Asia Pac. J. Math. 2024 **11**:17

MIXED TYPE OF ADDITIVE QUADRATIC QUARTIC ($\mathcal{AQ}_2\mathcal{Q}_4$) FUNCTIONAL EQUATION AND ITS STABILITY OVER NON-ARCHIMEDEAN NORMED SPACE

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Received Nov. 2, 2023

Abstract. In this paper, we discussed the general solution and the generalized Hyers-Ulam stability (GHUS) of the mixed type of an additive quadratic quartic ($\mathcal{AQ}_2\mathcal{Q}_4$) functional equation

$$2\left[\mathfrak{f}(y+\nu z)+\mathfrak{f}(y-\nu z)\right]=\nu^2\left[\mathfrak{f}(y+z)+\mathfrak{f}(y-z)\right]+\nu^2\left[\mathfrak{f}(z-y)+\mathfrak{f}(-y-z)\right]$$
$$-(2\nu^2-4)\mathfrak{f}(y)-2\nu^2\mathfrak{f}(-y)+2\left[\mathfrak{f}(\nu z)+\mathfrak{f}(-\nu z)\right]-2\nu^2\left[\mathfrak{f}(z)+\mathfrak{f}(-z)\right]$$

for a fixed integer $\nu \neq 0, \pm 1$ in two variables over non-Archimedean normed space with some suitable counter examples.

2020 Mathematics Subject Classification. 39B62; 39B72; 12J25.

Key words and phrases. Hyers-Ulam stability (HUS); additive function; quadratic function; quartic function; non-Archimedean normed (NAN) space.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [20] in 1940, concerning the stability of group homomorphisms. "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". In 1941, Hyers [12] gave the very first positive response to the question of Ulam for Banach spaces. Aoki [1] generalized the Hyers theorem for additive mappings. In 1978, Rassias [18] provided a generalization of Hyers theorem which allows the Cauchy difference to be unbounded. In 1991, Gajada [6] answered the question for the case $p > 1$, which was raised by Rassias. In 2007, Moslehian and Rassias [16] proved GHUS of the Cauchy functional equation and the quadratic functional equation in NAN spaces.

DOI: 10.28924/APJM/11-17

The quadratic quartic functional equation was introduced by Lee et al., [14]. Later, Gordji et al., [7] generalized quadratic-quartic functional equation in quasi Banach space. There are a number of references available that provide a detailed description of mixed type functional equations [3–5,8–10,17].

Mohamadi et al., in [15] introduced the additive quadratic quartic ($\mathcal{A}\mathcal{Q}_2\mathcal{Q}_4$) functional equation in complete random normed spaces.

In this article, we discuss the GHUS of the mixed type of an additive quadratic quartic ($\mathcal{A}\mathcal{Q}_2\mathcal{Q}_4$) functional equation

$$\mathcal{Q}\mathfrak{f}(y,z) : 2[\mathfrak{f}(y+\nu z)+\mathfrak{f}(y-\nu z)] - \nu^2\left[\mathfrak{f}(y+z)+\mathfrak{f}(y-z)\right] - \nu^2\left[\mathfrak{f}(z-y)+\mathfrak{f}(-y-z)\right] + (2\nu^2-4)\mathfrak{f}(y) + 2\nu^2\mathfrak{f}(-y) - 2\left[\mathfrak{f}(\nu z)+\mathfrak{f}(-\nu z)\right] + 2\nu^2\left[\mathfrak{f}(z)+\mathfrak{f}(-z)\right] \quad (1)$$

in non-Archimedean normed space.

2. Preliminaries

In 1897, Hensel [11] has introduced a normed space which does not have the Archimedean property. It turnout that non-Archimedean spaces have many nice applications [13,19,21,22]. The basic definition and properties of non-Archimedean space are as follows.

Definition 2.1. [2] *A non-Archimedean field is a field $\mathcal{K}$ equipped with a function (valuation) $|.|$ from $\mathcal{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r+s| \leq \max\{|r|,|s|\}$ for all $r,s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|\eta| \leq 1$ for all $\eta \in \mathcal{N}$. An example of a non-Archimedean valuation is the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$ this valuation is called trivial.*

Definition 2.2. [11] *Let $\mathcal{Y}$ be a linear space over a non-Archimedean field $\mathcal{K}$ with a non-trivial valuation $|.|$. A function norm from $\mathcal{Y}$ to $\mathcal{R}$ is a non-Archimedean norm if it satisfies the following conditions:*

(NA1) $\|r\| \geq 0$ *and* $= 0$ *iff* $r = 0$,

(NA2) $\|\alpha r\| = |\alpha|\|r\|, \alpha \in \mathcal{K}, r \in \mathcal{Y}$,

(NA3) $\|r+s\| \leq \max\{\|r\|,\|s\|\}$, *for all* $r,s \in \mathcal{Y}$.

Then $(\mathcal{Y},\|.\|)$ is called a non-Archimedean normed space.

It follows from (NA3) that

$$\|y_p - y_q\| \leq \max\{\|y_{r+1}-y_r\| : q \leq r \leq p-1\}\ (p > q).$$

Therefore a sequence $\{y_p\}$ is Cauchy in $\mathcal{Y}$ iff $\{y_{p+1}-y_p\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for $x,y > 0$, there exists $\eta \in \mathcal{N}$ such that $x < \eta y$.

Throughout this article, we assume that $\mathcal{Y}$ is a non-Archimedean normed space and let $\mathcal{Z}$ be a complete non-Archimedean normed space. The generalized additive quadratic quartic $(\mathcal{A}\mathcal{Q}_2\mathcal{Q}_4)$ functional inequality is defined as below.

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \xi(y,z) \qquad \forall\, y,z \in \mathcal{Y}.$$

3. Main Results

To achieve our goal in this article, we need the following two lemmas.

Lemma 1. *An odd function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ that satisfy (1) is additive.*

Proof. Replacing z by y and put $\mathfrak{f}(0) = 0$ in (1), using oddness

$$\mathfrak{f}((\nu+1)y) - \mathfrak{f}((\nu-1)y) = 2\mathfrak{f}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{2}$$

Using(2) by induction method, we get

$$\mathfrak{f}(\nu y) = \nu \mathfrak{f}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{3}$$

Substituting y by νy in (1), using oddness we get

$$\nu[\mathfrak{f}(y+z) + \mathfrak{f}(y-z) - 2\mathfrak{f}(y)] = 0 \qquad \forall\, y,z \in \mathcal{Y}. \tag{4}$$

Since $\nu \neq 0$ from (4), we get

$$\mathfrak{f}(y+z) + \mathfrak{f}(y-z) = 2\mathfrak{f}(y) \qquad \forall\, y,z \in \mathcal{Y}. \tag{5}$$

Substituting y by z and z by y in (5), using oddness we get

$$\mathfrak{f}(y+z) - \mathfrak{f}(y-z) = 2\mathfrak{f}(z) \qquad \forall\, y,z \in \mathcal{Y}. \tag{6}$$

Adding (4) and (5) gives

$$\mathfrak{f}(y+z) = \mathfrak{f}(y) + \mathfrak{f}(z) \qquad \forall\, y,z \in \mathcal{Y}. \tag{7}$$

Therefore $\mathfrak{f}$ is an additive mapping. □

Lemma 2. *An even function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ that satisfy (1) is quadratic-quartic.*

Proof. Using evenness in (1), we get

$$\mathfrak{f}(\nu z+y) + \mathfrak{f}(\nu z-y) = \nu^2[\mathfrak{f}(y+z) + \mathfrak{f}(y-z)] - (2\nu^2-2)\mathfrak{f}(y) + 2\mathfrak{f}(\nu z) - 2\nu^2\mathfrak{f}(z) \qquad \forall\, y,z \in \mathcal{Y}. \tag{8}$$

Replacing z=y, $2y$, $3y$ in (8), we get

$$\mathfrak{f}\big((\nu+1)y\big) + \mathfrak{f}\big((\nu-1)y\big) = 2\mathfrak{f}(\nu y) + \nu^2\mathfrak{f}(2y) - (4\nu^2-2)\mathfrak{f}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{9}$$

$$\mathfrak{f}\big((2\nu+1)y\big) + \mathfrak{f}\big((2\nu-1)y\big) = 2\mathfrak{f}(2\nu y) + \nu^2\mathfrak{f}(3y) - 2\nu^2\mathfrak{f}(2y) - (\nu^2-2)\mathfrak{f}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{10}$$

$$\mathfrak{f}\big((3\nu+1)y\big)+\mathfrak{f}\big((3\nu-1)y\big)=2\mathfrak{f}(3\nu y)+\nu^2\mathfrak{f}(4y)-2\nu^2\mathfrak{f}(3y)+\nu^2\mathfrak{f}(2y)\ \ -(2\nu^2-2)\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{11}$$

In (9), replacing y with $2y$, we get

$$\mathfrak{f}\big((2\nu+2)y\big)+\mathfrak{f}\big((2\nu-2)y\big)=2\mathfrak{f}(2\nu y)+\nu^2\mathfrak{f}(4y)-(4\nu^2-2)\mathfrak{f}(2y)\quad \forall\, y\in\mathcal{Y}. \tag{12}$$

Replacing y by $2y, (\nu+1)y, (\nu-1)y, (2\nu+1)y, (2\nu-1)y, \nu y, 2\nu y$ and z by y in (8), respectively, we get

$$\mathfrak{f}\big((\nu+2)y\big)+\mathfrak{f}\big((\nu-2)y\big)=2\mathfrak{f}(\nu y)+\nu^2\mathfrak{f}(3y)-(2\nu^2-2)\mathfrak{f}(2y)-\nu^2\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{13}$$

$$\mathfrak{f}\big((2\nu+1)y\big)=\nu^2\mathfrak{f}\big((\nu+2)y\big)-(2\nu^2-2)\mathfrak{f}\big((\nu+1)y\big)+(\nu^2+2)\mathfrak{f}(\nu y)-(2\nu^2+1)\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{14}$$

$$\mathfrak{f}\big((2\nu-1)y\big)=\nu^2\mathfrak{f}\big((\nu-2)y\big)-(2\nu^2-2)\mathfrak{f}((\nu-1)y)+(\nu^2+2)\mathfrak{f}(\nu y)-(2\nu^2+1)\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{15}$$

$$\begin{aligned}\mathfrak{f}\big((3\nu+1)y\big)=-\mathfrak{f}\big((\nu+1)y\big)+\nu^2\mathfrak{f}((2\nu+2)y)-(2\nu^2-2)\mathfrak{f}((2\nu+1)y)+\nu^2\mathfrak{f}(2\nu y)&\\ +2\mathfrak{f}(\nu y)-2\nu^2\mathfrak{f}(y)&\quad \forall\, y\in\mathcal{Y}.\end{aligned} \tag{16}$$

$$\begin{aligned}\mathfrak{f}\big((3\nu-1)y\big)=-\mathfrak{f}\big((\nu-1)y\big)+\nu^2\mathfrak{f}((2\nu-2)y)-(2\nu^2-2)\mathfrak{f}((2\nu-1)y)+\nu^2\mathfrak{f}(2\nu y)&\\ +2\mathfrak{f}(\nu y)-2\nu^2\mathfrak{f}(y)&\quad \forall\, y\in\mathcal{Y}.\end{aligned} \tag{17}$$

$$2\mathfrak{f}(2\nu y)=8\mathfrak{f}(\nu y)+2\nu^4\mathfrak{f}(2y)-8\nu^4\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{18}$$

$$2\mathfrak{f}(3\nu y)=4\mathfrak{f}(2\nu y)+2\mathfrak{f}(\nu y)+2\nu^4\mathfrak{f}(3y)-4\nu^4\mathfrak{f}(2y)-2\nu^4\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}. \tag{19}$$

Combining (14) and (15), we obtain

$$\begin{aligned}&\mathfrak{f}\big((2\nu+1)y\big)+\mathfrak{f}\big((2\nu-1)y\big)=\nu^2[\mathfrak{f}((\nu+2)y)+\mathfrak{f}((\nu-2)y)]\\ &-(2\nu^2-2)[\mathfrak{f}((\nu+1)y)+\mathfrak{f}(\nu-1)y]+2(\nu^2+2)\mathfrak{f}(\nu y)-2(2\nu^2+1)\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}.\end{aligned} \tag{20}$$

Using (9) and (13) in (20), we get

$$\begin{aligned}\mathfrak{f}\big((2\nu+1)y\big)+\mathfrak{f}\big((2\nu-1)y\big)=&8\mathfrak{f}(\nu y)+\nu^4\mathfrak{f}(3y)+(4\nu^2-4\nu^4)\mathfrak{f}(2y)\\ &+(7\nu^4-16\nu^2+2)\mathfrak{f}(y)\quad \forall\, y\in\mathcal{Y}.\end{aligned} \tag{21}$$

Equating (10) and (21), we get

$$2\mathfrak{f}(2\nu y)-8\mathfrak{f}(\nu y)+(\nu^2-\nu^4)\mathfrak{f}(3y)+(4\nu^4-6\nu^2)\mathfrak{f}(2y)+(15\nu^2-7\nu^4)\mathfrak{f}(y)=0\ \forall\, y\in\mathcal{Y}. \tag{22}$$

Substituting (18) in (22), we get

$$4\mathfrak{f}(3y)-24\mathfrak{f}(2y)+60\mathfrak{f}(y)=0\quad \forall\, y\in\mathcal{Y}. \tag{23}$$

Combining (16) and (17), we get

$$\begin{aligned}\mathfrak{f}\big((3\nu+1)y\big)+\mathfrak{f}\big((3\nu-1)y\big) &= -[\mathfrak{f}\big((\nu+1)y\big)+\mathfrak{f}\big((\nu-1)y\big)]+\nu^2[\mathfrak{f}\big((2\nu+2)y\big)\\ &+\mathfrak{f}\big((2\nu-2)y\big)]-(2\nu^2-2)[\mathfrak{f}\big((2\nu+1)y\big)+\mathfrak{f}\big((2\nu-2)y\big)]+2\nu^2\mathfrak{f}(2\nu y)+4\mathfrak{f}(\nu y)-4\nu^2\mathfrak{f}(y)\end{aligned} \tag{24}$$

Substituting (9), (10) and (12) in (24), we get

$$\begin{aligned}\mathfrak{f}\big((3\nu+1)y\big)+\mathfrak{f}\big((3\nu-1)y\big) =& 4\mathfrak{f}(2\nu y)+2\mathfrak{f}(\nu y)+\nu^4\mathfrak{f}(4y)+(2\nu^2-2\nu^4)\mathfrak{f}(3y)-3\nu^2\mathfrak{f}(2y)\\ &+(2\nu^4-6\nu^2+2)\mathfrak{f}(y)\ \forall\, y\in\mathcal{Y}.\end{aligned} \tag{25}$$

Equating (11) and (25), we get

$$\begin{aligned}2\mathfrak{f}(3\nu y)-4\mathfrak{f}(2\nu y)-2\mathfrak{f}(\nu y)+(\nu^2-\nu^4)\mathfrak{f}(4y)+(-4\nu^2+2\nu^4)\mathfrak{f}(3y)+4\nu^2\mathfrak{f}(2y)&\\ +(4\nu^2-2\nu^4)\mathfrak{f}(y)=0 &\qquad \forall\, y\in\mathcal{Y}.\end{aligned} \tag{26}$$

Substituting (19) in (26), we get

$$\mathfrak{f}(4y)-4\mathfrak{f}(3y)+4\mathfrak{f}(2y)+4\mathfrak{f}(y)=0 \qquad \forall\, y\in\mathcal{Y}. \tag{27}$$

Substituting (23) in (27), we get

$$\mathfrak{f}(4y)-20\mathfrak{f}(2y)+64\mathfrak{f}(y)=0 \qquad \forall\, y\in\mathcal{Y}. \tag{28}$$

The desired results can be attained from relation (28). □

Theorem 3. *Let a function ξ from $\mathcal{Y}^2\to[0,\infty)$ be such that*

$$\lim_{\eta\to\infty}\frac{\xi\big((\nu+1)^\eta y,(\nu+1)^\eta z\big)}{|\nu+1|^\eta}=0 \qquad \forall\, y,z\in\mathcal{Y}. \tag{29}$$

$$\lim_{\eta\to\infty}\frac{1}{|2(\nu+1)^\eta|}\xi\big((\nu+1)^{\eta-1}y,(\nu+1)^{\eta-1}y\big)=0 \qquad \forall\, y,z\in\mathcal{Y}. \tag{30}$$

The limit exists for each $y\in\mathcal{Y}$

$$\lim_{\eta\to\infty}\max\left\{\frac{1}{|\nu+1|^\jmath}\xi\big((\nu+1)^\jmath y,(\nu+1)^\jmath y\big):0\le\jmath<\eta\right\}, \tag{31}$$

indicated by $\psi_{\mathcal{A}}(y)$. Assume that $\mathfrak{f}:\mathcal{Y}\to\mathcal{Z}$ is an odd function that satisfies the inequality.

$$\|\mathcal{Q}\mathfrak{f}(y,z)\|\le\xi(y,z) \qquad \forall\, y,z\in\mathcal{Y}. \tag{32}$$

Then there exist an additive mapping $\mathcal{A}(y):\mathcal{Y}\to\mathcal{Z}$ such that

$$\|\mathfrak{f}(y)-\mathcal{A}(y)\|\le\frac{1}{|2(\nu+1)|}\psi_{\mathcal{A}}(y) \qquad \forall\, y\in\mathcal{Y}. \tag{33}$$

Moreover, if

$$\lim_{\iota\to\infty}\lim_{\eta\to\infty}\frac{1}{|\nu+1|^\iota}\max\left\{\max\frac{1}{|\nu+1|^\iota}\xi\big((\nu+1)^\jmath y,(\nu+1)^\jmath y\big):0\le\jmath<\eta+\iota\right\}=0 \qquad \forall\, y\in\mathcal{Y}, \tag{34}$$

then $\mathcal{A}(y)$ is the unique additive mapping satisfies (33).

Proof. By taking $z{=}y$ in (32), we get

$$\|\mathfrak{f}\big((\nu+1)y\big)-(\nu+1)\mathfrak{f}(y)\| \leq \frac{1}{|2|}\xi(y,y) \qquad \forall y \in \mathcal{Y}. \tag{35}$$

Dividing (35) by $|\nu+1|$, we get

$$\left\|\frac{\mathfrak{f}\big((\nu+1)y\big)}{\nu+1}-\mathfrak{f}(y)\right\| \leq \frac{1}{|2(\nu+1)|}\xi(y,y) \qquad \forall y \in \mathcal{Y}. \tag{36}$$

Replacing y by $(\nu+1)^{\eta-1}y$ in (36), we obtain

$$\left\|\frac{\mathfrak{f}\big((\nu+1)^{\eta}y\big)}{(\nu+1)^{\eta}}-\frac{\mathfrak{f}\big((\nu+1)^{\eta-1}y\big)}{(\nu+1)^{\eta-1}}\right\| \leq \frac{1}{|2(\nu+1)^{\eta}|}\xi\big((\nu+1)^{\eta-1}y,(\nu+1)^{\eta-1}y\big) \qquad \forall y \in \mathcal{Y}. \tag{37}$$

It follows from (30) and (37) the sequence $\left\{\frac{\mathfrak{f}\left((\nu+1)^{\eta}y\right)}{(\nu+1)^{\eta}}\right\}$ is Cauchy. Since $\mathcal{Y}$ is complete, we conclude that $\left\{\frac{\mathfrak{f}\left((\nu+1)^{\eta}y\right)}{(\nu+1)^{\eta}}\right\}$ is convergent.
Let $\mathcal{A}(y)=\lim_{\eta\to\infty}\left\{\frac{\mathfrak{f}\left((\nu+1)^{\eta}y\right)}{(\nu+1)^{\eta}}\right\}$.

By using induction one can show that

$$\left\|\frac{\mathfrak{f}\big((\nu+1)^{\eta}y\big)}{(\nu+1)^{\eta}}-\mathfrak{f}(y)\right\| \leq \frac{1}{|2(\nu+1)|}\max\left\{\frac{1}{|\nu+1|^{\iota}}\xi\big((\nu+1)^{\jmath}y,(\nu+1)^{\jmath}y\big):0\leq \jmath<\eta\right\}$$
$$\forall\, \eta \in \mathcal{N}, y \in \mathcal{Y}. \tag{38}$$

By taking η to approach infinity in (38) and using (31) we get (33). From (29) and (32), we get

$$\begin{aligned}\|D\mathcal{A}(y,z)\| &= \lim_{\eta\to\infty}\frac{1}{|\nu+1|^{\eta}}\|D\mathfrak{f}\big((\nu+1)^{\eta}y,(\nu+1)^{\eta}z\big)\|\\ &\leq \lim_{\eta\to\infty}\frac{1}{|\nu+1|^{\eta}}\|\xi\big((\nu+1)^{\eta}y,(\nu+1)^{\eta}z\big)\|\\ &= 0\ \forall y,z \in \mathcal{Y}.\end{aligned}$$

Therefore the function $\mathcal{A}(y):\mathcal{Y}\to\mathcal{Z}$ satisfies (1).

Uniqueness: Let there exist another additive function $\mathcal{A}'(y)$

$$\begin{aligned}\|\mathcal{A}(y)-\mathcal{A}'(y)\| &= \lim_{\iota\to\infty}|\nu+1|^{-\iota}\|\mathcal{A}\big((\nu+1)^{\iota}y\big)-\mathcal{A}\big((\nu+1)^{\iota}y\big)\|\\ &= \lim_{\iota\to\infty}|\nu+1|^{-\iota}\max\left\{\|\mathcal{A}\big((\nu+1)^{\iota}y\big)-\mathfrak{f}\big((\nu+1)^{\iota}y\big)\|,\|\mathfrak{f}\big((\nu+1)^{\iota}y\big)-\mathcal{A}\big((\nu+1)^{\iota}y\big)\|\right\}\\ &= \frac{1}{|2(\nu+1)|}\lim_{\iota\to\infty}\lim_{\eta\to\infty}\frac{1}{|\nu+1|^{\iota}}\max\left\{\max\frac{1}{|\nu+1|^{\iota}}\xi\big((\nu+1)^{\jmath}y,(\nu+1)^{\jmath}y\big):\leq \jmath<\eta+\iota\right\}=0\end{aligned}$$

for all $y \in \mathcal{Y}$. Hence $\mathcal{A}(y)=\mathcal{A}'(y)$. This completes the proof of uniqueness. □

Corollary 4. *Let r, s and δ are positive real numbers and let $r+s<1$. Define a function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ and if $\mathfrak{f}$ is a additive mapping satisfies the inequality*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \delta(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^r\|z\|^s) \qquad \forall\, y, z \in \mathcal{Y}.$$

Then, there is a unique additive function $\mathcal{A}(y) : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|\mathfrak{f}(y) - \mathcal{A}(y)\| \leq \frac{3\delta\|y\|^{r+s}}{|2(\nu+1)|} \qquad \forall\, y \in \mathcal{Y}.$$

For the case $r+s=1$, we have the following counterexample.

Example 3.1. *Let $p>2$ be a prime number and $\mathfrak{f} : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $\mathfrak{f}(y) = y+1$. Since $|2^\eta|_p = 1$ for all $\eta \in \mathcal{N}$. Then for $\delta > 0$,*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| = |\nu^2 - 1|_p \leq \delta\Big(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^r\|z\|^s\Big) \qquad \forall\, y, z \in \mathcal{Y}.$$

and

$$\Big\|\frac{\mathfrak{f}((\nu+1)^\eta y)}{(\nu+1)^\eta} - \frac{\mathfrak{f}((\nu+1)^{\eta-1} y)}{(\nu+1)^{\eta-1}}\Big\| = \frac{|2|_p|\nu|_p}{|\nu+1|_p} \neq 0.$$

Hence $\left\{\frac{\mathfrak{f}((\nu+1)^\eta y)}{(\nu+1)^\eta}\right\}$ is not a Cauchy sequence.

Theorem 5. *Let a function ξ from $\mathcal{Y}^2 \to [0,\infty)$ be such that*

$$\lim_{\nu\to\infty} \frac{1}{|2|^{2\nu}}\bar{\varphi}(2^\nu y) = 0 = \lim_{\nu\to\infty} \frac{1}{|2|^{2\nu}} \max\Big\{\xi(2^{\nu+1}y, 2^{\nu+1}z), |16|\xi(2^\nu y, 2^\nu z)\Big\} \qquad \forall\, y, z \in \mathcal{Y}. \quad (39)$$

Let $\mathfrak{f}$ be an even function from $\mathcal{Y} \to \mathcal{Z}$ that satisfies the inequality

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \xi(y,z) \quad (40)$$

and $\mathfrak{f}(0) = 0$. Then, uniqueness of quadratic function $\mathcal{Q}_2(y) : \mathcal{Y} \to \mathcal{Z}$ exists and

$$\|\mathfrak{f}(2y) - 16\mathfrak{f}(y) - \mathcal{Q}_2(y)\| \leq \frac{1}{|2|^2}\psi_{q_2}(y) \qquad \forall\, y \in \mathcal{Y}, \quad (41)$$

where

$$\psi_{q_2}(y) = \lim_{\nu\to\infty} \max\Big\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^\jmath y) : 0 \leq \jmath < \nu\Big\}, \quad (42)$$

$$\begin{aligned}
\bar{\Phi}(y) = \max\Bigg\{ & \frac{1}{|\nu^2 - \nu^4|} \max\Big\{ \max\{\frac{1}{|2|}\xi((2\nu+1)y, y), \frac{1}{|2|}\xi((2\nu-1)y, y)\}, \\
& \frac{1}{|2|}\xi(y, 3y), \frac{1}{|2|}\xi(y,y), \frac{1}{|2|}\xi(2y, 2y), \frac{1}{|2|}\xi(y, 2y), \xi(2\nu y, y)\Big\}, \\
& \frac{4}{|\nu^2 - \nu^4|} \max\Big\{ \max\{\frac{1}{|2|}\xi\big((\nu+1)y, y\big), \frac{1}{|2|}\xi\big((\nu-1)y, y\big)\}, \\
& \frac{1}{|2|}\xi(y, 2y), \frac{1}{|2|}\xi(2y, y), \frac{1}{|2|}\xi(y,y), \frac{1}{|2|}\xi(\nu y, y)\Big\}\Bigg\} \qquad \forall\, y \in \mathcal{Y}. \quad (43)
\end{aligned}$$

Moreover, if

$$\lim_{\iota\to\infty}\lim_{\nu\to\infty}\max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}y):\iota\le\jmath<\nu+\iota\right\}=0 \qquad \forall\, y\in\mathcal{Y}, \tag{44}$$

then $\mathcal{Q}_2(y)$ is the unique quadratic function satisfying (41).

Proof. Using evenness in (40), we get

$$\|[\mathfrak{f}(\nu z+y)+\mathfrak{f}(\nu z-y)]-\nu^2[\mathfrak{f}(y+z)+\mathfrak{f}(y-z)]+(2\nu^2-2)\mathfrak{f}(y)-2\mathfrak{f}(\nu z)+2\nu^2\mathfrak{f}(z)\|\le\frac{1}{|2|}\xi(y,z) \tag{45}$$

Substituting z by $y, 2y$ and $3y$ in (45), we get

$$\|\mathfrak{f}\big((\nu+1)y\big)+\mathfrak{f}\big((\nu-1)y\big)-2\mathfrak{f}(\nu y)-\nu^2\mathfrak{f}(2y)+(4\nu^2-2)\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi(y,y) \tag{46}$$

$$\|\mathfrak{f}\big((2\nu+1)y\big)+\mathfrak{f}\big((2\nu-1)y\big)-2\mathfrak{f}(2\nu y)-\nu^2\mathfrak{f}(3y)+2\nu^2\mathfrak{f}(2y)+(\nu^2-2)\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi(y,2y) \tag{47}$$

$$\begin{aligned}\|\mathfrak{f}\big((3\nu+1)y\big)+\mathfrak{f}\big((3\nu-1)y\big)-2\mathfrak{f}(3\nu y)-\nu^2\mathfrak{f}(4y)+2\nu^2\mathfrak{f}(3y)\\ -\nu^2\mathfrak{f}(2y)+(2\nu^2-2)\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi(y,3y)\end{aligned} \tag{48}$$

In (46), replace y with $2y$ and z with $2y$, we get

$$\|\mathfrak{f}\big((2\nu+2)y\big)+\mathfrak{f}\big((2\nu-2)y\big)-2\mathfrak{f}(2\nu y)-\nu^2\mathfrak{f}(4y)+(4\nu^2-2)\mathfrak{f}(2y)\|\le\frac{1}{|2|}\xi(2y,2y) \tag{49}$$

Replacing y by $2y, (\nu+1)y, (\nu-1)y, (2\nu+1)y, (2\nu-1)y$ and z by y in (45), we get

$$\|\mathfrak{f}\big((\nu+2)y\big)+\mathfrak{f}\big((\nu-2)y\big)-2\mathfrak{f}(2\nu y)-\nu^2\mathfrak{f}(3y)+(2\nu^2-2)\mathfrak{f}(2y)+\nu^2(\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi(2y,y) \tag{50}$$

$$\begin{aligned}\|\mathfrak{f}\big((2\nu+1)y\big)-\nu^2\mathfrak{f}\big((\nu+2)y\big)+(2\nu^2-2)\mathfrak{f}\big((\nu+1)y\big)-(\nu^2+2)\mathfrak{f}(\nu y)+(2\nu^2+1)\mathfrak{f}(y)\|\\ \le\frac{1}{|2|}\xi((\nu+1)y,y)\end{aligned} \tag{51}$$

$$\begin{aligned}\|\mathfrak{f}\big((2\nu-1)y\big)-\nu^2\mathfrak{f}\big((k-2)y\big)+(2\nu^2-2)\mathfrak{f}((\nu-1)y)-(\nu^2+2)\mathfrak{f}(\nu y)+(2\nu^2+1)\mathfrak{f}(y)\|\\ \le\frac{1}{|2|}\xi((\nu-1)y,y)\end{aligned} \tag{52}$$

$$\begin{aligned}\|\mathfrak{f}\big((3\nu+1)y\big)+\mathfrak{f}\big((\nu+1)y\big)-\nu^2\mathfrak{f}\big((2\nu+2)y\big)+(2\nu^2-2)\mathfrak{f}((2\nu+1)y)-\nu^2\mathfrak{f}(2\nu y)\\ -2\mathfrak{f}(\nu y)+2\nu^2\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi((2\nu+1)y,y)\end{aligned} \tag{53}$$

$$\begin{aligned}\|\mathfrak{f}\big((3\nu-1)y\big)+\mathfrak{f}\big((\nu-1)y\big)-\nu^2\mathfrak{f}\big((2\nu-2)y\big)+(2\nu^2-2)\mathfrak{f}((2\nu-1)y)-\nu^2\mathfrak{f}(2\nu y)\\ -2\mathfrak{f}(\nu y)+2\nu^2\mathfrak{f}(y)\|\le\frac{1}{|2|}\xi((2\nu-1)y,y)\end{aligned} \tag{54}$$

Replacing y by νy and z by y in (45) and using (46), we get

$$\|2\mathfrak{f}(2\nu y) - 8\mathfrak{f}(\nu y) - 2\nu^4\mathfrak{f}(2y) + 8\nu^4\mathfrak{f}(y)\| \le \xi(\nu y, y) \tag{55}$$

Replace y by $2\nu y$ and z by y in (45) and using (47), we get

$$\|2\mathfrak{f}(3\nu y) - 4\mathfrak{f}(2\nu y) - 2\mathfrak{f}(\nu y) - 2\nu^4\mathfrak{f}(3y) + 4\nu^4\mathfrak{f}(2y) + 2\nu^4\mathfrak{f}(y)\| \le \xi(2\nu y, y) \tag{56}$$

Combining (51) and (52), we get

$$\begin{aligned}\|\mathfrak{f}\big((2\nu+1)y\big) + \mathfrak{f}((2\nu-1)y) - \nu^2[\mathfrak{f}((\nu+2)y) + \mathfrak{f}((\nu-2)y)]\\ + (2\nu^2-2)[\mathfrak{f}\big((\nu+1)y\big) + \mathfrak{f}((\nu-1)y)] - 2(\nu^2+2)\mathfrak{f}(\nu x)\\ + 2(2\nu^2+1)\mathfrak{f}(y)\| \le \max\left\{\frac{1}{|2|}\xi((\nu+1)y, y), \frac{1}{|2|}\xi((\nu-1)y, y)\right\}\end{aligned} \tag{57}$$

Using (46),(47),(50) and (55) in (57), we get

$$\begin{aligned}\|4\mathfrak{f}(3y) - 24\mathfrak{f}(2y) + 60\mathfrak{f}(y)\| \le \frac{4}{|\nu^2-\nu^4|}\max\Big\{\max\{\frac{1}{|2|}\xi((\nu+1)y, y),\\ \frac{1}{|2|}\xi((\nu-1)y, y)\}, \frac{1}{|2|}\xi(y, 2y), \xi(\nu y, y), \frac{1}{|2|}\xi(2y, y), \frac{1}{|2|}\xi(y, y)\Big\}\end{aligned} \tag{58}$$

Combining (53) and (54), we get

$$\begin{aligned}\|\mathfrak{f}\big((3\nu+1)y\big) + \mathfrak{f}\big((3\nu-1)y\big) + \mathfrak{f}\big((\nu+1)y\big) + \mathfrak{f}\big((\nu-1)y\big) - \nu^2[\mathfrak{f}((2\nu+2)y)\\ - \mathfrak{f}((2\nu-2)y)] + (2\nu^2-2)[\mathfrak{f}((2\nu+1)y) + \mathfrak{f}((2\nu-1)y)] - 2\nu^2\mathfrak{f}(2\nu y)\\ - 4(\nu y) + 4\nu^2\mathfrak{f}(y)\| \le \max\left\{\frac{1}{|2|}\xi((2\nu+1)y, y), \frac{1}{|2|}\xi((2\nu-1)y, y)\right\}\end{aligned} \tag{59}$$

Using (46),(47),(48),(49) and (56) in (58), we get

$$\begin{aligned}\|\mathfrak{f}(4y) - 4\mathfrak{f}(3y) + 4\mathfrak{f}(2y) + 4\mathfrak{f}(y)\| \le \frac{1}{|\nu^2-\nu^4|}\max\Big\{\max\{\frac{1}{|2|}\xi((2\nu+1)y, y),\\ \frac{1}{|2|}\xi\big((2\nu-1)y, y\big)\}, \frac{1}{|2|}\xi(y, 3y), \frac{1}{|2|}\xi(2\nu y, y), \frac{1}{|2|}\xi(y, y), \frac{1}{|2|}\xi(2y, 2y), \frac{1}{|2|}\xi(y, 2y)\Big\}\end{aligned} \tag{60}$$

Using (58) in (60)

$$\|\mathfrak{f}(4y) - 20\mathfrak{f}(2y) + 64\mathfrak{f}(y)\| \le \max\{4\Phi_1(y), \Phi_2(y)\}$$

$$\|\mathfrak{f}(4y) - 20\mathfrak{f}(2y) + 64\mathfrak{f}(y)\| \le \bar{\Phi}(y) \tag{61}$$

where $\bar{\Phi}(y) = \max\{4\Phi_1(y), \Phi_2(y)\}$.

The rest of the proof is similar to Theorem 3.2 in [17]. □

Corollary 6. *Let r, s and δ are positive real numbers and let $r+s<2$. Define a function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ and if $\mathfrak{f}$ is a quadratic mapping satisfies the inequality*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \delta(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^{r}\|z\|^{s}) \; \forall\, y,z \in \mathcal{Y}.$$

Then, there is a unique quadratic function $\mathcal{Q}_2(y):\mathcal{Y}\to\mathcal{Z}$ such that

$$\|\mathfrak{f}(2y) - 16\mathfrak{f}(y) - \mathcal{Q}_2(y)\| \leq \frac{1}{|2|^2}\psi_{q_2}(y) \qquad \forall\, y \in \mathcal{Y}.$$

where

$$\psi_{q_2}(y) = \lim_{\nu\to\infty} \max\Big\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}y) : 0 \leq \jmath < \nu\Big\}, \tag{62}$$

$\bar{\Phi}(y) = \max\{4\Phi_1(y), \Phi_2(y)\}$

$$\begin{aligned}\Phi_1(y) =& \frac{1}{|\nu^2-\nu^4|}\max\Big\{\max\Big\{\frac{1}{|2|}\delta\Big(|2\nu+1|^{r+s}+1+|2\nu+1|^{r}\Big),\\ &\frac{1}{|2|}\delta\Big(|2\nu-1|^{r+s}+1+|2\nu-1|^{r}\Big)\Big\}, \frac{1}{|2|}\delta\Big(1+|3|^{r+s}+|3|^{s}\Big),\\ &\frac{1}{|2|}3\delta, \frac{1}{|2|}3\delta|2|^{r+s}, \frac{1}{|2|}\delta(1+|2|^{r+s}+|2|^{s}), \delta(|2\nu|^{r+s}+1+|2|^{r})\Big\}\|2^{j}y\|^{r+s}\end{aligned}$$

$$\begin{aligned}\Phi_2(y) =& \frac{1}{|\nu^2-\nu^4|}\max\Big\{\max\Big\{\frac{1}{|2|}\delta\Big(|\nu+1|^{r+s}+1+|\nu+1|^{r}\Big),\\ &\frac{1}{|2|}\delta\Big(|\nu-1|^{r+s}+1+|\nu-1|^{r}\Big)\Big\}, \frac{1}{|2|}\delta\Big(1+|2|^{r+s}+|2|^{s}\Big),\\ &\frac{1}{|2|}\delta\Big(|2|^{r+s}+1+|2|^{r}\Big), \frac{1}{|2|}3\delta, \frac{1}{|2|}\delta(|\nu|^{r+s}+1+|\nu|^{r})\Big\}\|2^{j}y\|^{r+s}.\end{aligned}$$

For the case $r+s=2$, we have the following counter example.

Example 3.2. *Let $p>2$ be a prime number and $\mathfrak{f}:\mathcal{Q}_p\to\mathcal{Q}_p$ be defined by $\mathfrak{f}(y)=y^2+1$. Since $|2^{\eta}|_p=1$ for all $\eta\in\mathcal{N}$. Then for $\delta>0$,*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| = |\nu^2-1|_p \leq \delta\Big(\|y\|^{r+s}+\|z\|^{r+s}+\|y\|^{r}\|z\|^{s}\Big) \qquad \forall y,z\in\mathcal{Y}.$$

and

$$\Big\|\frac{h(2^{\nu}y)}{2^{2\nu}} - \frac{h(2^{\nu-1}x)}{2^{2(\nu-1)}}\Big\| = \frac{|180|_p}{|2^{2\nu}|_p} \neq 0.$$

Hence $\big\{2^{-2\nu}h(2^{\nu}y)\big\}$ is not a Cauchy sequence. Where $h(y)=\mathfrak{f}(2y)-16f(y)$.

Theorem 7. *Let a function ξ from $\mathcal{Y}^2\to[0,\infty)$ be such that*

$$\lim_{\nu\to\infty}\frac{1}{|2|^{4\nu}}\bar{\varphi}(2^{\nu}y) = 0 = \lim_{\nu\to\infty}\frac{1}{|2|^{4\nu}}\max\Big\{\xi(2^{\nu+1}y,2^{\nu+1}z), |4|\xi(2^{\nu}y,2^{\nu}z)\Big\} \qquad \forall\, y,z\in\mathcal{Y}. \tag{63}$$

Let $\mathfrak{f}$ be an even function from $\mathcal{Y} \to \mathcal{Z}$ that satisfies the inequality

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \xi(y,z) \tag{64}$$

and $\mathfrak{f}(0) = 0$. Then, uniqueness of quartic function $\mathcal{Q}_4(y) : \mathcal{Y} \to \mathcal{Z}$ exists and

$$\|\mathfrak{f}(2y) - 4\mathfrak{f}(y) - \mathcal{Q}_4(y)\| \leq \frac{1}{|2|^4}\psi_{q_4}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{65}$$

where

$$\psi_{q_4}(y) = \lim_{\nu\to\infty} \max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\jmath y) : 0 \leq \jmath < \nu\right\} \tag{66}$$

$\bar{\Phi}(y)$ is defined in Theorem (5).

Moreover, if

$$\lim_{\iota\to\infty}\lim_{\nu\to\infty} \max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\jmath y) : \iota \leq \jmath < \nu+\iota\right\} = 0 \qquad \forall\, y \in \mathcal{Y}, \tag{67}$$

then $\mathcal{Q}_4(y)$ is the unique quartic function satisfying (65).

Proof. Similar to the proof of Theorem 5, we obtain,

$$\|\mathfrak{f}(4y) - 20\mathfrak{f}(2y) + 64\mathfrak{f}(y)\| \leq \bar{\Phi}(y) \qquad \forall\, y \in \mathcal{Y}. \tag{68}$$

The rest of the proof is similar to Theorem 3.3 in [17]. □

Corollary 8. *Let r, s and δ are positive numbers and let $r+s<4$. Define a function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ and if $\mathfrak{f}$ is a quartic mapping satisfies the inequality*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \delta(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^r\|z\|^s) \qquad \forall\, y, z \in \mathcal{Y}.$$

Then, there is a unique quartic function $\mathcal{Q}_2(y) : \mathcal{Y} \to \mathcal{Z}$ such that

$$\|\mathfrak{f}(2y) - 4\mathfrak{f}(y) - \mathcal{Q}_2(y)\| \leq \frac{1}{|2|^4}\psi_{q_4}(y) \qquad \forall\, y \in \mathcal{Y}.$$

where

$$\psi_{q_4}(y) = \lim_{\nu\to\infty} \max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^\jmath y) : 0 \leq \jmath < \nu\right\}, \tag{69}$$

$\bar{\Phi}(y) = \max\{4\Phi_1(y), \Phi_2(y)\}$,

$$\begin{aligned}\Phi_1(y) =& \frac{1}{|\nu^2-\nu^4|}\max\Bigg\{\max\Bigg\{\frac{1}{|2|}\delta\Big(|2\nu+1|^{r+s}+1+|2\nu+1|^r\Big),\\ &\frac{1}{|2|}\delta\Big(|2\nu-1|^{r+s}+1+|2\nu-1|^r\Big)\Bigg\}, \frac{1}{|2|}\delta\Big(1+|3|^{r+s}+|3|^s\Big),\\ &\frac{1}{|2|}3\delta, \frac{1}{|2|}3\delta|2|^{r+s}, \frac{1}{|2|}\delta(1+|2|^{r+s}+|2|^s), \delta(|2\nu|^{r+s}+1+|2|^r)\Bigg\}\|2^\jmath y\|^{r+s}\end{aligned}$$

$$\Phi_2(y) = \frac{1}{|\nu^2 - \nu^4|} \max\Big\{ \max\Big\{ \frac{1}{|2|}\delta\Big(|\nu+1|^{r+s} + 1 + |\nu+1|^r\Big), \frac{1}{|2|}\delta\Big(|\nu-1|^{r+s} + 1 + |\nu-1|^r\Big)\Big\}, \frac{1}{|2|}\delta\Big(1 + |2|^{r+s} + |2|^s\Big), \frac{1}{|2|}\delta\Big(|2|^{r+s} + 1 + |2|^r\Big), \frac{1}{|2|}3\delta, \frac{1}{|2|}\delta(|\nu|^{r+s} + 1 + |\nu|^r)\Big\} \|2^j y\|^{r+s}.$$

For the case $r+s=4$, we have the following counter example.

Example 3.3. *Let $p > 2$ be a prime number and $\mathfrak{f} : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $\mathfrak{f}(y) = y^4 + 1$. Since $|2^\eta|_p = 1$ for all $\eta \in \mathcal{N}$. Then for $\delta > 0$,*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| = |\nu^2 - 1|_p \le \delta\Big(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^r\|z\|^s\Big) \qquad \forall\, y,z \in \mathcal{Y}.$$

and

$$\Big\|\frac{h(2^\nu y)}{2^{4\nu}} - \frac{h(2^{\nu-1}x)}{2^{4(\nu-1)}}\Big\| = \frac{|720|_p}{|2^{4\nu}|_p} \neq 0.$$

Hence $\{2^{-2\nu}h(4^\nu y)\}$ is not a Cauchy sequence. Where $h(y) = \mathfrak{f}(2y) - 4\mathfrak{f}(y)$.

Theorem 9. *Let a function ξ from $\mathcal{Y}^2 \to [0,\infty)$ be such that*

$$\lim_{\nu\to\infty} \frac{1}{|2|^{2\nu}}\bar{\varphi}(2^\nu y) = 0 = \lim_{\nu\to\infty} \frac{1}{|2|^{4\nu}}\bar{\varphi}(2^\nu y)\ \forall\, y,z \in \mathcal{Y}. \tag{70}$$

$$\lim_{\nu\to\infty} \frac{1}{|2|^{2\nu}} \max\Big\{\xi(2^{\nu+1}y, 2^{\nu+1}z), |16|\xi(2^\nu y, 2^\nu z)\Big\} = 0 = \lim_{\nu\to\infty} \frac{1}{|2|^{4\nu}} \max\Big\{\xi(2^{\nu+1}y, 2^{\nu+1}z), |4|\xi(2^\nu y, 2^\nu z)\Big\} \qquad \forall\, y,z \in \mathcal{Y}. \tag{71}$$

Let $\mathfrak{f}$ be an even function from $\mathcal{Y} \to \mathcal{Z}$ that satisfies the inequality

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \le \xi(y,z) \tag{72}$$

and $\mathfrak{f}(0) = 0$. Then, uniqueness of quadratic and quartic function exists and

$$\|\mathfrak{f}(y) - \mathcal{Q}_2'(y) - \mathcal{Q}_4'(y)\| \le \frac{1}{|192|}\max\{\psi_{q_4}(y), |4|\psi_{q_2}(y)\} \qquad \forall\, y \in \mathcal{Y}, \tag{73}$$

where $\psi_{q_2}(y)$ is defined in(69) & $\psi_{q_4}(y)$ is defined in (66).

Proof. The proof is similar to Theorem 3.4 in [17]. □

Theorem 10. *Let a function ξ from $\mathcal{Y}^2 \to [0,\infty)$ be such that*

$$\lim_{\eta\to\infty} \frac{\xi\big((\nu+1)^\eta y, (\nu+1)^\eta y\big)}{|\nu+1|^\eta} = \lim_{\nu\to\infty} \frac{1}{|2|^{2\nu}} \max\Big\{\xi(2^{\nu+1}y, 2^{\nu+1}z), |16|\xi(2^\nu y, 2^\nu z)\Big\} = \lim_{\nu\to\infty} \frac{1}{|2|^{4\nu}} \max\Big\{\xi(2^{\nu+1}y, 2^{\nu+1}z), |4|\xi(2^\nu y, 2^\nu z)\Big\} = 0 \qquad \forall\, y \in \mathcal{Y}. \tag{74}$$

The limit exists for each $y \in \mathcal{Y}$,

$$\psi_{\mathcal{A}(y)} = \lim_{\eta\to\infty} \max\left\{\frac{1}{|\nu+1|^{\jmath}}\xi\big((\nu+1)^{\jmath}y,(\nu+1)^{\jmath}y\big) : 0 \le \jmath < \eta\right\}, \tag{75}$$

$$\psi_{q_2}(y) = \lim_{\nu\to\infty} \max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}y) : 0 \le \jmath < \nu\right\} \tag{76}$$

$$\psi_{q_4}(y) = \lim_{\nu\to\infty} \max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^{\jmath}y) : 0 \le \jmath < \nu\right\} \tag{77}$$

where $\bar{\Phi}(y)$ is defined in (43) for all $y \in \mathcal{Y}$. Suppose that $\mathfrak{f} : \mathcal{Y} \to \mathcal{Z}$ is a function satisfies the inequality

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \le \xi(y,z)\ \forall\, y,z \in \mathcal{Y}. \tag{78}$$

Then, there is a unique additive function $\mathcal{A}(y) : \mathcal{Y} \to \mathcal{Z}$, a unique quadratic function $\mathcal{Q}_2(y) : \mathcal{Y} \to \mathcal{Z}$ and a unique quartic function $\mathcal{Q}_4(y) : \mathcal{Y} \to \mathcal{Z}$ such that

$$\begin{aligned}\|\mathfrak{f}(y) - \mathcal{A}(y) - \mathcal{Q}_2{}'(y) - \mathcal{Q}_4{}'(y)\| \le \max\Big\{&\frac{1}{|4(\nu+1)|}\max\{\psi\mathcal{A}(y),\psi\mathcal{A}(-y)\},\\ &\frac{1}{|384|}\max\{\psi_{q_4}(y),|4|\psi_{q_2}(y)\},\max\{\psi_{q_4}(-y),|4|\psi_{q_2}(-y)\}\Big\} \qquad \forall y \in \mathcal{Y}.\end{aligned} \tag{79}$$

Moreover if,

$$\lim_{\iota\to\infty}\lim_{\eta\to\infty}\frac{1}{|\nu+1|^{\iota}}\max\left\{\max\frac{1}{|\nu+1|^{\iota}}\xi\big((\nu+1)^{\jmath}y,(\nu+1)^{\jmath}y\big) : 0 \le \jmath < \eta+\iota\right\} = 0, \tag{80}$$

$$\lim_{\iota\to\infty}\lim_{\nu\to\infty}\max\left\{\frac{1}{|2|^{2\jmath}}\bar{\varphi}(2^{\jmath}y) : \iota \le \jmath < \nu+\iota\right\} = 0 \qquad \forall\, y \in \mathcal{Y}, \tag{81}$$

$$\lim_{\iota\to\infty}\lim_{\nu\to\infty}\max\left\{\frac{1}{|2|^{4\jmath}}\bar{\varphi}(2^{\jmath}y) : \iota \le \jmath < \nu+\iota\right\} = 0 \qquad \forall\, y \in \mathcal{Y}. \tag{82}$$

then $\mathcal{A}(y)$, $\mathcal{Q}_2{}'(y)$ and $\mathcal{Q}_4{}'(y)$ are the unique additive, quadratic and quartic function from $\mathfrak{f} : \mathcal{Y} \to \mathcal{Z}$ respectively.

Proof. Consider $F_0(y)=\frac{1}{2}[f(y) - f(-y)] \qquad \forall y \in \mathcal{Y}$. Then $F_0(0) = 0$, $F_0(-y) = -F_0(y)$ and

$$\|\mathcal{Q}F_0(y,z)\| \le \frac{1}{|2|}\max\Big\{\|\mathcal{Q}\mathfrak{f}(y,z)\|,\|\mathcal{Q}\mathfrak{f}(-y,-z)\|\Big\}$$

$$\le \frac{1}{|2|}\max\Big\{\xi(y,z),\xi(-y,-z)\Big\} \qquad \forall\, y,z \in \mathcal{Y}.$$

By Theorem (3), a unique additive function $\mathcal{A}(y) : \mathcal{Y} \to \mathcal{Z}$ satisfies

$$\|F_0(y) - \mathcal{A}(y)\| \le \max\left\{\frac{1}{|2|}\Big\|\mathfrak{f}(y) - \mathcal{A}(y)\Big\|,\frac{1}{|2|}\Big\|\mathfrak{f}(-y) - \mathcal{A}(-y)\Big\|\right\}$$

$$\|F_0(y) - \mathcal{A}_0(y)\| \le \frac{1}{|4(\nu+1)|}\max\Big\{\psi_{\mathcal{A}}(y),\psi_{\mathcal{A}}(-y)\Big\} \qquad \forall y \in \mathcal{Y}. \tag{83}$$

Let $F_e(y) = \frac{1}{2}[\mathfrak{f}(y) + \mathfrak{f}(-y)]$ for all $y \in \mathcal{Y}$. Then $F_e(0)$=0, $F_e(-y)$=$F_e(y)$ and

$$\begin{aligned}\|\mathcal{Q}F_e(y,z)\| &\leq \frac{1}{|2|}\max\left\{\|\mathcal{Q}\mathfrak{f}(y,z)\|, \|\mathcal{Q}\mathfrak{f}(-y,-z)\|\right\}\\ &\leq \frac{1}{|2|}\max\left\{\xi(y,z), \xi(-y,-z)\right\} \qquad \forall\, y,z \in \mathcal{Y}.\end{aligned}$$

From Theorem (9), a unique quadratic and quartic function satisfies

$$\|F_e(y) - \mathcal{Q}_2'(y) - \mathcal{Q}_4'(y)\| \leq \left\{\frac{1}{|2|}\|\mathfrak{f}(y) - \mathcal{Q}_2'(y) - \mathcal{Q}_4'(y)\|, \frac{1}{|2|}\|\mathfrak{f}(-y) - \mathcal{Q}_2'(-y) - \mathcal{Q}_4'(-y)\|\right\}$$

$$\|F_e(y) - \mathcal{Q}_2'(y) - \mathcal{Q}_4'(y)\| \leq \frac{1}{|384|}\max\left\{\max\{\psi_{q_4}(y), |4|\psi_{q_2}(y)\}, \max\{\psi_{q_4}(y), |4|\psi_{q_2}(y)\}\right\}\forall\, y \in \mathcal{Y}. \tag{84}$$

Hence by using (83) and (84), we get

$$\begin{aligned}\|\mathfrak{f}(y) - \mathcal{A}(y) - \mathcal{Q}_2{}'(y) - \mathcal{Q}_4{}'(y)\| \leq \max\Big\{&\frac{1}{|4(\nu+1)|}\max\{\psi\mathcal{A}(y), \psi\mathcal{A}(-y)\},\\ &\frac{1}{|384|}\max\{\psi_{q_4}(y), |4|\psi_{q_2}(y)\}, \max\{\psi_{q_4}(-y), |4|\psi_{q_2}(-y)\}\Big\}\forall\, y \in \mathcal{Y}.\end{aligned} \tag{85}$$

Uniqueness can be proved as similar way in the Theorem 5 □

Corollary 11. *Let r, s and δ are positive real numbers and let $1 < r + s < 4$. Define a function $\mathfrak{f}$ from $\mathcal{Y}$ to $\mathcal{Z}$ and if $\mathfrak{f}$ is a mapping satisfies the inequality*

$$\|\mathcal{Q}\mathfrak{f}(y,z)\| \leq \delta(\|y\|^{r+s} + \|z\|^{r+s} + \|y\|^r\|z\|^s) \qquad \forall\, y,z \in \mathcal{Y}.$$

Then, there is a unique additive $\mathcal{A}(y)$, quadratic $\mathcal{Q}_2(y)$ and quartic $\mathcal{Q}_4(y)$ function such that

$$\begin{aligned}\|\mathfrak{f}(y) - \mathcal{A}(y) - \mathcal{Q}_2{}'(y) - \mathcal{Q}_4{}'(y)\| \leq \max\Big\{&\frac{1}{|4(\nu+1)|}\max\{\psi\mathcal{A}(y), \psi\mathcal{A}(-y)\},\\ &\frac{1}{|384|}\max\{\psi_{q_4}(y), |4|\psi_{q_2}(y)\}, \max\{\psi_{q_4}(-y), |4|\psi_{q_2}(-y)\}\Big\}\forall\, y \in \mathcal{Y}.\end{aligned} \tag{86}$$

where $\psi_{\mathcal{A}(y)}, \psi_{q_2}(y), \psi_{q_4}(y)$ are defined in (75),(76),(77) respectively.

4. Conclusion

Many authors discussed the GHUS of mixed type functional equation in non-Archimedean spaces in recent years. In this article, we have studied a new mixed type of an additive quadratic quartic ($\mathcal{A}\mathcal{Q}_2\mathcal{Q}_4$) functional equation (1) in (NAN) space. And also given some suitable counter examples.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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