Asia Pac. J. Math. 2023 **10**:9

NON-DEGENERATE ROTATIONAL SURFACES OF COORDINATE FINITE II-TYPE

HASSAN AL-ZOUBI

Department of Mathematics, Al-Zaytoonah University of Jordan, Jordan

dr.hassanz@zuj.edu.jo

Received Feb. 18, 2023

Abstract. We study rotational surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Furthermore, We classify non-degenerate rotational surfaces in $\mathbb{E}^3$ in terms of its finite Chen type Gauss map. We show that the only rotational surfaces in $\mathbb{E}^3$ whose Gauss map is of coordinate finite type are those of constant mean curvature.

2020 Mathematics Subject Classification. 53A05; 49Q05; 53A45.

Key words and phrases. rotational surfaces; Gauss map; surfaces of coordinate finite type; Beltrami operator.

1. Introduction

The idea of surfaces of finite type was born in the early 1970s by B-Y. Chen [12], and since then, this field of research has been spread widely and became a source of research for many geometers up to this moment.

Let M be a surface in the Euclidean 3-space $\mathbb{E}^3$ with $\boldsymbol{z}$ is its isometric immersion. Denote by Δ^I the Laplacian operator of M acting on the space of regular functions $C^\infty(S)$. For any function h referred to the system of coordinates of M, if $\Delta^I h = \lambda h$, then we say that h is an eigenfunction of Δ^I corresponding to the eigenvalue $\lambda \in R$. When $\lambda = 0$, then we say h is harmonic. Following this, we say that a surface M is of finite I-type, if its position vector $\boldsymbol{z}$ can be decomposed as a finite sum of eigenvectors of Δ^I of M, that is

$$\boldsymbol{z} = \boldsymbol{c} + \boldsymbol{z}_1 + \boldsymbol{z}_2 + \ldots + \boldsymbol{z}_k, \tag{1}$$

DOI: 10.28924/APJM/10-9

where $\boldsymbol{c}$ is a fixed vector, and

(2) $$\Delta^I \boldsymbol{z}_i = \lambda_i \boldsymbol{z}_i, \quad i = 1, \ldots, k,$$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigenvalues of the operator Δ^I.

A special case followed in [23] when T. Takahashi showed that a surface M for which $\Delta^I \boldsymbol{z} = \lambda \boldsymbol{z}$ i.e. for which all coordinate functions are eigenfunctions of Δ^I with the same eigenvalue λ are either the minimals with $\lambda = 0$ or the spheres with eigenvalue $\lambda = 2$. As a generalization of this, one can study surfaces in $\mathbb{E}^3$ whose position vector $\boldsymbol{z}$ satisfies

(3) $$\Delta^I \boldsymbol{z} = P\boldsymbol{z},$$

where $P \in \mathbb{R}^{3\times 3}$. Surfaces satisfying the above condition are called of coordinate finite type.

This kind of research can be also applied to any smooth map, not only for the position vector of the surface, i.e., the normal vector $\boldsymbol{\xi}$ of a surface [13].

The theory of Gauss map is always one of the interesting topics in a Euclidean and pseudo-Euclidean space, see for example [7,14–17,19]. Considering condition (3), we ask:

"Find all surfaces in $\mathbb{E}^3$ whose normal vector $\boldsymbol{\xi}$ satisfies

(4) $$\Delta^I \boldsymbol{\xi} = P\boldsymbol{\xi}, \quad P \in \mathbb{R}^{3\times 3}\text{''}.$$

According to relation (4), tubes, quadric surfaces, ruled surfaces, cyclides of Dupin, helicoidal surfaces, and surfaces of revolution were investigated in [2,4,5,9–11,18] respectively. In [6] authors studied the class of tubular surfaces according to (4) by applying the Laplacian to the second fundamental form of M. Recently, in [3] similar study has been done for the classes of ruled and quadric surfaces.

2. Fundamentals

Let M be a regular surface in $\mathbb{E}^3$ given by a patch $\boldsymbol{z} = \boldsymbol{z}(u^1, u^2)$ on a region $U := (a, b) \times \mathbb{R}$ of $\mathbb{R}^2$. We define the second fundamental form II of M by

(5) $$II = \mathfrak{b}_{\mathfrak{ij}} du^i du^j.$$

For any function h referred to the system of coordinates of M, the second Laplace operator is defined by

(6) $$\Delta^{II} h = -\mathfrak{b}^{\mathfrak{ij}} \nabla^{II}_i h_{/j},$$

where ∇_i^{II} is the covariant derivative in the u^i direction regarding the second fundamental form [1].

In [8] authors classified the rotational surfaces in the Lorentz-Minkowski space in terms of coordinate finite type regarding the second fundamental form. An interesting study can be drawn by applying relation (6) using the definition of the fractional vector operators [20].

Our main result is the following

Theorem 1. *The only rotational surfaces in $\mathbb{E}^3$ whose Gauss map of coordinate finite type are those whose mean curvature H is constant.*

3. Non-degenerate rotational surfaces

Let $\mathcal{C}$ be a regular curve lies on the x_1x_3-plane parametrized by

$$\boldsymbol{r}(s) = (a(s), 0, b(s)), \quad s \in J, (J \subset \mathbb{R}),$$

where a, b are regular functions and $a > 0$. When $\mathcal{C}$ is revolved about the x_3-axis, the resulting point set M is called the rotational surface or surface of revolution generated by $\mathcal{C}$ which is called the profile curve of M. In this case, the x_3-axis is called the axis of revolution of M. Then the position vector of M is given by [18]

$$\boldsymbol{r}(s,t) = \big(a(s)\cos t, a(s)\sin t, b(s)\big), \quad s \in J, \quad 0 \le t < 2\pi. \tag{7}$$

Suppose that $\mathcal{C}$ is arc-length parametrized, then

$$(a\prime)^2 + (b\prime)^2 = 1, \tag{8}$$

where $\prime := \frac{d}{ds}$. Moreover, if $a\prime b\prime = 0$, then one of the functions a or b is constant, and M would be part of a plane or a circular cylinder. In this case, M would consist only of parabolic points, a case that has been excluded [22].

The partial derivatives of (7) are

$$\boldsymbol{r_s} = (a\prime(s)\cos t, a\prime(s)\text{sin}t, b\prime(s)),$$

and

$$\boldsymbol{r_t} = (-a(s)\sin t, b(s)\cos t, 0)$$

Computing the components $\mathfrak{g}_{ij}$ of I, we get

$$\mathfrak{g}_{11} =<\boldsymbol{r_s}, \boldsymbol{r_s}>= 1, \quad \mathfrak{g}_{12} =<\boldsymbol{r_s}, \boldsymbol{r_t}>= 0, \quad \mathfrak{g}_{22} =<\boldsymbol{r_t}, \boldsymbol{r_t}>= a^2.$$

Let κ denotes the curvature of the curve $\mathcal{C}$. Then the Gauss and the mean curvature of M are respectively

$$K = \frac{\kappa b\prime}{a} = -\frac{a\prime\prime}{a}, \quad 2H = \kappa + \frac{b\prime}{a}.$$

The Gauss map $\boldsymbol{\xi}$ of M is given by

$$\boldsymbol{\xi}(s,t) = \frac{\boldsymbol{r_s} \times \boldsymbol{r_t}}{\sqrt{\mathfrak{g}}} = \big(- b\prime \cos t, -b\prime \sin t, a\prime \big), \tag{9}$$

where $\mathfrak{g} := \det(\mathfrak{g}_{\mathrm{ij}})$.

The components $\mathfrak{b}_{ij}$ of II are given as follows

$$\mathfrak{b}_{11} = \kappa, \quad \mathfrak{b}_{12} = 0, \quad \mathfrak{b}_{22} = ab\prime.$$

The Laplacian Δ in terms of local coordinates (s,t) of M can be expressed as follows

$$\Delta = -\frac{1}{\kappa}\frac{\partial^2}{\partial s^2} - \frac{1}{ab\prime}\frac{\partial^2}{\partial t^2} + \frac{1}{2}\left(\frac{\kappa\prime}{\kappa^2} - \frac{a\prime b\prime + \kappa a a\prime}{\kappa a b\prime}\right)\frac{\partial}{\partial s}. \tag{10}$$

On account of (8) we put

$$a\prime = \cos\beta, \quad b\prime = \sin\beta, \tag{11}$$

where $\beta = \beta(s)$. Then $\kappa = \beta\prime$ and the parametric representation (9) of the normal vector $\boldsymbol{\xi}$ of M becomes

$$\boldsymbol{\xi}(s,t) = \big(- sin\beta cost, -sin\beta sint, cos\beta \big). \tag{12}$$

Also relation (10) takes the following form

$$\begin{aligned}\Delta = &-\frac{1}{\beta\prime}\frac{\partial^2}{\partial s^2} - \frac{1}{asin\beta}\frac{\partial^2}{\partial t^2}\\ &+\frac{1}{2}\left(\frac{\beta\prime\prime}{\beta\prime^2} - \frac{cos\beta sin\beta + a\beta\prime cos\beta}{a\beta\prime sin\beta}\right)\frac{\partial}{\partial s}.\end{aligned} \tag{13}$$

The mean and the Gaussian curvature become

$$2H = \beta\prime + \frac{\sin\beta}{a}, \tag{14}$$

$$K = \frac{\beta\prime \sin\beta}{a}. \tag{15}$$

Let (ξ_1, ξ_2, ξ_3) be the coordinate functions of (9). From (13), one can find

$$\Delta\xi_1 = \left(\frac{\beta\prime\prime cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{sin\beta} - \beta\prime sin\beta\right)cost, \tag{16}$$

$$\Delta\xi_2 = \left(\frac{\beta\prime\prime cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{sin\beta} - \beta\prime\sin\beta\right)\sin t, \tag{17}$$

$$\Delta\xi_3 = \frac{\beta\prime\prime}{2\beta\prime}sin\beta + \beta\prime cos\beta + Hcos\beta. \tag{18}$$

Now we will study surfaces of revolution in $\mathbb{E}^3$ whose Gauss map satisfies the relation

$$\Delta\boldsymbol{\xi} = P\boldsymbol{\xi}, \tag{19}$$

where $P \in \mathbb{R}^{3\times3}$. We denote by $\lambda_{uv}, u, v = 1, 2, 3$ be the entries of the matrix P. From (19) and taking into account relations (16-18), we have

$$\begin{aligned} &\left(\frac{\beta\prime\prime\cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta\prime sin\beta\right)cost \\ = &-\mu_{11}sin\beta\cos t - \mu_{12}sin\beta\sin t + \mu_{13}cos\beta, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left(\frac{\beta\prime\prime}{2\beta\prime}cos\beta - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta\prime sin\beta\right)sint \\ = &-\mu_{21}sin\beta cost - \mu_{22}sin\beta\sin t + \mu_{23}cos\beta, \end{aligned} \tag{21}$$

$$\begin{aligned} &\frac{\beta\prime\prime}{2\beta\prime}sin\beta + \beta\prime cos\beta + H\cos\beta \\ = &\ \mu_{31}sin\beta\cos t - \mu_{32}sin\beta sint + \mu_{33}cos\beta. \end{aligned} \tag{22}$$

Differentiating (20) and (21) twice with respect to v we obtain

$$\begin{aligned} &\left(\frac{\beta\prime\prime\cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta\prime\sin\beta\right)\cos t \\ = &-\mu_{11}\sin\beta\cos t - \mu_{12}\sin\beta\sin t, \end{aligned}$$

$$\begin{aligned} &\left(\frac{\beta\prime\prime\cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta\prime\sin\beta\right)\sin t \\ = &-\mu_{21}\sin\beta\cos t - \mu_{22}\sin\beta\sin t. \end{aligned}$$

Thus we have $\mu_{13} = \mu_{23} = 0$. From (22) it can be easily verified that $\mu_{31} = \mu_{32} = 0$. Also the functions $\sin t$, $\cos t$ are linearly independent of the variable t, so finally we get $\mu_{11} = \mu_{22}$ and $\mu_{12} = \mu_{21} = 0$. For simplicity we put $\mu_{11} = \mu_{22} = m$ and $\mu_{33} = n$. Then the system of equations (20), (21) and (22) reduces to the following two equations

$$\frac{\beta\prime\prime\cos\beta}{2\beta\prime} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta\prime\sin\beta = -m\sin\beta, \tag{23}$$

$$(24) \qquad \frac{\beta'' \sin\beta}{2\beta'} + \beta' \cos\beta + H\cos\beta = n\cos\beta.$$

We obtain the following cases:

Case I. $m = n = 0$. Equations (23) and (24) become

$$(25) \qquad \frac{\beta'' \cos\beta}{2\beta'} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta' \sin\beta = 0,$$

$$(26) \qquad \frac{\beta'' \sin\beta}{2\beta'} + \beta' \cos\beta + H\cos\beta = 0.$$

We multiply (25) by $\sin\beta$ and (26) by $-\cos\beta$ then we add the resulting of these two equations it follows $\beta' + \frac{\sin\beta}{a} = 2H = 0$. Consequently, M, being a minimal surface of revolution, is a catenoid.

Case II. $m = n \neq 0$. Equations (23) and (24) become

$$(27) \qquad \frac{\beta'' \cos\beta}{2\beta'} - \frac{1}{a} + \frac{H\cos^2\beta}{\sin\beta} - \beta' \sin\beta = -m\sin\beta,$$

$$(28) \qquad \frac{\beta'' \sin\beta}{2\beta'} + \beta' \cos\beta + H\cos\beta = m\cos\beta.$$

We multiply (27) by $sin\beta$ and (28) by $-cos\beta$ then we add the resulting of these two equations it follows $\beta' + \frac{\sin\beta}{a} = 2H = m$. Consequently, M has a none zero constant mean curvature.

Case III. $m \neq 0, n = 0$. Equations (23) and (24) become

$$(29) \qquad \frac{\beta'' \cos\beta}{2\beta'} - \frac{1}{a} + \frac{H\cos^2\beta}{sin\beta} - \beta' \sin\beta = -msin\beta,$$

$$(30) \qquad \frac{\beta'' \sin\beta}{2\beta'} + \beta' \cos\beta + Hcos\beta = 0.$$

We multiply (29) by $sin\beta$ and (30) by $-cos\beta$ then we add the resulting of these two equations which follow

$$\beta' + \frac{sin\beta}{a} = m\sin^2\beta,$$

or

$$(31) \qquad \beta' = m\sin^2\beta - \frac{sin\beta}{a}.$$

Differentiating the last equation we have

$$(32) \qquad \beta'' = 2m\beta' sin\beta cos\beta - \frac{\beta' cos\beta}{a} + \frac{sin\beta cos\beta}{a^2}.$$

On account of (3) relation (32) becomes

$$\beta'' = \big(2m^2 \sin^2 \beta - \frac{3msin\beta}{a} + \frac{2}{a^2}\big) sin\beta cos\beta. \tag{33}$$

Taking into account (14) We write (30) as follows

$$\beta'' = -\frac{\beta'}{sin\beta}\big(3\beta' cos\beta + \frac{1}{a} sin\beta cos\beta\big), \tag{34}$$

where because of (3) becomes

$$\beta'' = \big(\frac{1}{a} - msin\beta\big)\big(3msin\beta - \frac{2}{a}\big) sin\beta cos\beta. \tag{35}$$

From (33) and (35) we have

$$5m^2 a^2 \sin^2 \beta + 8ma \sin \beta + 4 = 0. \tag{36}$$

The last equation is a second-order polynomial of the variable $ma \sin \beta$ which has no solution. So in this case condition (19) cannot be satisfied.

Case IV. $n \neq 0, m = 0$. Equations (23) and (24) become

$$\frac{\beta'' \cos \beta}{2\beta'} - \frac{1}{a} + \frac{H \cos^2 \beta}{sin\beta} - \beta' \sin \beta = 0, \tag{37}$$

$$\frac{\beta'' sin\beta}{2\beta'} + \beta' cos\beta + H \cos \beta = n \cos \beta. \tag{38}$$

We multiply (37) by $sin\beta$ and (38) by $-cos\beta$ then we add the resulting of these two equations which follow

$$\beta' + \frac{1}{a} sin\beta = ncos^2\beta,$$

or

$$\beta' = n \cos^2 \beta - \frac{1}{a} sin\beta. \tag{39}$$

Differentiating the last equation we have

$$\beta'' = -2n\beta' sin\beta cos\beta - \frac{}{a}\beta' cos\beta + \frac{1}{a^2} sin\beta cos\beta. \tag{40}$$

From (39) and (40) relation (38) becomes

$$n^2(5 \sin^4 \beta - 6 \sin^2 \beta + 1)a^2 + 4n \sin \beta (2 \sin^2 \beta - 1)a + 4 \sin^2 \beta = 0. \tag{41}$$

Equation (41) as a second-degree polynomial of a, cannot be satisfied for all values of $a(s)$ unless only when all coefficients equal zero. A contradiction since we will have $n = 0$.

Case IV. $n \neq 0, m \neq 0, m \neq n$.

We multiply (23) by $sin\beta$ and (24) by $-cos\beta$ then we add the resulting of these two equations it follows

$$\beta\prime + \frac{1}{a}sin\beta = msin^2\beta + n\cos^2\beta,$$

or

$$\beta\prime = m\sin^2\beta + n\cos^2\beta - \frac{\sin\beta}{a}. \tag{42}$$

Differentiating the last equation we have

$$\beta\prime\prime = 2(m-n)\beta\prime sin\beta cos\beta - \frac{1}{a}\beta\prime\cos\beta + \frac{1}{a^2}sin\beta cos\beta. \tag{43}$$

On account of (42) and (43) relation (24) becomes

$$\begin{aligned}&(5(m-n)^2\sin^4\beta + 6n(m-n)\sin^2\beta + n^2)a^2\\&-4sin\beta(2(m-n)\sin^2\beta + n)a + 4\sin^2\beta = 0.\end{aligned} \tag{44}$$

In the same way, it can be easily verified that equation (44), as a second-degree polynomial of a, cannot be satisfied for all values of $a(s)$ unless only when all coefficients equal zero. A contradiction, and so our theorem is proved.

References

[1] H. Al-Zoubi, A. Dababneh, M. Al-Sabbagh, Ruled surfaces of finite II-type, WSEAS Trans. Math. 18 (2019), 1-5.

[2] H. Al-Zoubi, T. Hamadneh, M. Abu Hammad, M. Al-Sabbagh, Tubular surfaces of finite type Gauss map, J. Geom. Graph. 25 (2021), 45-52.

[3] H. Al-Zoubi, T. Hamadneh, M. Abu Hammad, M. Al-Sabbagh, M. Ozdemir, Ruled and quadric surfaces satisfying $\triangle^{II}\mathbf{N} = \lambda\mathbf{N}$, Symmetry. 15 (2023), 300. https://doi.org/10.3390/sym15020300.

[4] H. Al-Zoubi, H. Alzaareer, T. Hamadneh, M. Al Rawajbeh, Tubes of coordinate finite type Gauss map in the Euclidean 3-space, Indian J. Math. 62 (2020), 171-182.

[5] H. Al-Zoubi, T. Hamadneh, A. Alkhatib, Quadric surfaces of coordinate finite type Gauss map in the Euclidean 3-space, Indian J. Math. 64 (2022), 385–399.

[6] H. Al-Zoubi, T. Hamadneh, H. Alzaareer, M. Al-Sabbagh, Tubes in the Euclidean 3-space with coordinate finite type Gauss map, in: 2021 International Conference on Information Technology (ICIT), IEEE, Amman, Jordan, 2021: pp. 85-88. https://doi.org/10.1109/ICIT52682.2021.9491118.

[7] H. Al-Zoubi, M. Al-Sabbagh, Anchor rings of finite type Gauss map in the Euclidean 3-space, Int. J. Math. Comput. Methods. 5 (2020), 9-13.

[8] H. Al-Zoubi, A.K. Akbay, T. Hamadneh, M. Al-Sabbagh, Classification of surfaces of coordinate finite type in the Lorentz-Minkowski 3-space, Axioms. 11 (2022), 326. https://doi.org/10.3390/axioms11070326.

[9] C. Baikoussis, D.E. Blair, On the Gauss map of ruled surfaces, Glasgow Math. J. 34 (1992), 355-359. https://doi.org/10.1017/s0017089500008946.

[10] C. Baikoussis, F. Denever, P. Emprechts, L. Verstraelen, On the Gauss map of the cyclides of Dupin, Soochow J. Math. 19 (1993), 417-428.

[11] C. Baikoussis, L. Verstraelen, On the Gauss map of helicoidal surfaces, Rend. Sem. Mat. Messina Ser. II. 2 (1993), 31-42.

[12] B.Y. Chen, Total mean curvature and submanifolds of finite type, Second edition, World Scientific, Singapore, (2015).

[13] B. Chen, P. Piccinni, Submanifolds with finite type Gauss map, Bull. Austral. Math. Soc. 35 (1987), 161-186. https://doi.org/10.1017/s0004972700013162.

[14] M.K. Choi, D.S. Kim, Y.H. Kim, Helicoidal surfaces with pointwise 1-type gauss map, J. Korean Math. Soc. 46 (2009), 215-223. https://doi.org/10.4134/JKMS.2009.46.1.215.

[15] M. Choi, D.S. Kim, Y.H. Kim, D.W. Yoon, Circular cone and its Gauss map, Colloq. Math. 129 (2012), 203-210. https://doi.org/10.4064/cm129-2-4.

[16] S.M. Choi, On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space, Tsukuba J. Math. 19 (1995), 351–367. https://www.jstor.org/stable/43685932.

[17] S.M. Choi, On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space, Tsukuba J. Math. 19 (1995), 285–304. https://www.jstor.org/stable/43685928.

[18] F. Dillen, J. Pass, and L. Verstraelen, On the Gauss map of surfaces of revolution, Bull. Inst. Math. Acad. Sinica 18 (1990), 239-246.

[19] U. Dursun, Flat surfaces in the Euclidean space E^3 with pointwise 1-type Gauss map, Bull. Malays. Math. Sci. Soc. (2) 33 (2010), 469–478.

[20] M. Mhailan, M. Abu Hammad, M. Al Horani, R. Khalil, On fractional vector analysis, J. Math. Comput. Sci. 10 (2020), 2320-2326. https://doi.org/10.28919/jmcs/4863.

[21] Z.M. Nemrawi, Levels of understanding of trigonometric functions (sine, cosine, tangent) by mathematics students at Al-Zaytoonah University of Jordan, Sohag Univ. Int. J. Educ. Res. 74 (2020), 176-203.

[22] S. Stamatakis, H. Al-Zoubi, Surfaces of revolution satisfying $\triangle^{III}\mathbf{x} = A\mathbf{x}$, J. Geom. Graph. 14 (2010), 181-186.

[23] T. Takahashi, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan 18 (1966), 380-385. https://doi.org/10.2969/jmsj/01840380.