|,\ \forall\tau_1,\tau_2\in[0,\tau_0]\}.$$

It is clear that $C(\rho,\lambda)$ is a bounded, closed and convex subset of $C([0,\tau_0],\mathbb{R})$.

Moreover, we assume that f is globally Lipschitz in υ_1, υ_2, ..., υ_n, that is, there are positive constants η_1, η_2, ..., η_n such that

$$|f(\tau,\upsilon_1,...,\upsilon_n) - f(\tau,y_1,...,y_n)| \leq \sum_{i=1}^{n}\eta_i|\upsilon_i-y_i|. \tag{2}$$

Definition 1 ([12]). *The fractional integral of order $\alpha>0$ of a function $\upsilon:\mathbb{R}^+\longrightarrow\mathbb{R}$ is defined by*

$$I_{0+}^{\alpha}\upsilon(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}\upsilon(s)\,ds,$$

where Γ is the gamma function.

Definition 2 ([12]). *The Caputo fractional derivative of order $\alpha>0$ of a function $\upsilon:\mathbb{R}^+\longrightarrow\mathbb{R}$ is defined by*

$${}^{C}D_{0+}^{\alpha}\upsilon(\tau) = I_{0+}^{n-\alpha}\upsilon^{(n)}(\tau) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\tau}(\tau-s)^{n-\alpha-1}\upsilon^{(n)}(s)\,ds,$$

where $n=[\alpha]+1$.

Lemma 1 ([12]). *Suppose that $\upsilon\in C^{n-1}([0,+\infty))$ and $\upsilon^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}\upsilon\right)(\tau) = \upsilon(\tau) - \sum_{k=0}^{n-1}\frac{\upsilon^{(k)}(0)}{k!}\tau^k.$$

In particular, when $\alpha\in(1,2)$, $\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}\upsilon\right)(\tau) = \upsilon(\tau)-\upsilon(0)-\upsilon'(0)\tau$.

From the above Lemma 1, we deduce the following lemma.

Lemma 2. *Let $\upsilon\in C([0,\tau_0],\mathbb{R})$, then υ is a mild solution of (1) if*

$$\upsilon(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-s)^{\alpha-1}p(s)\,\upsilon^m(s)\,f\left(\tau,\upsilon^{[1]}(s),...,\upsilon^{[n]}(s)\right)ds. \tag{3}$$

Lemma 3 ([16]). *If $\varphi, \psi \in C(\rho, \lambda)$, then*

$$\left\|\varphi^{[i]} - \psi^{[i]}\right\| \leq \sum_{j=0}^{i-1} \lambda^j \|\varphi - \psi\|, \ i = 1, 2, ...$$

Lemma 4 ([7]). *If $\varphi, \psi \in C(\rho, \lambda)$, then*

$$\|\varphi^m - \psi^m\| \leq m\rho^{m-1} \|\varphi - \psi\|, \ m \geq 0.$$

Theorem 1 (Schauder fixed point theorem [14]). *Let $\mathbb{D}$ be a compact convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Then, the continuous mapping $\mathcal{A} : \mathbb{D} \to \mathbb{D}$ has a fixed point.*

3. Main results

Theorem 1 is employed in this section to demonstrate the existence of mild nonnegative solutions of (1). We will also give the sufficient conditions for the uniqueness of the mild nonnegative solution.

To applicable the Schauder fixed point, we define the mapping $\mathcal{A} : C(\rho, \lambda) \to C([0, \tau_0], \mathbb{R})$ by

$$(\mathcal{A}\varphi)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} p(s) \varphi^m(s) f\left(s, \varphi^{[1]}(s), .., \varphi^{[n]}(s)\right) ds. \tag{4}$$

Since $C(\rho, \lambda)$ is a closed, uniformly bounded and equicontinuous subset of the space $C([0, \tau_0], \mathbb{R})$, then $C(\rho, \lambda)$ is a compact set. To demonstrate that the mapping $\mathcal{A}$ has a fixed point, we will show that $\mathcal{A}$ is well defined, continuous and $\mathcal{A}(C(\rho, \lambda)) \subset C(\rho, \lambda)$, i.e.

$$\mathcal{A}\varphi \in C(\rho, \lambda) \text{ for all } \varphi \in C(\rho, \lambda).$$

Now, we need to introduce the following constants

$$\theta = \sup_{\tau \in [0, \tau_0]} |p(\tau)|, \ \sigma = \sup_{\tau \in [0, \tau_0]} |f(\tau, 0, ..., 0)|.$$

Lemma 5. *Suppose that condition (2) holds. Then, the mapping $\mathcal{A} : C(\rho, \lambda) \to C([0, \tau_0], \mathbb{R})$ given by (4) is well defined and continuous.*

Proof. Let $\mathcal{A}$ be defined by (4). Obviously, $\mathcal{A}$ is well defined. To prove the continuity of $\mathcal{A}$, for $\varphi, \psi \in C(\rho, \lambda)$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(\tau) - (\mathcal{A}\psi)(\tau)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} p(s) \left|\varphi^m(s) f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - \psi^m(s) f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right)\right. \\
&\left. + \psi^m(s) f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - \psi^m(s) f\left(s, \psi^{[1]}(s), ..., \psi^{[n]}(s)\right)\right| ds
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} p(s) \left|\varphi^m(s) - \psi^m(s)\right| \left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right)\right| ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} p(s) \left|\psi^m(s)\right| \left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \psi^{[1]}(s), ..., \psi^{[n]}(s)\right)\right| ds.$$

It follow from (2) that

$$\begin{aligned}
&\left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right)\right| \\
&= \left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - f(s, 0, ..., 0) + f(s, 0, ..., 0)\right| \\
&\leq \left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - f(s, 0, ..., 0)\right| + \left|f(s, 0, ..., 0)\right| \\
&\leq \sigma + \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \|\varphi\| \\
&\leq \sigma + \rho \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j.
\end{aligned}$$

Also, from (2) and Lemma 3 we have

$$\begin{aligned}
&\left|f\left(s, \varphi^{[1]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \psi^{[1]}(s), ..., \psi^{[n]}(s)\right)\right| \\
&\leq \sum_{i=1}^{n} \eta_i \left\|\varphi^{[i]} - \psi^{[i]}\right\| \leq \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \|\varphi - \psi\|,
\end{aligned}$$

and from Lemma 4 we have

$$\left|\varphi^m(s) - \psi^m(s)\right| \leq m\rho^{m-1} \|\varphi - \psi\|.$$

So, we obtain

$$\begin{aligned}
&|(\mathcal{A}\varphi)(\tau) - (\mathcal{A}\psi)(\tau)| \\
&\leq \theta \left(\sigma + \rho \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j\right) m\rho^{m-1} \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} ds \|\varphi - \psi\| \\
&+ \theta\rho^m \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1} ds \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \|\varphi - \psi\| \\
&\leq \frac{\theta\tau_0^\alpha}{\Gamma(\alpha+1)} \left(m\rho^{m-1}\sigma + (m+1)\rho^m \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j\right) \|\varphi - \psi\|.
\end{aligned}$$

Then, the mapping $\mathcal{A}$ is continuous. □

Lemma 6. *Assume that the condition (2) holds. If*

$$\frac{\theta\rho^m\tau_0^\alpha}{\Gamma(\alpha+1)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right)\leq\rho, \tag{5}$$

and

$$\frac{\theta\rho^m\tau_0^{\alpha-1}}{\Gamma(\alpha)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right)\leq\lambda, \tag{6}$$

then, $\mathcal{A}(C(\rho,\lambda))\subset C(\rho,\lambda)$.

Proof. For $\varphi\in C(\rho,\lambda)$, we have

$$|(\mathcal{A}\varphi)(\tau)|\leq\frac{1}{\Gamma(\alpha)}\int_0^\tau(\tau-s)^{\alpha-1}p(s)\,|\varphi^m(s)|\left|f\left(s,\varphi^{[1]}(s),...,\varphi^{[n]}(s)\right)\right|ds.$$

But

$$\left|f\left(s,\varphi^{[1]}(s),...,\varphi^{[n]}(s)\right)\right|\leq\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j,$$

and

$$|\varphi^m(s)|\leq\rho^m.$$

So

$$\begin{aligned}|(\mathcal{A}\varphi)(\tau)| &\leq \frac{\theta\rho^m}{\Gamma(\alpha)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right)\int_0^\tau(\tau-s)^{\alpha-1}ds\\ &\leq \frac{\theta\rho^m\tau_0^\alpha}{\Gamma(\alpha+1)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right).\end{aligned}$$

From (5), we get

$$0\leq(\mathcal{A}\varphi)(\tau)\leq|(\mathcal{A}\varphi)(\tau)|\leq\rho. \tag{7}$$

Let $\tau_1,\tau_2\in[0,\tau_0]$ with $\tau_1<\tau_2$, we have

$$\begin{aligned}&|(\mathcal{A}\varphi)(\tau_2)-(\mathcal{A}\varphi)(\tau_1)|\\ &\leq\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}\left|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right|p(s)\,|\varphi^m(s)|\left|f\left(s,\varphi^{[1]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\ &+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}p(s)\,|\varphi^m(s)|\left|f\left(s,\varphi^{[1]}(s),...,\varphi^{[n]}(s)\right)\right|ds\\ &\leq\frac{\theta\rho^m}{\Gamma(\alpha)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right)\left(\int_0^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right)\\ &\leq\frac{\theta\rho^m}{\Gamma(\alpha+1)}\left(\sigma+\rho\sum_{i=1}^n\eta_i\sum_{j=0}^{i-1}\lambda^j\right)(\tau_2^\alpha-\tau_1^\alpha)\end{aligned}$$

$$\leq \frac{\theta \rho^m \tau_0^{\alpha-1}}{\Gamma(\alpha)} \left(\sigma + \rho \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \right) |\tau_2 - \tau_1| .$$

By using (6), we obtain

$$|(\mathcal{A}\varphi)(\tau_2) - (\mathcal{A}\varphi)(\tau_1)| \leq \lambda |\tau_2 - \tau_1| , \ \forall \tau_1, \tau_2 \in [0, \tau_0] . \tag{8}$$

From (7) and (8), we have $\mathcal{A}\varphi \in C(\rho, \lambda)$ for all $\varphi \in C(\rho, \lambda)$. So, we conclude that $\mathcal{A}(C(\rho, \lambda)) \subset C(\rho, \lambda)$. □

Theorem 2. *Assume that the conditions (2), (5) and (6) hold. Then, (1) admits at least one mild nonnegative solution $\upsilon \in C(\rho, \lambda)$.*

Proof. From Lemma 2, (1) has a mild nonnegative solution $\upsilon \in C(\rho, \lambda)$ if the mapping $\mathcal{A}$ defined by (4) has a fixed point. From Lemmas 5 and 6 all hypotheses of the Schauder fixed point theorem are satisfied. Hence, the mapping $\mathcal{A}$ has at least one fixed point in $C(\rho, \lambda)$ which is a mild nonnegative solution of the problem (1). □

Theorem 3. *Assume that all the hypotheses of Theorem 2 hold. If*

$$\gamma = \frac{\theta \tau_0^{\alpha}}{\Gamma(\alpha+1)} \left(m\rho^{m-1}\sigma + (m+1)\rho^m \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \right) < 1, \tag{9}$$

then, (1) admits a unique mild nonnegative solution in $C(\rho, \lambda)$.

Proof. It follow from Theorem 2 that $\mathcal{A}$ has at least one fixed point in $C(\rho, \lambda)$ which is a mild nonnegative solution of the problem (1). So, we need only to prove that the mapping $\mathcal{A}$ defined in (4) is a contraction on $C(\rho, \lambda)$. For any $\varphi, \psi \in C(\rho, \lambda)$, we have

$$\begin{aligned} &|(\mathcal{A}\varphi)(\tau) - (\mathcal{A}\psi)(\tau)| \\ &\leq \frac{\theta \tau_0^{\alpha}}{\Gamma(\alpha+1)} \left(m\rho^{m-1}\sigma + (m+1)\rho^m \sum_{i=1}^{n} \eta_i \sum_{j=0}^{i-1} \lambda^j \right) \|\varphi - \psi\| . \end{aligned}$$

Now using (9) we deduce that

$$\|\mathcal{A}\varphi - \mathcal{A}\psi\| \leq \gamma \|\varphi - \psi\| .$$

Therefore, we arrive at $\mathcal{A}$ is a contraction. By applying the contraction mapping principle, we conclude that $\mathcal{A}$ has a unique fixed point which is the unique mild nonnegative solution of (1). □

Example 1. *Consider the fractional iterative differential equation*

$$(10) \qquad \begin{cases} {}^{C}D_{0^{+}}^{\frac{3}{2}}\left(\upsilon\left(\tau\right)\right)=\frac{1}{12}\left(\cos^{2}\tau\right)\left(\frac{\upsilon^{[1]}(\tau)}{1+\upsilon^{[1]}(\tau)}+\frac{\upsilon^{[2]}(\tau)}{1+\upsilon^{[2]}(\tau)}\right),\ \tau\in\left[0,1\right], \\ \upsilon\left(0\right)=0,\ \upsilon'\left(0\right)=0, \end{cases}$$

where $\tau_0 = 1$, $\alpha = \frac{3}{2}$, $m = 0$, $n = 2$, $f(\tau, \upsilon, y) = \frac{\upsilon}{1+\upsilon} + \frac{y}{1+y}$ *and* $p(\tau) = \frac{1}{12}(\cos^2 \tau)$. *If* $\rho = \frac{1}{6}$ *and* $\lambda = \frac{1}{4}$ *in the definition of* $C(\rho, \lambda)$, *then the function* f *is positive continuous, and for* $\upsilon_1, \upsilon_2, y_1, y_2 \in \mathbb{R}$ *we have*

$$\left|f\left(\tau,\upsilon_1,\upsilon_2\right)-f\left(\tau,y_1,y_2\right)\right|\leq\left|\upsilon_1-y_1\right|+\left|\upsilon_2-y_2\right|$$

so

$$\left|f\left(\tau,\upsilon_1,\upsilon_2\right)-f\left(\tau,y_1,y_2\right)\right|\leq\sum_{i=1}^{2}\eta_i\left|\upsilon_i-y_i\right|.$$

with $\eta_1 = \eta_2 = 1$, *Also, we have* $\sigma = \sup_{\tau\in[0,1]} |f(\tau, 0, 0)| = 0$ *and* $\theta = \sup_{\tau\in[0,1]} |p(\tau)| = \frac{1}{12}$. *Therefore, we obtain*

$$\frac{\theta\rho^{m}\tau_0^{\alpha}}{\Gamma\left(\alpha+1\right)}\left(\sigma+\rho\sum_{i=1}^{n}\eta_i\sum_{j=0}^{i-1}\lambda^{j}\right)=2.3508\times10^{-2}\leq\rho=\frac{1}{6},$$

and

$$\frac{\theta\rho^{m}\tau_0^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(\sigma+\rho\sum_{i=1}^{n}\eta_i\sum_{j=0}^{i-1}\lambda^{j}\right)=3.5262\times10^{-2}\leq\lambda=\frac{1}{4}.$$

Also

$$\frac{\theta\tau_0^{\alpha}}{\Gamma\left(\alpha+1\right)}\left(m\rho^{m-1}\sigma+\left(m+1\right)\rho^{m}\sum_{i=1}^{n}\eta_i\sum_{j=0}^{i-1}\lambda^{j}\right)=0.14105<1.$$

Then, the problem (9) *has a unique mild nonnegative solution in* $C\left(\frac{1}{6}, \frac{1}{4}\right)$.

References

[1] S. Abbas, Existence of solutions to fractional order ordinary and delay differential equations and applications, Electron. J. Differ. Equ. 2011 (2011), 9.

[2] A. Ardjouni, Existence and uniqueness of positive solutions for nonlinear Caputo-Hadamard fractional differential equations, Proyecciones. 40 (2021), 139-152. https://doi.org/10.22199/issn.0717-6279-2021-01-0009.

[3] A. Ardjouni, Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions, AIMS Math. 4 (2019), 1101-1113. https://doi.org/10.3934/math.2019.4.1101.

[4] A. Ardjouni, A. Djoudi, Positive solutions for first-order nonlinear Caputo-Hadamard fractional relaxation differential equations, Kragujevac J. Math. 45 (2021), 897-908. https://doi.org/10.46793/KgJMat2106.897A.

[5] A. Ardjouni, A. Djoudi, Existence and uniqueness of positive solutions for first-order nonlinear Liouville-Caputo fractional differential equations, São Paulo J. Math. Sci. 14 (2020), 381-390. https://doi.org/10.1007/s40863-019-00147-2.

[6] A. Ardjouni, A. Djoudi, Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle, Ural Math. J. 5(1) 2019, 3-12. https://doi.org/10.15826/umj.2019.1.001.

[7] A. Bouakkaz, Positive periodic solutions for a class of first-order iterative differential equations with an application to an hematopoiesis model, Carpathian J. Math. 38 (2022), 347-355. https://doi.org/10.37193/CJM.2022.02.07.

[8] H. Boulares, A. Ardjouni, Y. Laskri, Positive solutions for nonlinear fractional differential equations, Positivity. 21 (2017), 1201-1212. https://doi.org/10.1007/s11117-016-0461-x.

[9] A. Guerfi, A. Ardjouni, Periodic solutions for totally nonlinear iterative differential equations, Bull. Int. Math. Virtual Inst. 12 (2022), 69-82. https://doi.org/10.7251/BIMVI2201069G.

[10] A. Guerfi, A. Ardjouni, Existence, uniqueness, continuous dependence and Ulam stability of mild solutions for an iterative Caputo fractional differential equation, CUBO, 24 (2022), 83-94. https://doi.org/10.4067/S0719-06462022000100083.

[11] A. Guerfi, A. Ardjouni, Periodic solutions for second order totally nonlinear iterative differential equations, J. Anal. 30 (2022), 353-367. https://doi.org/10.1007/s41478-021-00347-0.

[12] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[13] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, Nonlinear Anal. 74 (2011), 5975-5986. https://doi.org/10.1016/j.na.2011.05.074.

[14] D.R. Smart, Fixed Point Theorems, Cambridge Tracts in Mathematics, no. 66, Cambridge University Press, London-New York, 1974.

[15] S. Niyom, S. K. Ntouyas, S. Laoprasittichok, J. Tariboon, Boundary value problems with four orders of Riemann-Liouville fractional derivatives, Adv. Differ. Equ. 2016 (2016), 165. https://doi.org/10.1186/s13662-016-0897-0.

[16] H.Y. Zhao, J. Liu, Periodic solutions of an iterative functional differential equation with variable coefficients, Math. Meth. Appl. Sci. 40(1) (2017), 286-292. https://doi.org/10.1002/mma.3991.

[17] S. Zhang, Existence results of positive solutions to boundary value problem for fractional differential equation, Positivity 13(3) (2009), 583-599. https://doi.org/10.1007/s11117-008-2260-5.

[18] S. Zhang, The existence of a positive solution for a fractional differential equation, J. Math. Anal. Appl. 252 (2000), 804-812. https://doi.org/10.1006/jmaa.2000.7123.