Asia Pac. J. Math. 2024 **11**:15

UNILATERAL PROBLEM FOR NON-COERCIVE NEUMANN ELLIPTIC EQUATIONS IN $p(x)-$ANISOTROPIC SOBOLEV SPACES

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Received Oct. 21, 2023

Abstract. In this paper, we investigate the existence of entropy solutions for the unilateral problem associated to the Neumann degenerate anisotropic elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{r(x)-2}u = f(x,u) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u) n_i = g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

where the right-hand side term $f(x,s)$ satisfies only some growth condition, while $g(x)$ belongs to $L^1(\partial\Omega)$.

2020 Mathematics Subject Classification. 35J62, 35J20.

Key words and phrases. unilateral problems; anisotropic Sobolev space with variable exponents; degenerate coercivity; Neumann boundary conditions.

1. Introduction

Let Ω be a bounded open subset of $\mathbf{R}^N$ $(N \geq 2)$.

A multitude of obstacle problem models have been studied : In [28], Porretta have studied the existence of solution for the unilateral problem associated to the elliptic equation

$$\begin{cases} Au + g(u)|\nabla u|^p = \mu & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where the right-hand side is a bounded Radon measure on Ω. For more results regarding unilateral problems, we refer the reader to [2], [3], [7] and [20].

DOI: 10.28924/APJM/11-15

Akdim et al. have established in [4] the existence of solution for the unilateral problem associated to the degenerate quasilinear elliptic equation

$$\begin{cases} Au + g(x,u,\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where A is a Leray-Lions operator acted from $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega)$, and the nonlinear term $g(x,s,\xi)$ satisfies some growth and sign conditions.

In the recent years, there has been a growing interest in the study of elliptic and parabolic problems in the anisotropic variable exponents Sobolev spaces. The advancement of a theory, primarily attributed to Ruzicka [29], aimed at describing the behavior of electrorheological fluids, which belong to a significant category of non-Newtonian fluids, that greatly energized the ongoing effort to explore and make sense of nonlinear PDE's involving variable exponents. There are other application areas like image processing [21], elasticity [1], the flow in porous media [10], and mathematical problems in the field of calculus of variations involving variational integrals with nonstandard growth [33].

Recently, Ayadi, has studied in [11] the quasilinear anisotropic elliptic equation

$$\begin{cases} -\sum_{i=1}^{N} D^i\Big(\dfrac{a_i(x,\nabla u)}{(1+|u|)^{\gamma_i(x)}}\Big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

he has proved the existence of entropy solutions to the obstacle problem associated to the nonlinear degenerate anisotropic elliptic equations with variable exponents and L^1- data, we refer the reader to [13,16,17] and [34] for more results.

The aim of this paper is to study the existence of entropy solutions for the unilateral problem associated to the degenerated quasilinear Neumann elliptic equation :

$$\begin{cases} Au + |u|^{r(x)-2}u = f(x,u) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = g(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

where $Au = \sum_{i=1}^{N} D^i a_i(x,u,\nabla u))$ is a Leray-Lions operator acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into it's dual, such that $a_i(x,u,\nabla u)$ are Carathéodory functions that satisfying some nonstandard conditions, and $f(x,s)$ verifying only some growth condition.

This paper is organized as follows: the second section is devoted to recalling some definitions and properties concerning the anisotropic Sobolev spaces with variable exponent. In the section 3, we present the assumptions on the Carathéodory functions $a_i(x,u,\nabla u)$ under which our problem has at least one solution. We study in the section 4 the existence of weak solutions for the unilateral problem associated to our equation with right-hand side $F(x,s) \in L^\infty(\Omega)$ and $G(x) \in L^\infty(\partial\Omega)$. In the last section, we show the existence of entropy solutions for the unilateral problem associated to the noncoercive elliptic equation (4) with the right-hand side $f(x,s) \in L^1(\Omega)$ and $g(x) \in L^1(\partial\Omega)$.

2. Preliminary

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function} \quad p(\cdot) : \Omega \longmapsto I\!R \ \text{ such that } \ 1 < p^- \leq p^+ < N\},$$

where

$$p^- = ess\inf\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = ess\sup\{p(x) \,/\, x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto I\!R$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\, dx\right| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{5}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result :

Proposition 2.1. (*see* [24], [32])

If $u_n, u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:

- **(i):** $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho_{p(\cdot)}(u) < 1 \quad (resp, = 1, > 1)$,
- **(ii):** $\|u\|_{p(\cdot)} > 1 \implies \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$ *and* $\|u\|_{p(\cdot)} < 1 \implies \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$,
- **(iii):** $\|u_n\|_{p(\cdot)} \to 0 \iff \rho_{p(\cdot)}(u_n) \to 0$, *and* $\|u_n\|_{p(\cdot)} \to \infty \iff \rho_{p(\cdot)}(u_n) \to \infty$,

which implies that the norm convergence and the modular convergence are equivalent.

Now, we present the anisotropic variable exponent Sobolev space, used in the study of our quasilinear elliptic problem (4).

Let $p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot)), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

and we define

$$p_M = \max\{p_1^+, \dots, p_N^+\} \quad \text{and} \quad \underline{p} = \min\{p_1^-, \dots, p_N^-\} \quad \text{then} \quad \underline{p} > 1. \tag{6}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{7}$$

The space $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\big)$ is a reflexive Banach space (cf. [27]).

Lemma 2.1. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for* $q \in [\underline{p}, \underline{p}^*[$, *where* $\underline{p}^* = \dfrac{N\underline{p}}{N - \underline{p}}$,

- *if* $\underline{p} = N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, +\infty[$,

- *if* $\underline{p} > N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorems for Sobolev spaces.

Definition 2.1. *Let* $k > 0$, *we consider the truncation function* $T_k(\cdot) : I\!R \longmapsto I\!R$, *given by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.2. *For any* $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$, *there exists a unique measurable function* $v_i : \Omega \mapsto I\!R$ *for any* $i \in \{1, \dots, N\}$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \quad \text{a.e.} \quad x \in \Omega,$$

where χ_E *denotes the characteristic function of a measurable set* E. *The functions* v_i *are called the weak partial derivatives of* u *and are still denoted* $D^i u$. *Moreover, if* u *belongs to* $W^{1,1}(\Omega)$, *then* v_i *coincides with the standard distributional derivative of* u, *that is,* $v_i = D^i u$.

The proof of the Proposition 2.2 follows the usual techniques developed in [20] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [9,18,22,23].

Definition 2.2. *We introduce the set* $T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$ *as a subset of* $T^{1,\vec{p}(\cdot)}(\Omega)$ *for which a generalized notion of trace may be defined (see also [8] for the case of constant exponent). More precisely,* $T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$ *is the set of function* u *in* $T^{1,\vec{p}(\cdot)}(\Omega)$*, such that : there exists a sequence* $(u_n)_n$ *in* $W^{1,\vec{p}(\cdot)}(\Omega)$ *and a measurable function* v *on* $\partial\Omega$ *verifying*

- **(a):** $u_n \longrightarrow u$ *a.e. in* Ω,
- **(b):** $D^i T_k(u_n) \longrightarrow D^i T_k(u)$ *in* $L^1(\Omega)$ *for every* $k > 0$.
- **(c):** $u_n \longrightarrow v$ *a.e. on* $\partial\Omega$.

The function v *is the trace of* u *in the generalized sense introduced in [8].*

Proposition 2.3. *Let* $u \in W^{1,\vec{p}(\cdot)}(\Omega)$*, the trace of* u *on* $\partial\Omega$ *will be denoted by* $\tau(u)$*.*
For any $u \in T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$*, the trace of* u *on* $\partial\Omega$ *will be denoted by* $tr(u)$ *or* u*, the operator* $tr(\cdot)$ *satisfied the following properties*

- **(i):** *if* $u \in T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$*, then* $\tau(T_k(u)) = T_k(tr(u))$ *for any* $k > 0$.
- **(ii):** *if* $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega)$*, then, for any* $u \in T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$*, we have* $u - \varphi \in T^{1,\vec{p}(\cdot)}_{tr}(\Omega)$ *and* $tr(u-\varphi) = tr(u) - \tau(\varphi)$.

In the case where $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, $tr(u)$ *coincides with* $\tau(u)$*. Obviously, we have*

$$W^{1,\vec{p}(\cdot)}(\Omega) \subset T^{1,\vec{p}(\cdot)}_{tr}(\Omega) \subset T^{1,\vec{p}(\cdot)}(\Omega).$$

Lemma 2.2. *(see [25], Theorem 13.47) Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

- **(i):** $u_n \to u$ *a.e. in* Ω,
- **(ii):** $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,
- **(iii):** $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ *strongly in* $L^1(\Omega)$.

3. Essential Assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$. and we consider $r(\cdot) \in C_+(\Omega)$ and $p_i(\cdot) \in C_+(\Omega)$ for $i = 1, \ldots, N$.
We consider the Neumann degenerate anisotropic elliptic equation

$$\begin{cases} Au + |u|^{r(x)-2}u = f(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i = g(x) & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where A is a Leray-Lions operator acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$, defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

such that $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions for $i = 1, \ldots, N$ (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfy the following conditions :

$$|a_i(x,s,\xi)| \leq \beta(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}) \qquad \text{for} \quad i = 1, \ldots, N, \tag{9}$$

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i(x)} \qquad \text{with} \quad b(|s|) \geq \frac{b_0}{(1+|s|)^{\lambda(x)}} \qquad \text{for} \quad i = 1, \ldots, N, \tag{10}$$

where β and b_0 are two positive constants. The nonnegative functions $K_i(\cdot)$ are assumed to be in $L^{p_i'(\cdot)}(\Omega)$ and $0 \leq \lambda(x) < \min\left(1, p_i(x)-1, \frac{1}{p_i(x)-1}\right)$ for $i = 1, \ldots, N$.

$$\sum_{i=1}^{N} (a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \qquad \text{for} \quad \xi_i \neq \xi_i', \tag{11}$$

for almost every $x \in \Omega$ and any (s,ξ) in $I\!R \times I\!R^N$.

We consider the obstacle function $\psi(\cdot) : \Omega \longmapsto \overline{I\!R}$ such that $\psi^+ \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and we define the following convex set

$$K_\psi = \{v \in W^{1,\vec{p}(\cdot)}(\Omega) \qquad \text{such that} \qquad v \geq \psi \ a.e. \ in \ \Omega\}.$$

We are going now to recall the following technical Lemma, useful to prove our main results.

Lemma 3.1. *(see [17]) Let $k > 0$, assuming that (9) – (11) hold true, and let $(u_n)_{n\in I\!N}$ be a sequence in $W^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\,dx \\ &+\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla u_n) - a_i(x,T_k(u_n),\nabla u))(D^i u_n - D^i u)\,dx \to 0 \quad as \quad n\to\infty, \end{aligned} \tag{12}$$

then $u_n \to u$ strongly in $W^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.

4. Existence of Weak Solutions for $L^\infty-$ data

We consider the quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i(a_i(x,T_n(u),\nabla u)) + |u|^{r(x)-2}u = F(x,u) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,T_n(u),\nabla u)\cdot n_i = G(x) & \text{on } \partial\Omega \end{cases} \tag{13}$$

with

$$G(x) \in L^{\infty}(\partial\Omega) \qquad \text{and} \qquad |F(x,s)| \leq C_0 \quad \text{for any} \quad (x,s) \in \Omega \times \mathbb{R}, \tag{14}$$

where C_0 is a positive constant.

Definition 4.1. *A measurable function u is called weak solution for the unilateral problem associated to the quasilinear anisotropic elliptic equation (13), if $u \in K_\psi$ and $|u|^{r(x)} \in L^1(\Omega)$, such that u verifies the following equality*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v)\, dx + \int_\Omega |u|^{r(x)-2} u(u-v)\, \mathrm{d}x \\ &\leq \int_\Omega F(x,u)(u-v)\, \mathrm{d}x + \int_{\partial\Omega} G(u-v)\, \mathrm{d}\sigma, \end{aligned} \tag{15}$$

for any $v \in K_\psi$.

Theorem 4.1. *Assuming that (9) − (11) and (14) hold true. Then there exists at least one weak solution for the unilateral problem associate to the quasilinear elliptic equation (13).*

Proof of Theorem 4.1.

Step 1 : Approximate problem. We consider the following approximate problem for our quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + |T_m(u_m)|^{r(x)-2} T_m(u_m) + \dfrac{1}{m} |u_m|^{\underline{p}-2} u_m = F(x, u_m) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u_m), \nabla u_m).n_i = G(x) & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We consider the two operators A_m and H acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$, defined by

$$\langle A_m u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v\, dx + \int_\Omega |T_m(u)|^{r(x)-2} T_m(u) v\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} uv\, dx, \tag{17}$$

and

$$\langle Hu, v\rangle = -\int_\Omega F(x,u) v\, dx - \int_{\partial\Omega} G(x) v\, d\sigma \;\; \text{for any} \;\; u, v \in W^{1,\vec{p}(\cdot)}(\Omega). \tag{18}$$

Lemma 4.1. *The operator $B_m = A_m + H$ acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$ is bounded and pseudo-monotone. Moreover B_m is coercive in the following sense : There exists $v_0 \in K_\psi$ such that*

$$\frac{\langle B_m v, v - v_0\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}(\cdot)} \longrightarrow \infty, \quad \text{for} \quad v \in K_\psi. \tag{19}$$

Proof of Lemma 4.1. We have

$$\begin{aligned}
&|\langle A_m u, v\rangle| \\
&\le \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u),\nabla u)||D^i v|\,dx + \int_\Omega |T_m(u)|^{r(x)-1}|v|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-1}|v|\,dx \\
&\le \beta\sum_{i=1}^{N}\int_\Omega \left(K_i(x) + n^{p_i(x)-1} + |D^i u|^{p_i(x)-1}\right)|D^i v|\,dx \\
&\quad + m^{r^+-1}\int_\Omega |v|\,dx + \frac{1}{m}\|u\|_{L^{\underline{p}}(\Omega)}^{\underline{p}-1}\|v\|_{L^{\underline{p}}(\Omega)} \\
&\le 2\beta\sum_{i=1}^{N}\left(\|K_i(x)\|_{L^{p_i'(\cdot)}(\Omega)} + n^{p_i^+-1} + \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}^{p_i^+-1}\right)\|v\|_{1,\vec{p}(\cdot)} \\
&\quad + m^{r^+-1}\|v\|_{1,\vec{p}(\cdot)} + \frac{C_1}{m}\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}-1}\|v\|_{1,\vec{p}(\cdot)} \\
&\le C_2\Big(1 + \|u\|_{1,\vec{p}(\cdot)}^{p_M-1}\Big)\|v\|_{1,\vec{p}(\cdot)}.
\end{aligned} \tag{20}$$

Thus, the operator A_m is bounded. Moreover, we have

$$\begin{aligned}
|\langle Hu, v\rangle| &= \Big| -\int_\Omega F(x,u)v\,dx - \int_{\partial\Omega} G(x)v d\sigma\Big| \\
&\le \int_\Omega |F(x,u)|\,|v|\,dx + \int_{\partial\Omega} |G(x)|\,|v|\,d\sigma \\
&\le C_0\int_\Omega |v|\,dx + \|G(\cdot)\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|v|\,d\sigma \\
&\le C_0\|v\|_{L^1(\Omega)} + \|G(\cdot)\|_{L^\infty(\partial\Omega)}\|v\|_{L^1(\partial\Omega)} \\
&\le C_3\|v\|_{1,\vec{p}(\cdot)} \qquad\qquad \text{for any } u,v\in W^{1,\vec{p}(\cdot)}(\Omega).
\end{aligned} \tag{21}$$

We conclude that the operator B_m bounded. For coercivity, we have

$$\begin{aligned}
&\langle B_m u, u\rangle \\
&= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i u\,dx + \int_\Omega |T_m(u)|^{r(x)-1}|u|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx \\
&\quad - \int_\Omega F(x,u)u\,dx - \int_{\partial\Omega} G(x)u d\sigma \\
&\ge \sum_{i=1}^{N}\int_\Omega \frac{b_0|D^i u|^{p_i(x)}}{(1+|T_n(u)|)^{\lambda(x)}}\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx - C_0\|u\|_{L^1(\Omega)} - \|G(\cdot)\|_{L^\infty(\partial\Omega)}\|u\|_{L^1(\partial\Omega)} \\
&\ge \frac{b_0}{(1+n)^{\lambda_+}}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx + \frac{C_4}{m}\|u\|_{L^1(\Omega)}^{\underline{p}} - C_0\|u\|_{L^1(\Omega)} - \|G(\cdot)\|_{L^\infty(\partial\Omega)}\|u\|_{L^1(\partial\Omega)} \\
&\ge \frac{b_0}{2(1+n)^{\lambda_+}}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx + C_5\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - C_6\|u\|_{1,\vec{p}(\cdot)},
\end{aligned} \tag{22}$$

Furthermore, using (20) and (21) we conclude that

$$\begin{aligned}
&|\langle B_m u, u_0\rangle| \\
&\le \beta\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)-1}|D^i u_0|\,dx + C_2\Big(1+\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}-1}\Big)\|u_0\|_{1,\vec{p}(\cdot)} + C_3\|u_0\|_{1,\vec{p}(\cdot)}
\end{aligned} \tag{23}$$

$$\leq \frac{b_0}{2(1+n)^{\lambda_+}}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx + C_7\sum_{i=1}^{N}\int_\Omega |D^i u_0|^{p_i(x)}\,dx$$
$$+C_2\Big(1+\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}-1}\Big)\|u_0\|_{1,\vec{p}(\cdot)} + C_3\|u_0\|_{1,\vec{p}(\cdot)}$$
$$\leq \frac{b_0}{2(1+n)^{\lambda_+}}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i(x)}\,dx + C_8\Big(1+\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}-1}\Big)\Big(1+\|u_0\|_{1,\vec{p}(\cdot)}^{p_M}\Big),$$

then we obtain

$$\begin{aligned}\frac{\langle B_m u, u-u_0\rangle}{\|u\|_{1,\vec{p}(\cdot)}} &\geq \frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} - \frac{|\langle B_m u, u_0\rangle|}{\|u\|_{1,\vec{p}(\cdot)}}\\ &\geq \frac{C_2\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - C_3\|u\|_{1,\vec{p}(\cdot)}}{\|u\|_{1,\vec{p}(\cdot)}} - \frac{C_8\Big(1+\|u\|_{1,\vec{p}(\cdot)}^{\underline{p}-1}\Big)\Big(1+\|u_0\|_{1,\vec{p}(\cdot)}^{p_M}\Big)}{\|u\|_{1,\vec{p}(\cdot)}}\\ &\longrightarrow \infty \quad \text{as} \quad \|u\|_{1,\vec{p}(\cdot)} \to \infty.\end{aligned} \tag{24}$$

Now, we will prove that B_m is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } W^{1,\vec{p}(\cdot)}(\Omega),\\ B_m u_k \rightharpoonup \chi_m & \text{weakly in } (W^{1,\vec{p}(\cdot)}(\Omega))',\\ \limsup_{k\to\infty}\langle B_m u_k, u_k\rangle \leq \langle \chi_m, u\rangle. \end{cases} \tag{25}$$

We will show that

$$\chi_m = B_m u \quad \text{and} \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle \chi_m, u\rangle \text{ as } k\to+\infty.$$

In view of the compact embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, there exists a subsequence still denoted $(u_k)_{k\in\mathbb{N}^*}$ such that $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$.
As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition (9), it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}^*}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a measurable function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \qquad \text{weakly in} \quad L^{p_i'(\cdot)}(\Omega) \text{ as } k\to\infty. \tag{26}$$

We have $(F(x,u_k))_{k\in\mathbb{N}^*}$ is uniformly bounded in $L^\infty(\Omega) \subset L^{\underline{p}'}(\Omega)$, and $F(x,u_k) \to F(x,u)$ almost everywhere in Ω, in view of Lebesgue dominated convergence theorem we conclude that

$$F(x,u_k) \to F(x,u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{27}$$

Similarly, we obtain

$$|T_m(u_k)|^{r(x)-2}T_m(u_k) \longrightarrow |T_m(u)|^{r(x)-2}T_m(u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega), \tag{28}$$

Moreover, since $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\frac{1}{m}|u_k|^{\underline{p}-2}u_k \longrightarrow \frac{1}{m}|u|^{\underline{p}-2}u \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega), \tag{29}$$

Thus, for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$ we have

$$\begin{aligned}
\langle \chi_n, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v\, dx + \lim_{k\to\infty} \int_\Omega |T_m(u_k)|^{r(x)-2} T_m(u_k) v\, dx \\
&\quad + \lim_{k\to\infty} \frac{1}{m} \int_\Omega |u_k|^{\underline{p}-2} u_k v\, dx - \lim_{k\to\infty} \int_\Omega F(x, u_k) v\, dx - \int_{\partial\Omega} G\, v\, d\sigma \\
&= \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i v\, dx + \int_\Omega |T_m(u)|^{r(x)-2} T_m(u) v\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} u v\, dx \\
&\quad - \int_\Omega F(x, u) v\, dx - \int_{\partial\Omega} G v\, d\sigma.
\end{aligned} \tag{30}$$

In view of (25) and (30), we conclude that

$$\begin{aligned}
\limsup_{k\to\infty} \langle B_m(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big(\int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx + \int_\Omega |T_m(u_k)|^{r(x)-1} |u_k|\, dx \\
&\qquad + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}}\, dx - \int_\Omega F(x, u_k) u_k\, dx - \int_{\partial\Omega} G\, u_k\, d\sigma \Big) \\
&\le \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i u\, dx + \int_\Omega |T_m(u)|^{r(x)-1} |u|\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}}\, dx \\
&\qquad - \int_\Omega F(x, u) u\, dx - \int_{\partial\Omega} G\, u\, d\sigma.
\end{aligned} \tag{31}$$

Thanks to (27) − (29) we have

$$\int_\Omega |T_m(u_k)|^{r(x)-1} |u_k|\, dx + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}}\, dx \longrightarrow \int_\Omega |T_m(u)|^{r(x)-1} |u|\, dx + \frac{1}{m} \int_\Omega |u|^{\underline{p}}\, dx \ \text{ as } \ k \to \infty, \tag{32}$$

and

$$\int_\Omega F(x, u_k) u_k\, dx \longrightarrow \int_\Omega F(x, u) u\, dx \quad \text{as} \quad k \to \infty, \tag{33}$$

Having in mind that $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^1(\partial\Omega)$ then $u_k \rightharpoonup u$ weakly in $L^1(\partial\Omega)$, and since $G \in L^\infty(\partial\Omega)$ then

$$\int_{\partial\Omega} G\, u_k\, d\sigma \longrightarrow \int_{\partial\Omega} G u\, d\sigma \qquad \text{as} \quad k \to \infty. \tag{34}$$

It follows that

$$\limsup_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \le \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i u\, dx. \tag{35}$$

On the other hand, in view of (11) we have

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u)\, dx \ge 0, \tag{36}$$

then

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx &\ge \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u\, dx \\
&\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u)\, dx.
\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ strongly in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ strongly in $L^{p_i'(x)}(\Omega)$, and using (26) we get

$$\liminf_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{37}$$

Having in mind (35), we conclude that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx = \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx. \tag{38}$$

Therefore, having in mind (32), (33) and (34) we obtain

$$\langle B_m u_k, u_k \rangle \longrightarrow \langle \chi_m, u \rangle \quad \text{as} \quad k \to \infty. \tag{39}$$

On the other hand, thanks to (38) we can show that

$$\lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx = 0.$$

We have $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2} u_n - |u|^{\underline{p}-2} u)(u_k - u) \, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u)(D^i u_n - D^i u) \, dx \to 0, \end{aligned} \tag{40}$$

in view of Lemma 3.1, we conclude that

$$u_k \to u \quad \text{in} \quad W^{1,\vec{p}(\cdot)}(\Omega) \qquad \text{and} \qquad D^i u_k \to D^i u \quad \text{a.e. in} \quad \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \qquad \text{weakly in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N.$$

Having in mind (27) – (29) we obtain $\chi_m = B_m u$. Thus, the proof of the Lemma 4.1 is concluded. In view of Lemma 4.1 (cf. [26], Theorem 8.2) there exists at least one weak solution $u_m \in K_\psi$ for the problem (16), i.e.

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v) \, dx + \int_\Omega |T_m(u_m)|^{r(x)-2} T_m(u_m)(u_m - v) \, dx \\ &+ \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m (u_m - v) \, dx = \int_\Omega F(x, u_m)(u_m - v) \, dx + \int_{\partial\Omega} G(x)(u_m - v) \, d\sigma, \end{aligned} \tag{41}$$

for any $v \in K_\psi$.

Step 2 : Weak convergence of the sequence $(u_n)_n$.

By taking $v = \psi^+ \in K_\psi$ as a test function for the approximate problem (16), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)(D^iu_m - D^i\psi^+)\,dx + \int_\Omega |T_m(u_m)|^{r(x)-2}T_m(u_m)\,(u_m-\psi^+)\,dx\\&+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-\psi^+)\,dx = \int_\Omega F(x,u_m)(u_m-\psi^+)\,dx + \int_{\partial\Omega} G\,(u_m-\psi^+)\,d\sigma.\end{aligned} \tag{42}$$

Since $u_m - \psi^+$ have the same sign as u_n. Thus, in view of (10) and (14) we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega b(|T_n(u_m)|)|D^iu_m|^{p_i(x)}\,dx + \int_\Omega |T_m(u_m)|^{r(x)-1}\,|u_m-\psi^+|\,dx\\&\quad+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}|u_m-\psi^+|\,dx\\&\le \int_\Omega |F(x,u_m)|\,|u_m-\psi^+|\,dx + \int_{\partial\Omega}|G(x)|\,|u_m-\psi^+|\,d\sigma\\&\quad+\sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx\\&\le C_0\int_\Omega |u_m|+|\psi^+|\,dx + \|G\|_{L^\infty(\Omega)}\int_{\partial\Omega}|u_m|+|\psi^+|\,d\sigma\\&\quad+\sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx\\&\le C_1(\|u_m\|_{1,1}+\|\psi^+\|_{1,1}) + \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx.\end{aligned} \tag{43}$$

with C_1 is a constant that doesn't depend on m and n.

Concerning the second term on the left-hand side of (43), using Young's inequality we have

$$\begin{aligned}&\int_\Omega |T_m(u_m)|^{r(x)-1}\,|u_m-\psi^+|\,dx\\&\ge \int_\Omega |T_m(u_m)|^{r(x)-1}\,|u_m|\,dx - \int_\Omega |T_m(u_m)|^{r(x)-1}\,\psi^+\,dx\\&\ge \int_\Omega |T_m(u_m)|^{r(x)-1}\,|u_m|\,dx - \frac{1}{2}\int_\Omega |T_m(u_m)|^{r(x)}\,dx - C_2\int_\Omega |\psi^+|^{r(x)}\,dx\\&\ge \frac{1}{2}\int_\Omega |T_m(u_m)|^{r(x)-1}\,|u_m|\,dx - C_3.\end{aligned} \tag{44}$$

For the last term on the right-hand side of (43), in view of (9) we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u_m)|\,|D^i\psi^+|\,dx\\&\le \beta\sum_{i=1}^{N}\int_\Omega (K_i(x)+n^{p_i(x)-1}+|D^iu_m|^{p_i(x)-1})\,|D^i\psi^+|\,dx\\&\le \beta\sum_{i=1}^{N}\int_\Omega (|K_i(x)|^{p_i'(x)}+n^{p_i(x)})\,dx + 2\beta\sum_{i=1}^{N}\int_\Omega |D^i\psi^+|^{p_i(x)}\,dx\\&\quad+C_4\sum_{i=1}^{N}\int_\Omega \frac{|D^i\psi^+|^{p_i(x)}}{b(|T_n(u_m)|)^{p_i(x)-1}}\,dx + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega b(|T_n(u_m)|)|D^iu_m|^{p_i(x)}\,dx\\&\le C_5 + \frac{1}{2}\sum_{i=1}^{N}\int_\Omega b(|T_n(u_m)|)|D^iu_m|^{p_i(x)}\,dx.\end{aligned} \tag{45}$$

By combining (43) and (44) – (45), it follows that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_{\Omega} b(|T_n(u_m)|)|D^i u_m|^{p_i(x)}\,dx + \frac{1}{2}\int_{\Omega}|T_m(u_m)|^{r(x)-1}\,|u_m|\,dx \\
&\quad + \frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}|u_m-\psi^+|\,dx \\
&\leq C_1\|u_m\|_{1,1} + C_6 \\
&= C_1\Big(\int_{\Omega}|u_m|\,dx + \sum_{i=1}^{N}\int_{\Omega}|D^i u_m|\,dx\Big) + C_6 \\
&\leq \frac{1}{4}\int_{\Omega}|T_m(u_m)|^{r(x)-1}\,|u_m|\,dx + \frac{1}{4}\sum_{i=1}^{N}\int_{\Omega} b(|T_n(u_m)|)|D^i u_m|^{p_i(x)}\,dx + C_7.
\end{aligned} \tag{46}$$

We conclude that

$$\begin{aligned}
&\frac{b_0}{4(1+n)^{\lambda_+}}\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i(x)}\,dx + \frac{1}{4}\int_{\Omega}|T_m(u_m)|^{r(x)-1}\,|u_m|\,dx \\
&\quad + \frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}|u_m-\psi^+|\,dx \leq C_7.
\end{aligned} \tag{47}$$

Furthermore, we deduce that

$$\begin{aligned}
\|u_m\|_{1,\vec{p}(\cdot)} &= \|u_m\|_{1,1} + \sum_{i=1}^{N}\|D^i u_m\|_{L^{p_i(\cdot)}(\Omega)} \\
&\leq \|u_m\|_{L^1(\Omega)} + 2\sum_{i=1}^{N}\int_{\Omega}|D^i u_m|^{p_i(x)}\,dx + 2N \\
&\leq C_8.
\end{aligned} \tag{48}$$

with C_7 and C_8 are two constants that doesn't depend on m. Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted by $(u_m)_m$ such that

$$\begin{cases}
u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}(\cdot)}(\Omega), \\
u_m \longrightarrow u & \text{strongly in } \quad L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \\
u_m \rightharpoonup u & \text{weakly in } \quad L^1(\partial\Omega).
\end{cases} \tag{49}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{50}$$

Moreover, in view of (47) we conclude that $(T_m(u_m))_m$ is bounded in $L^{r(\cdot)}(\Omega)$, and since $T_m(u_m) \to u$ almost everywhere in Ω, we get

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{r(\cdot)}(\Omega). \tag{51}$$

Having in mind (14) and the fact that $u_m \to u$ a.e. in Ω, thanks to Lebesgue dominated convergence theorem we conclude that

$$F(x,u_m) \longrightarrow F(x,u) \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{52}$$

Step 3 : The convergence almost everywhere of the gradient.

By taking $v = u$ as a test function for the approximated problem (16) we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_m),\nabla u_m)(D^i u_m - D^i u)\,dx + \frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-2}u_m(u_m-u)\,dx\\&+\int_{\Omega}|T_m(u_m)|^{r(x)-2}T_m(u_m)\,(u_m-u)\,dx\\&\leq \int_{\Omega}F(x,u_m)(u_m-u)\,dx + \int_{\partial\Omega}G\,(u_m-u)\,d\sigma,\end{aligned}\tag{53}$$

it follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m - D^i u)\,dx\\&+\int_{\Omega}\Big(|T_m(u_m)|^{r(x)-2}T_m(u_m)-|T_m(u)|^{r(x)-2}T_m(u)\Big)(u_m-u)\,dx\\&\leq \sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m - D^i u|\,dx + \int_{\Omega}|T_m(u)|^{r(x)-1}\,|u_m-u|\,dx\\&+\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\,|u_m-u|\,dx + \int_{\Omega}|F(x,u_m)|\,|u_m-u|\,dx + \int_{\partial\Omega}|G|\,|u_m-u|\,d\sigma.\end{aligned}\tag{54}$$

For the first term on the right-hand side of (54), we have $T_n(u_m) \to T_n(u)$ strongly in $L^{p_i(\cdot)}(\Omega)$ then

$$|a_i(x,T_n(u_m),\nabla u)| \longrightarrow |a_i(x,T_n(u),\nabla u)| \quad \text{strongly in} \quad L^{p_i'(\cdot)}(\Omega),$$

and since $D^i u_m \rightharpoonup D^i u$ weakly in $L^{p_i(\cdot)}(\Omega)$, it follows that

$$\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m - D^i u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty.\tag{55}$$

Concerning the second and third terms on the right-hand side of (54), in view of (50) and (51) we conclude that

$$\int_{\Omega}|T_m(u)|^{r(x)-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty,\tag{56}$$

and

$$\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty.\tag{57}$$

Moreover, we have $|F(x,u_m)| \to |F(x,u)|$ strongly in $L^{\underline{p}'}(\Omega)$, then

$$\int_{\Omega}|F(x,u_m)|\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty.\tag{58}$$

For the last term on the right-hand side of (54), we have $G(x) \in L^{\infty}(\partial\Omega)$ and $u_m \rightharpoonup u$ weakly in $L^1(\partial\Omega)$, then

$$\int_{\partial\Omega}|G|\,|u_m-u|\,d\sigma \longrightarrow 0 \qquad \text{as} \quad m\to\infty.\tag{59}$$

By combining (54) and (55) – (59) we conclude that

$$\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m - D^i u)\,dx = 0,\tag{60}$$

and since $u_m \to u$ strongly in $L^{\underline{p}}(\Omega)$. Thus, in view of Lemma 3.1, we conclude that

$$\begin{cases} u_m \to u & \text{strongly in} \quad W^{1,\vec{p}(\cdot)}(\Omega), \\ D^i u_m \to D^i u & \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \ldots, N. \end{cases} \tag{61}$$

It follows that $a_i(x, T_n(u_n), \nabla u_n) \longrightarrow a_i(x, T_n(u), \nabla u)$ almost everywhere in Ω, then

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in} \quad L^{p_i'(\cdot)}(\Omega) \quad \text{for} \quad i = 1, \ldots, N. \tag{62}$$

Step 4 : Passage to the limit.

By taking $v \in K_\psi$ as a test function for the approximate problem (16) we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i v)\, dx + \int_\Omega |T_m(u_m)|^{r(x)-2} T_m(u_m)(u_m - v)\, dx \\ &+ \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m (u_m - v)\, dx \le \int_\Omega F(x, u_m)(u_m - v)\, dx + \int_{\partial\Omega} G\,(u_m - v)\, d\sigma, \end{aligned} \tag{63}$$

Thanks to Fatou's lemma we have

$$\begin{aligned} &\liminf_{m\to\infty} \int_\Omega |T_m(u_m)|^{r(x)-2} T_m(u_m)(u_m - v)\, dx \\ &= \liminf_{m\to\infty} \int_\Omega (|T_m(u_m)|^{r(x)-2} T_m(u_m) - |T_m(v)|^{r(x)-2} T_m(v))(u_m - v)\, dx \\ &\quad + \int_\Omega |v|^{r(x)-2} v(u - v)\, dx \\ &\ge \int_\Omega |u|^{r(x)-2} u(u - v)\, dx \end{aligned} \tag{64}$$

In view of (50), (51) and (62), by letting m tends to infinity we conclude that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)(D^i u - D^i v)\, dx + \int_\Omega |u|^{r(x)-2} u(u - v)\, dx \\ &\le \int_\Omega F(x, u)(u - v)\, dx + \int_{\partial\Omega} G\,(u - v)\, d\sigma. \end{aligned} \tag{65}$$

Thus, the proof of the theorem 4.1 is concluded.

5. Main Result

Now, we consider the nonlinear Carathéodory function $f(x,s)$ that verifying the growth condition

$$|f(x,s)| \le f_0(x) + c(x)|s|^{\gamma(x)}, \tag{66}$$

where $f_0(\cdot) \in L^1(\Omega)$, with $c(x) \in L^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}(\Omega)$ and $0 < \gamma(x) < r(x) - 1$.

Definition 5.1. *A measurable function u is an entropy solution of the unilateral problem associated to the quasilinear elliptic equation (8) if $T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0$, with $u \ge \psi$ a.e. in Ω such that u verifying the inequality*

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - v)\, \mathrm{d}x + \int_\Omega |u|^{r(x)-2} u T_k(u - v)\, dx \\ &\le \int_\Omega f(x, u) T_k(u - v)\, dx + \int_{\partial\Omega} g_n(x) T_k(u - v)\, \mathrm{d}\sigma \end{aligned} \tag{67}$$

for any $v \in K_\psi \cap L^\infty(\Omega)$.

Theorem 5.1. *Assume that* (9) − (11) *and* (66) *hold true, then there exists at least one entropy solution* u *for the unilateral problem associated to the quasilinear anisotropic elliptic Neumann equation* (8).

Proof of Theorem 5.1.

Step 1: Approximate problem. Let $f_n(x,s) = T_n(f(x,T_n(s)))$ and $g_n(x) = T_n(g(x))$. We consider the sequence of approximate problem :

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,T_n(u_n),\nabla u_n)) + |u_n|^{r(x)-2}u_n = f_n(x,u_n) & \text{in} \quad \Omega, \\ \sum_{i=1}^{N} a_i(x,T_n(u),\nabla u)\cdot n_i = g_n(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{68}$$

In view of theorem 4.1, there exists at least one weak solution $u_n \in K_\psi$ for the unilateral problem associated to the quasilinear elliptic Neumann equation (68), i.e.

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i(u_n-v)\,\mathrm{d}x + \int_\Omega |u_n|^{r(x)-2}u_n(u_n-v)\,\mathrm{d}x \\ &\leq \int_\Omega f_n(x,u_n)(u_n-v)\,\mathrm{d}x + \int_{\partial\Omega} g_n(x)(u_n-v)\,\mathrm{d}\sigma. \end{aligned} \tag{69}$$

for any $v \in K_\psi$.

Step 2: Weak convergence of truncations

Let $k \geq \max(1,\|\psi^+\|_\infty)$, and $v = u_n - \eta T_k(u_n-\psi^+)$, since $v \in W^{1,\vec{p}(\cdot)}(\Omega)$, and for $\eta > 0$ small enough we have $v \geq \psi$. Thus v is an admissible test function for the approximate problem (69), and we have

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq k\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\psi^+)\,\mathrm{d}x + \int_\Omega |u_n|^{r(x)-2}u_nT_k(u_n-\psi^+)\,\mathrm{d}x \\ &\leq \int_\Omega f_n(x,u_n)T_k(u_n-\psi^+)\,\mathrm{d}x + \int_{\partial\Omega} g_n(x)T_k(u_n-\psi^+)\,\mathrm{d}\sigma. \end{aligned} \tag{70}$$

Since $T_k(u_n-\psi^+)$ have the same sign as u_n. Thus, using (10) and the growth condition (66) we obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq k\}} b(|T_n(u_n)|)|D^iu_n|^{p_i(x)}\,dx + \int_\Omega |u_n|^{r(x)-1}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\ &\leq \int_\Omega |f_0(x)||T_k(u_n-\psi^+)|\,\mathrm{d}x + \int_\Omega |c(x)||u_n|^{\gamma(x)}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} |g_n(x)||T_k(u_n-\psi^+)|\,\mathrm{d}\sigma + \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq k\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x \\ &\leq k(\|f_0(x)\|_{L^1(\Omega)} + \|g(x)\|_{L^1(\partial\Omega)}) + \frac{1}{2}\int_\Omega |u_n|^{r(x)-1}|T_k(u_n-\psi^+)|\,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} |c(x)|^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}|T_k(u_n-\psi^+)|\,dx + \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq k\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x \\ &\leq C_1k + \frac{1}{2}\int_\Omega |u_n|^{r(x)-1}|T_k(u_n-\psi^+)|\,\mathrm{d}x + \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq k\}} |a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x. \end{aligned} \tag{71}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i(x)}\,dx+\frac{1}{2}\int_\Omega |u_n|^{r(x)-1}|T_k(u_n-\psi^+)|\,dx\\&\le C_1k+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x.\end{aligned}\tag{72}$$

For the second term on the left-hand side of (72), in view of Young's inequality we obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_n|^{r(x)-1}|T_k(u_n-\psi^+)|\,dx\\&=\frac{1}{2}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)-1}|u_n-\psi^+|\,dx+\frac{k}{2}\int_{\{|u_n-\psi^+|>k\}}|u_n|^{r(x)-1}\,dx\\&=\frac{1}{2}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)}\,dx-\frac{1}{2}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)-1}\psi^+\,dx\\&\quad+\frac{k}{2}\int_{\{|u_n-\psi^+|>k\}}|u_n|^{r(x)-1}\,dx\\&\ge\frac{1}{2}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)}\,dx-\frac{1}{4}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)}\,dx\\&\quad-2\int_{\{|u_n-\psi^+|\le k\}}|\psi^+|^{r(x)}\,dx+\frac{k}{2}\int_{\{|u_n-\psi^+|>k\}}|u_n|^{r(x)-1}\,dx\\&\ge\frac{1}{4}\int_{\{|u_n-\psi^+|\le k\}}|u_n|^{r(x)}\,dx+\frac{k}{2}\int_{\{|u_n-\psi^+|>k\}}|u_n|^{r(x)-1}\,dx-C_2.\end{aligned}\tag{73}$$

Concerning the last term on the right hand side of (72). We have $\lambda(x)(p_i(x)-1)<1$, and using (9) we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,\mathrm{d}x\\&\le\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}(K_i(x)+|T_n(u_n)|^{p_i(x)-1}+|D^iu_n|^{p_i(x)-1})|D^i\psi^+|\,\mathrm{d}x\\&\le\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|K_i(x)|^{p_i'(x)}\,dx+\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|T_n(u_n)|^{p_i(x)}\,dx\\&\quad+2\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i(x)}\mathrm{d}x+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i(x)}\,dx\\&\quad+\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}\frac{|D^i\psi^+|^{p_i(x)}}{b(|T_n(u_n)|)^{p_i(x)-1}}\mathrm{d}x\\&\le C_3+\frac{1}{8}\int_{\{|u_n-\psi^+|\le k\}}|T_n(u_n)|^{r(x)}\,dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i(x)}\,dx\\&\quad+C_4\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}|D^i\psi^+|^{p_i(x)}(1+|T_n(u_n)|)^{\lambda(x)(p_i(x)-1)}\,dx\\&\le C_5k+\frac{1}{8}\int_{\{|u_n-\psi^+|\le k\}}|T_n(u_n)|^{r(x)}\,dx+\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}}b(|T_n(u_n)|)|D^iu_n|^{p_i(x)}\,dx.\end{aligned}\tag{74}$$

By combining (72) and (73) – (74), we conclude that

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k\}} b(|T_n(u_n)|)|D^i u_n|^{p_i(x)}\,dx \\
&\quad+\frac{1}{8}\int_{\{|u_n-\psi^+|\le k\}} |u_n|^{r(x)}\,dx+\frac{k}{2}\int_{\{|u_n-\psi^+|>k\}} |u_n|^{r(x)-1}\,dx\le C_6 k.
\end{aligned} \tag{75}
$$

Since $\{|u_n|\le k\}\subset\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}$, we conclude that

$$
\begin{aligned}
\frac{b_0}{(1+k)^{\lambda^+}}\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,dx &= \frac{b_0}{(1+k)^{\lambda^+}}\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} |D^i u_n|^{p_i(x)}\,dx \\
&\le \sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le k+\|\psi^+\|_\infty\}} b(|T_n(u_n)|)|D^i u_n|^{p_i(x)}\,dx \\
&\le C_7 k.
\end{aligned} \tag{76}
$$

It follows that

$$
\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,dx\le C_8 k^{\lambda^+ +1}. \tag{77}
$$

Furthermore, we have

$$
\begin{aligned}
\|T_k(u_n)\|_{1,\vec{p}(\cdot)} &= \|T_k(u_n)\|_{1,1}+\sum_{i=1}^{N}\|D^i T_k(u_n)\|_{p_i(\cdot)} \\
&\le \int_\Omega |T_k(u_n)|\mathrm{d}x+\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|\mathrm{d}x+\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,dx+N \\
&\le k\ \mathrm{meas}(\Omega)+2\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,dx+N(1+\mathrm{meas}(\Omega)) \\
&\le C_9 k^{\lambda^+ +1}.
\end{aligned} \tag{78}
$$

where C_9 is a positive constant that does not depend on k and n. Thus the sequence $(T_k(u_n))_n$ is uniformly bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$ and there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $v_k\in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$
\begin{cases}
T_k(u_n)\rightharpoonup v_k & \text{weakly in } \ W^{1,\vec{p}(\cdot)}(\Omega),\\
T_k(u_n)\to v_k & \text{strongly in } \ L^1(\Omega) \ \text{ and } \ \text{a.e in } \ \Omega.
\end{cases} \tag{79}
$$

Moreover, in view of (75), we have

$$
\begin{aligned}
k^{r^- -1}\mathrm{meas}\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{r(x)-1}\mathrm{d}x \\
&\le \int_{\{|u_n-\psi^+|>k-\|\psi^+\|_\infty\}} |u_n|^{r(x)-1}\mathrm{d}x \\
&\le C_{10}.
\end{aligned} \tag{80}
$$

It follows that

$$
\limsup_{n\to\infty}\ \mathrm{meas}(\{|u_n|>k\})\le \frac{C_{10}}{k^{r^- -1}}\longrightarrow 0 \qquad \text{as} \quad k\to\infty. \tag{81}
$$

Now, we are going to show that $(u_n)_n$ is a Cauchy sequence in measure.

For all $\lambda > 0$, we have

$$\begin{aligned}\text{meas}\{|u_n - u_m| > \lambda\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &+\text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}.\end{aligned}$$

Let $\varepsilon > 0$, using (81) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{82}$$

In addition, thanks to (79) we have $T_k(u_n) \to v_k$ strongly in $L^1(\Omega)$ and a.e. in Ω. So, we may assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure, and for all $k > 0$ and $\varepsilon, \lambda > 0$, there exists $n_0 = n_0(k, \varepsilon, \lambda)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(k, \varepsilon, \lambda). \tag{83}$$

By combining (82) $-$ (83), we conclude that : for all $\varepsilon, \lambda > 0$ there exists $n_0 = n_0(\varepsilon, \lambda)$ such that :

$$\text{meas}\{|u_n - u_m| > \lambda\} \leq \varepsilon \qquad \text{for any} \quad n, m \geq n_0(\varepsilon, \lambda).$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \to T_k(u) & \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega, \\ T_k(u_n) \to T_k(u) & \text{weakly in} \quad L^1(\partial\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega. \end{cases} \tag{84}$$

In view of Lebesgue's dominated convergence theorem, we conclude that

$$T_k(u_n) \to T_k(u) \qquad \text{in} \quad L^{p_i(\cdot)}(\Omega) \qquad \text{and} \qquad \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \ldots, N. \tag{85}$$

Moreover, in view of Young's inequality we have

$$\begin{aligned}\Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\partial\Omega)} &\leq \Big\|\frac{T_k(u_n)}{k}\Big\|_{1,1} \\ &\leq C_{11}\int_\Omega \frac{|T_k(u_n)|^{r(x)}}{k^{r(x)}}\,dx + C_{12}\sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i(x)}}{k^{p_i(x)}}\,dx \\ &\longrightarrow 0 \qquad \text{as} \quad k \to \infty.\end{aligned} \tag{86}$$

it follows that

$$\frac{T_k(u_n)}{k} \rightharpoonup 0 \qquad \text{weak} - * \quad L^\infty(\partial\Omega). \tag{87}$$

Step 3: Some regularity results

We will note by $\varepsilon_i(n) \quad i = 1, 2, ...$ some various functions of real numbers which converges to 0 as n

tends to infinity. Similarly, we define $\varepsilon_i(h)$ and $\varepsilon_i(n;h)$.

In this step, we are going to show this

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,\mathrm{d}x = 0. \tag{88}$$

Indeed, let $h\ge k\ge \max(1,\|\psi^+\|_\infty)$, we considering the function

$$v = u_n - \eta\frac{T_h(u_n-\psi^+)}{h}\in W^{1,\vec{p}(\cdot)}(\Omega),$$

we have $v\ge\psi$ for η small enough. Therefore, we have $v\in K_\psi$ is an admissible test function for the approximate problem (69), and we obtain

$$\begin{aligned}&\eta\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\frac{D^iT_h(u_n-\psi^+)}{h}\,\mathrm{d}x+\eta\int_\Omega |u_n|^{r(x)-2}u_n\frac{T_h(u_n-\psi^+)}{h}\,\mathrm{d}x\\ &\le \eta\int_\Omega f_n(x,u_n)\frac{T_h(u_n-\psi^+)}{h}\,\mathrm{d}x+\eta\int_{\partial\Omega} g_n(x)\frac{T_h(u_n-\psi^+)}{h}\,d\sigma.\end{aligned} \tag{89}$$

Since $T_h(u_n-\psi^+)$ have the same sign as u_n, thus, in view of (10), (66) and Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,\mathrm{d}x\\ &\quad+\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|u_n|)|D^iu_n|^{p_i(x)}\,dx+\frac{1}{2h}\int_\Omega |u_n|^{r(x)-1}|T_h(u_n-\psi^+)|\,dx\\ &\le\frac{1}{h}\int_\Omega |f_0(x)||T_h(u_n-\psi^+)|\,\mathrm{d}x+\frac{1}{h}\int_\Omega |c(x)||T_n(u_n)|^{\gamma(x)}|T_h(u_n-\psi^+)|\mathrm{d}x\\ &\quad+\frac{1}{h}\int_{\partial\Omega}|g_n(x)||T_h(u_n-\psi^+)|\,\mathrm{d}\sigma+\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^i\psi^+\,\mathrm{d}x\\ &\le\frac{1}{h}\int_\Omega |f_0(x)||T_h(u_n-\psi^+)|\,\mathrm{d}x+\frac{1}{2h}\int_\Omega |u_n|^{r(x)-1}|T_h(u_n-\psi^+)|\,dx\\ &\quad+\frac{C_0}{h}\int_\Omega |c(x)|^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}|T_h(u_n-\psi^+)|\,dx+\frac{1}{h}\int_{\partial\Omega}|g_n(x)||T_h(u_n-\psi^+)|\,\mathrm{d}\sigma\\ &\quad+\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx.\end{aligned} \tag{90}$$

Using the same argument as in (74) we conclude that

$$\begin{aligned}&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|a_i(x,T_n(u_n),\nabla u_n)|\,|D^i\psi^+|\,dx\\ &\le\frac{\beta}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|K_i(x)|^{p_i'(x)}\,dx+\frac{\beta}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|u_n|^{p_i(x)-1}|u_n-\psi^++\psi^+|\,dx\\ &\quad+2\beta\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i(x)}\mathrm{d}x+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}} b(|T_n(u_n)|)|D^iu_n|^{p_i(x)}\,dx\\ &\quad+\frac{C_1}{h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\le h\}}|D^i\psi^+|^{p_i(x)}(1+|T_n(u_n)|)^{\lambda(x)(p_i(x)-1)}\,dx\end{aligned} \tag{91}$$

$$\leq \frac{C_2}{h} + \frac{1}{4h}\int_{\{|u_n-\psi^+|\leq h\}} |u_n|^{r(x)-1}|T_h(u_n-\psi^+)|\,dx$$
$$+\frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} b(|T_n(u_n)|)|D^i u_n|^{p_i(x)}\,dx$$
$$+C_1\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} |D^i\psi^+|^{p_i(x)}\frac{(1+h+\|\psi^+\|_\infty)^{\lambda(x)(p_i(x)-1)}}{h}\,dx.$$

It follows that

$$\begin{aligned}
&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,\mathrm{d}x + \frac{1}{4h}\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} b(|u_n|)|D^i u_n|^{p_i(x)}\,dx\\
&+\frac{1}{4h}\int_\Omega |u_n|^{r(x)-1}|T_h(u_n-\psi^+)|\,dx\\
&\leq \frac{C_2}{h} + \frac{1}{h}\int_\Omega |f_0(x)||T_h(u_n-\psi^+)|\,dx + \frac{C_0}{h}\int_\Omega |c(x)|^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}|T_h(u_n-\psi^+)|\,dx\\
&+\frac{1}{h}\int_{\partial\Omega} |g_n(x)||T_h(u_n-\psi^+)|\,\mathrm{d}\sigma\\
&+C_1\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} |D^i\psi^+|^{p_i(x)}\frac{(1+h+\|\psi^+\|_\infty)^{\lambda(x)(p_i(x)-1)}}{h}\,dx.
\end{aligned} \tag{92}$$

For the second term on the right-hand side of (92), we have $\mathrm{meas}(\{|u_n|>k\}) \to 0$ as h tends to infinity and $\psi^+ \in L^\infty(\Omega)$, then $\dfrac{|T_h(u_n-\psi^+)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, it follows that

$$\frac{1}{h}\int_\Omega |f_0(x)||T_h(u_n-\psi^+)|\,dx \longrightarrow 0 \qquad \text{as} \quad h\to\infty. \tag{93}$$

Similarly, we have $c(x) \in L^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}(\Omega)$ then

$$\frac{C_0}{h}\int_\Omega |c(x)|^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}|T_h(u_n-\psi^+)|\,dx \longrightarrow 0 \qquad \text{as} \quad h\to\infty. \tag{94}$$

Moreover, in view of (87) we have $\dfrac{|T_h(u_n-\psi^+)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(\partial\Omega)$, then

$$\frac{1}{h}\int_{\partial\Omega} |g_n(x)||T_h(u_n-\psi^+)|\,\mathrm{d}\sigma \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{95}$$

Concerning the last term on the right-hand side of (92), we have $\lambda(x)(p_i(x)-1) < 1$ for any $i=1,\ldots,N$, it follows that

$$\sum_{i=1}^{N}\int_{\{|u_n-\psi^+|\leq h\}} |D^i\psi^+|^{p_i(x)}\frac{(1+h+\|\psi^+\|_\infty)^{\lambda(x)(p_i(x)-1)}}{h}\,dx \to 0 \text{ as } h\to\infty. \tag{96}$$

By combining (92) and (93) – (96), we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\,\mathrm{d}x = 0, \tag{97}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} b(|u_n|)|D^i u_n|^{p_i(x)}\,dx = 0. \tag{98}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{r(x)-1}\,dx=0. \tag{99}$$

Thus, for any $\varepsilon>0$ there exists $\beta>0$ such that : for any measurable subset $E\in\Omega$ with $meas(E)<\beta$ we have

$$\int_E|u_n|^{r(x)-1}\,dx\le\int_E|T_h(u_n)|^{r(x)-1}\,dx+\int_{\{|u_n|>h\}}|u_n|^{r(x)-1}\,dx\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon. \tag{100}$$

We conclude that, the sequence $(|u_n|^{r(x)-1})_n$ is uniformly equi-integrable, and since $|u_n|^{r(x)-1}\to|u|^{r(x)-1}$ almost everywhere in Ω. In view of Vitali's theorem we conclude that

$$|u_n|^{r(x)-1}\longrightarrow|u|^{r(x)-1}\qquad\text{strongly in}\quad L^1(\Omega). \tag{101}$$

Moreover, we have $f_n(x,T_n(u_n))$, converge to $f(x,u)$ almost everywhere in Ω, by using (66) and Young's inequality, we obtain

$$|f_n(x,u_n)|\le|f_0(x)|+c(x)|u_n|^{\gamma(x)}\le|f_0(x)|+|c(x)|^{\frac{r(x)-1}{r(x)-1-\gamma(x)}}+|u_n|^{r(x)-1}\qquad\text{a.e in}\quad\Omega, \tag{102}$$

Thus, the sequence $(f_n(x,u_n))_n$ is uniformly equi-integrable in Ω, and in view of Vitali's theorem we conclude that

$$f_n(x,u_n)\longrightarrow f(x,u)\qquad\text{strongly in}\quad L^1(\Omega). \tag{103}$$

Step 5: Almost everywhere convergence of the gradients

Let $h>k\ge\max(1,\|\psi^+\|)$, and we set

$$S_h(s)=1-\frac{|T_{2h}(s)-T_h(s)|}{h}$$

We have $v=u_n-\eta(T_k(u_n)-T_k(u))S_h(u_n)\in W^{1,\vec{p}(\cdot)}(\Omega)$ and $v\ge\psi$ for η small enough, then v is an admissible test function to the approximate problem (69), we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i((T_k(u_n)-T_k(u))S_h(u_n))\,\mathrm{d}x\\&+\int_\Omega|u_n|^{r(x)-2}u_n(T_k(u_n)-T_k(u))S_h(u_n)\,\mathrm{d}x\\&\le\int_\Omega f_n(x,u_n)(T_k(u_n)-T_k(u))S_h(u_n)\,\mathrm{d}x+\int_{\partial\Omega}g_n(x)(T_k(u_n)-T_k(u))S_h(u_n)\,\mathrm{d}\sigma.\end{aligned} \tag{104}$$

Then, we have

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))S_h(u_n)\,\mathrm{d}x\\&+\int_\Omega|u_n|^{r(x)-2}u_n(T_k(u_n)-T_k(u))S_h(u_n)\,\mathrm{d}x\\&\le\int_\Omega|f_n(x,u_n)|\,|(T_k(u_n)-T_k(u))|S_h(u_n)\,dx+\int_{\partial\Omega}|g_n(x)|\,|T_k(u_n)-T_k(u)|S_h(u_n)\,d\sigma\\&+\frac{1}{h}\sum_{i=1}^N\int_{\{h<|u_n|<2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n|(T_k(u_n)-T_k(u))|\,\mathrm{d}x,\end{aligned} \tag{105}$$

We have $S_h(u_n) = 1$ on the set $\{|u_n| \le k\}$, and $T_k(u_n) - T_k(u)$ has the same sign u_n on the set $\{|u_n| > k\}$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))\, dx \\
&+ \int_{\{|u_n| \le k\}} |T_k(u_n)|^{r(x)-2} T_k(u_n)(T_k(u_n) - T_k(u))\, \mathrm{d}x \\
&+ \int_{\{|u_n| > k\}} |u_n|^{r(x)-1} |T_k(u_n) - T_k(u)| S_h(u_n)\, \mathrm{d}x \\
&\le \int_{\Omega} |f_n(x, u_n)|\, |T_k(u_n) - T_k(u)|\, dx + \int_{\partial\Omega} |g_n(x)|\, |T_k(u_n) - T_k(u)|\, d\sigma \\
&+ \frac{2k}{h} \sum_{i=1}^{N} \int_{\{h < |u_n| < 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx \\
&+ \sum_{i=1}^{N} \int_{\{k < |u_n| \le 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\, |D^i T_k(u)|\, dx.
\end{aligned} \tag{106}$$

Thus, we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\, dx \\
&\qquad + \int_{\{|u_n| \le k\}} (|T_k(u_n)|^{r(x)-2} T_k(u_n) - |T_k(u)|^{r(x)-2} T_k(u))(T_k(u_n) - T_k(u))\, \mathrm{d}x \\
&\le \int_{\Omega} |f_n(x, u_n)|\, |T_k(u_n) - T_k(u)|\, dx + \int_{\{|u_n| \le k\}} |T_k(u)|^{r(x)-2} T_k(u)(T_k(u_n) - T_k(u))\, dx \\
&+ \int_{\partial\Omega} |g(x)|\, |T_k(u_n) - T_k(u)|\, d\sigma + \frac{2k}{h} \sum_{i=1}^{N} \int_{\{h < |u_n| < 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx \\
&+ \sum_{i=1}^{N} \int_{\Omega} |a_i(x, T_k(u_n), \nabla T_k(u))| |D^i T_k(u_n) - D^i T_k(u)|\, dx \\
&+ \sum_{i=1}^{N} \int_{\{k < |u_n| \le 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\, |D^i T_k(u)|\, dx,
\end{aligned} \tag{107}$$

For the first term on the right-hand side of (107), in view of (103) we have $f_n(x, u_n)$ tends to $f(x, u)$ strongly in $L^1(\Omega)$, and since $T_k(u_n) - T_k(u) \rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$ it follows that

$$\int_{\Omega} |f_n(x, u_n)|\, |(T_k(u_n) - T_k(u))|\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{108}$$

Also, we have $|T_k(u)|^{r(x)-2} T_k(u) \in L^1(\Omega)$ then

$$\int_{\{|u_n| \le k\}} |T_k(u)|^{r(x)-2} T_k(u)(T_k(u_n) - T_k(u))\, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{109}$$

Similarly, we have $g(x) \in L^1(\partial\Omega)$ and since $(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak$-*$ in $L^\infty(\partial\Omega)$, then

$$\int_{\partial\Omega} |g(x)|\, |T_k(u_n) - T_k(u)|\, d\sigma \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{110}$$

Concerning the fourth term of the right-hand side of (107), using (97) we have

$$\frac{2k}{h} \sum_{i=1}^{N} \int_{\{h < |u_n| < 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{111}$$

For the two last terms on the right-hand side of (107), we have $T_k(u_n) \to T_k(u)$ strongly in $L^{p_i(\cdot)}(\Omega)$, then $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and since $D^i T_k(u_n)$ tends to $D^i T_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we conclude that

$$\sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))||D^i T_k(u_n) - D^i T_k(u)|\, dx \to 0 \text{ as } n \to \infty. \tag{112}$$

Moreover, we have $(a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a measurable function $\xi_{2h} \in L^{p_i'(\cdot)}(\Omega)$ such that $a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n)) \rightharpoonup \xi_{2h}$ weakly in $L^{p_i'(\cdot)}(\Omega)$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{N} \int_{\{k<|u_n|\le 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|\, |D^i T_k(u)|\, dx \\
&= \sum_{i=1}^{N} \int_{\{k<|u|\le 2h\}} \xi_{2h}\, |D^i T_k(u)|\, dx = 0.
\end{aligned} \tag{113}$$

By combining (107) and (108) – (113), we conclude that

$$\sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\, dx \to 0 \tag{114}$$

as n tends to infinity, and since $T_k(u_n) \to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\, dx \\
&+ \int_\Omega (|T_k(u_n)|^{\underline{p}-2} T_k(u_n) - |T_k(u)|^{\underline{p}-2} T_k(u))(T_k(u_n) - T_k(u))\, dx \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{115}$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad W^{1,\vec{p}(\cdot)}(\Omega), \\ D^i u_n \to D^i u & \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \dots, N. \end{cases} \tag{116}$$

Step 6: Passage to the limit.

Let $\varphi \in K_\psi \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$. By taking $v = u_n - \eta T_k(u_n - \varphi)$ as a test function for the approximate problem (69), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\, \mathrm{d}x + \int_\Omega |u_n|^{r(x)-2} u_n T_k(u_n - \varphi)\, \mathrm{d}x \\
&\le \int_\Omega f_n(x, u_n) T_k(u_n - \varphi)\, \mathrm{d}x + \int_{\partial\Omega} g_n(x) T_k(u_n - \varphi)\, \mathrm{d}\sigma,
\end{aligned} \tag{117}$$

we have $\{|u_n - \varphi| \le k\} \subseteq \{|u_n| \le M\}$, then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u_n),\nabla T_M(u_n))-a_i(x,T_M(u_n),\nabla\varphi))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,\mathrm{d}x\\
&+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,\mathrm{d}x.
\end{aligned}\tag{118}$$

In view of Fatou's Lemma, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,\mathrm{d}x\\
&\ge\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))(D^iT_M(u)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,\mathrm{d}x\\
&\quad+\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,\mathrm{d}x.
\end{aligned}\tag{119}$$

Moreover, we have $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-\star$ in $L^\infty(\Omega)$. Having in mind (101) and (103), we conclude that

$$\int_{\Omega}|u_n|^{r(x)-2}u_nT_k(u_n-\varphi)\,\mathrm{d}x\longrightarrow\int_{\Omega}|u|^{r(x)-2}uT_k(u-\varphi)\,\mathrm{d}x.\tag{120}$$

and

$$\int_{\Omega}f_n(x,u_n)T_k(u_n-\varphi)\,\mathrm{d}x\longrightarrow\int_{\Omega}f(x,u)T_k(u-\varphi)\,\mathrm{d}x.\tag{121}$$

Also, since $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ weak$-\star$ in $L^\infty(\partial\Omega)$ we get

$$\int_{\partial\Omega}g_n(x)T_k(u_n-\varphi)\,\mathrm{d}\sigma\longrightarrow\int_{\partial\Omega}g(x)T_k(u-\varphi)\,\mathrm{d}\sigma.\tag{122}$$

Finally, by combining (117) and (119) $-$ (123), we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,\mathrm{d}x+\int_{\Omega}|u|^{r(x)-2}uT_k(u-\varphi)\,\mathrm{d}x\\
&\le\int_{\Omega}f(x,u)T_k(u-\varphi)\,\mathrm{d}x+\int_{\partial\Omega}g(x)T_k(u-\varphi)\,\mathrm{d}\sigma\quad\text{for all}\quad\varphi\in K_\psi\cap L^\infty(\Omega),
\end{aligned}\tag{123}$$

which complete the proof of the theorem 5.1.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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