Asia Pac. J. Math. 2024 **11**:12

EXPONENTIABLE OBJECT IN THE CATEGORY L-TOP

RANA NOOR*

Department of Mathematics, Banaras Hindu University, Varanasi-221 005, India

*Corresponding author: rananoor@bhu.ac.in

Received Oct. 27, 2023

Abstract. In this paper exponentiable object in the category of L-topological spaces have been characterized by using Sierpinski L-topological space.

2020 Mathematics Subject Classification. 54A40; 54B30.

Key words and phrases. splitting L-topology; conjoining L-topology; exponentiable L-topological space; Sierpinski L-topological space.

1. Introduction

The usual two-point Sierpinski topological space 2_S plays a significant role for characterizing the exponentiable object in the category Top of topological spaces (cf. [4,5]). Escardó and Heckmann [4] proved a result about function spaces in topology, which states that, given a topological space X, a topology τ on the set $C(X, 2_S)$ of all continuous maps from X to 2_S, is exponential if and only if for every topological space Y, the topology on the set $C(X, Y)$ of all continuous maps from X to Y, generated by the sets $N(H, V) = \{f \in C(X, Y) \mid \chi_{f^{-1}(V)} \in H\}$, where $H \in \tau$, V is open in Y and $\chi_{f^{-1}(V)}$ is the characteristic map of $f^{-1}(V)$, is exponential.

The counterpart of 2_S for L-topological spaces has been introduced in [10] as Sierpinski L-topological space. This paper provides a characterization of exponentiable object in the category L-Top of L-topological spaces by using Sierpinski L-topological space on the lines of Escardó and Heckmann [4].

2. Preliminaries

The necessary category-theoretic notions used in this paper can be ascertain from [1] and [11].

DOI: 10.28924/APJM/11-12

Definition 2.1. *A complete lattice L is called a* Frame *if it satisfies the infinite distributive law: $a \wedge (\vee b_i) = \vee(a \wedge b_i)$, for all $a \in L$ and $\{b_i \mid i \in I\} \subseteq L$. Furthermore, a* Frame map *between two frames is a lattice homomorphism which preserves finite meets and arbitrary joins.*

Definition 2.2. *A subset $A \subseteq L$ of a frame L is called a* subframe *if A is a frame under the partial order of L.*

Throughout this paper, L denotes a fixed frame with 0 and 1 being its least and largest elements respectively.

For a given set X, L^X is also a frame under the partial order induced by L. The least and largest element of L^X are the 0- and 1-valued constant maps from X to L denoted as $\bar{0}$ and $\bar{1}$ respectively. Members of L^X are known as *L-sets* in X [6].

Each map $f : X \to Y$ induces functions $f^{\leftarrow} : L^Y \to L^X$ given by $f^{\leftarrow}(\nu) = \nu \circ f$, for every $\nu \in L^Y$.

Definition 2.3. [7]

(1) *A family τ of L-sets in a set X is called an L-*topology *on X, and the pair (X, τ) an L-*topological space, *if τ is a subframe of L^X. Members of τ are known as* open *L-sets in X.*

(2) *A map $f : (X, \tau) \to (Y, \delta)$ between L-topological spaces is called* continuous *if $f^{\leftarrow}(\nu) \in \tau$, for every $\nu \in \delta$.*

Let L-Top denote the category of all L-topological spcaes and their continuous maps.

Definition 2.4. (1) *A subcollection β of an L-topology τ on X is said to be a* base *for τ (or for (X, τ)) if every member of τ is a join of some members of β.*

(2) *A subcollection β of an L-topology τ on X is said to be a* subbase *for τ (or for (X, τ)) if all finite meets of members of β form a base for τ.*

Remark 2.1. (1) *Let ζ be a family of L-sets in a set X. Let ξ be the collection of all finite meet of members of ζ. It can be verified that the collection of all join of members of ξ is an L-topology on X, to be denoted $\langle\zeta\rangle$, for which ζ is a subbase.*

(2) *Given a map $f : (X, \tau) \to (Y, \delta)$ between two L-topological spaces and a subbase β of (Y, δ), it can be easily verified that f is continuous if and only if $f^{\leftarrow}(\nu) \in \tau$, for every $\nu \in \beta$.*

Consider the frame L. Then $\langle\{id_L\}\rangle$, where id_L is the identity map on L. Call the L-topological space $(L, \langle\{id_L\}\rangle)$ as the $Sierpinski\ L\text{-}topological space$ and denote it as L_S (cf. [10]).

Here we recall a result from [10] which will be used further.

Proposition 2.1. *Let $(X, \tau) \in obL$-Top. Then $\mu \in \tau$ if and only if $\mu : (X, \tau) \to L_S$ is continuous.*

3. Exponentiable Topological Space

Given sets X, Y, Z and a map $g : Z \times X \to Y$, let $\bar{g} : Z \to Y^X$ be the map defined as $\bar{g}(z)(x) = g(z, x)$, for every $z \in Z$ and for every $x \in X$.

For topological spaces X and Y, let $C(X, Y)$ denote the set of all continuous maps from X to Y.

Definition 3.1. [4,5] *Given topological spaces X, Y, a topology on $C(X, Y)$ is called:*

(1) splitting *if for any topological space Z, the continuity of a map $g : Z \times X \to Y$ implies the continuity of the map $\bar{g} : Z \to C(X, Y)$.*

(2) conjoining *if for any topological space Z and a map $g : Z \times X \to Y$, the continuity of the map $\bar{g} : Z \to C(X, Y)$ implies the continuity of $g : Z \times X \to Y$.*

(3) exponential *if it is both splitting and conjoining.*

Definition 3.2. [4,5] *A topological space X is called* exponentiable *if the set $C(X, Y)$ admits an exponential topology, for every topological space Y.*

Let X, Y be topological spaces and $\mathcal{T}$ be a topology on $C(X, 2_S)$. Let $\mathcal{T}_Y^*$ denote the topology on $C(X, Y)$, generated by the sets $N(H, V) = \{f \in C(X, Y) \mid \chi_{f^{-1}(V)} \in H\}$, where $H \in \mathcal{T}$, V is open in Y and $\chi_{f^{-1}(V)}$ is the characteristic map of $f^{-1}(V)$.

Theorem 3.1. [4,5] *Let X be a topological space. Then a topology $\mathcal{T}$ on $C(X, 2_S)$ is:*

(1) *splitting iff $\mathcal{T}_Y^*$ is a splitting topology on $C(X, Y)$, for every topological space Y.*

(2) *conjoining iff $\mathcal{T}_Y^*$ is a conjoining topology on $C(X, Y)$, for every topological space Y.*

(3) *exponential iff $\mathcal{T}_Y^*$ is an exponential topology on $C(X, Y)$, for every topological space Y.*

Remark 3.1. *In view of Theorem 3.1, we can say that a topological space X is exponentiable if and only if the set $C(X, 2_S)$ admits an exponential topology.*

4. Exponentiable L-Topological Space

Function spaces in fuzzy topology have been studied by several authors, e.g., Alderton [2], Dang and Behera [3], Jäger [8], and Kohli and Prasannan [9]. In this section, the analogous result of Theorem 3.1 has been proved for L-topological spaces.

For L-topological spaces X and Y, let $C_L(X, Y)$ denote the set of all continuous maps from X to Y.

Definition 4.1. *Given L-topological spaces X, Y, an L-topology on $C_L(X, Y)$ is called:*

(1) splitting *if for any L-topological space Z, the continuity of a map $g : Z \times X \to Y$ implies the continuity of the map $\bar{g} : Z \to C_L(X, Y)$.*

(2) conjoining *if for any L-topological space Z and a map* $g : Z \times X \to Y$, *the continuity of the map* $\bar{g} : Z \to C_L(X, Y)$ *implies the continuity of* $g : Z \times X \to Y$.

(3) exponential *if it is both splitting and conjoining.*

Definition 4.2. *An L-topological space X is called* exponentiable *if the set* $C_L(X, Y)$ *admits an exponential L-topology, for every L-topological space Y.*

Let τ be an L-topology on $C_L(X, L_S)$. Let $O \in \tau$ and v be an open L-set in Y. So, $v : Y \to L_S$ is continuous. Define $[O, v] : C_L(X, Y) \to L$ as $[O, v](f) = O(v \circ f)$, for every $f \in C_L(X, Y)$. Let τ^*_Y denote the L-topology on $C_L(X, Y)$, whose subbase is $\{[O, v] \mid O \in \tau, v \text{ is an open } L\text{-set in } Y\}$.

Proposition 4.1. *Let X be an L-topological space. Then an L-topology* τ *on* $C_L(X, L_S)$ *is splitting iff* τ^*_Y *is a splitting L-topology on* $C_L(X, Y)$, *for every L-topological space Y.*

Proof: Let τ be a splitting L-topology on $C_L(X, L_S)$. Let Z be an L-topological space and let $g : Z \times X \to Y$ be a continuous map. We have to show that the map $\bar{g} : Z \to C_L(X, Y)$ is continuous. Let $O \in \tau$ and v be an open L-set in Y. Then for every $z \in Z$, $\bar{g}^{\leftarrow}([O, v])(z) = ([O, v] \circ \bar{g})(z) = [O, v](\bar{g}(z)) = O(v \circ \bar{g}(z))$. As v is an open L-set in Y, $g^{\leftarrow}(v)$ is an open L-set in $Z \times X$. So, $v \circ g : Z \times X \to L_S$ is continuous, whereby, $\overline{v \circ g} : Z \to C_L(X, L_S)$ is continuous. Hence $(\overline{v \circ g})^{\leftarrow}(O)$ is an open L-set in Z. For every $z \in Z$ and for every $x \in X$, $(\overline{v \circ g})(z)(x) = (v \circ g)(z, x) = v(g(z, x)) = v(\bar{g}(z)(x)) = (v \circ \bar{g}(z))(x)$, implying that $(\overline{v \circ g})(z) = v \circ \bar{g}(z)$, for every $z \in Z$. Now, for every $z \in Z$, $(\overline{v \circ g})^{\leftarrow}(O)(z) = (O \circ (\overline{v \circ g}))(z) = O((\overline{v \circ g})(z)) = O(v \circ \bar{g}(z)) = \bar{g}^{\leftarrow}([O, v])(z)$. So, $(\overline{v \circ g})^{\leftarrow}(O) = \bar{g}^{\leftarrow}([O, v])$, implying that $\bar{g}^{\leftarrow}([O, v])$ is an open L-set in Z. Hence $\bar{g}$ is continuous.

Conversely, let τ^*_Y be a splitting L-topology on $C_L(X, Y)$, for every L-topological space Y. So, $\tau^*_{L_S}$ is a splitting L-topology on $C_L(X, L_S)$. To show τ is a splitting L-topology on $C_L(X, L_S)$. It is easy to see that $\tau^*_{L_S} = \tau$ on $C_L(X, L_S)$ and so τ is a splitting L-topology on $C_L(X, L_S)$. □

Proposition 4.2. *Let X be an L-topological space. Then an L-topology* τ *on* $C_L(X, L_S)$ *is conjoining iff* τ^*_Y *is a conjoining L-topology on* $C_L(X, Y)$, *for every L-topological space Y.*

Proof: Let τ be a conjoining L-topology on $C_L(X, L_S)$. We show that τ^*_Y is a conjoining L-topology on $C_L(X, Y)$. Let Z be an L-topological space and $g : Z \times X \to Y$ be a map such that $\bar{g} : Z \to C_L(X, Y)$ is continuous. We have to show that the map $g : Z \times X \to Y$ is continuous. Let v be an open L-set in Y. Define a map $\hat{v} : C_L(X, Y) \to C_L(X, L_S)$ as $\hat{v}(f) = v \circ f$, for every $f \in C_L(X, Y)$. Let $O \in \tau$. Then for every $f \in C_L(X, Y)$, $(\hat{v}^{\leftarrow}(O))(f) = O(\hat{v}(f)) = O(v \circ f) = [O, v](f)$, implying that $\hat{v}^{\leftarrow}(O) = [O, v]$. Hence $\hat{v}^{\leftarrow}(O)$ is open in $C_L(X, Y)$, whereby $\hat{v}$ is continuous. Consider the continuous map $\hat{v} \circ \bar{g} : Z \to C_L(X, L_S)$. Then for every $z \in Z$, $(\hat{v} \circ \bar{g})(z) = \hat{v}(\bar{g}(z)) = v \circ \bar{g}(z) = (\overline{v \circ g})(z)$, showing that $\hat{v} \circ \bar{g} = \overline{v \circ g}$. Hence $\overline{v \circ g} : Z \to C_L(X, L_S)$ is continuous, whereby $v \circ g : Z \times X \to L_S$ is continuous.

Thus $g^{\leftarrow}(v)$ is an open L-set in $Z \times X$, showing that g is continuous.
Conversely, let τ_Y^* be a conjoining L-topology on $C_L(X, Y)$, for every L-topological space Y. So, $\tau_{L_S}^*$ is a conjoining L-topology on $C_L(X, L_S)$. To show τ is a conjoining L-topology on $C_L(X, L_S)$. It is easy to see that $\tau_{L_S}^* = \tau$ on $C_L(X, L_S)$ and so τ is a conjoining L-topology on $C_L(X, L_S)$. □

By Proposition 4.1 and Proposition 4.2, we have the following Theorem.

Theorem 4.1. *Let X be an L-topological space. Then an L-topology τ on $C_L(X, L_S)$ is exponential iff τ_Y^* is an exponential L-topology on $C_L(X, Y)$, for every L-topological space Y.*

The following result will characterize the exponentiable object in the category L-Top.

Theorem 4.2. *An L-topological space X is exponentiable iff the set $C_L(X, L_S)$ admits an exponential L-topology.*

Acknowledgment

The author would like to thank Institute of Eminence (IoE), Banaras Hindu University, Varanasi, India, for providing financial support through Seed Grant.

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and concrete categories, Wiley, New York, 1990.

[2] I.W. Alderton, Function spaces in fuzzy topology, Fuzzy Sets Syst. 32 (1989), 115–124. https://doi.org/10.1016/0165-0114(89)90092-4.

[3] S. Dang, A. Behera, On fuzzy compact-open topology, Fuzzy Sets Syst. 80 (1996), 377–381. https://doi.org/10.1016/0165-0114(95)00138-7.

[4] M. Escardó, R. Heckmann, Topologies on spaces of continuous functions, Topol. Proc. 26 (2001-2002), 545–564.

[5] M. Escardó, Synthetic topology, Elec. Notes Theor. Comp. Sci. 87 (2004), 21–156. https://doi.org/10.1016/j.entcs.2004.09.017.

[6] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145–174. https://doi.org/10.1016/0022-247x(67)90189-8.

[7] U. Höhle, Locales and L-topologies, in: Mathematik-Arbeitspapiere 48, Univ. Bremen (1997), 223–250.

[8] G. Jäger, On fuzzy function spaces, Int. J. Math. Math. Sci. 22 (1999), 727–737. https://doi.org/10.1155/s0161171299227275.

[9] J.K. Kohli, A.R. Prasannan, Fuzzy topologies on function spaces, Fuzzy Sets Syst. 116 (2000), 415–420. https://doi.org/10.1016/s0165-0114(98)00383-2.

[10] R. Noor, A.K. Srivastava, The categories L-Top_0 and L-Sob as epireflective hulls, Soft Comp. 18 (2014), 1865–1871. https://doi.org/10.1007/s00500-014-1315-8.

[11] G. Preuss, Theory of topological structures: an approach to categorical topology, D. Reidel Publishing Company, Dordrecht, 1988.