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ALGEBRAS OF GENERALIZED TERMS INDUCED BY SOME CLASSES OF TRANSFORMATIONS

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Abstract. Based on transformations with restricted range on a finite set $\{1, \ldots, n\}$, in this paper, the set $W^{GT(\bar{n},Y)}_{\tau_n}(X)$ of full terms generated by transformations with restricted range on a finite set consisting elements greater than n are defined. Applying the generalized superposition operations of full terms, the algebra $clone_{GT(\bar{n},Y)}(\tau_n)$ of such terms which satisfies the superassociativity is constructed and it generating systems is proposed. We then show that there is a mapping that takes any element of the generating system to our generalized full terms, called generalized clone substitutions. Properties of the freeness of $clone_{GT(\bar{n},Y)}(\tau_n)$ in a variety of Menger algebras are examined. To find a method for classifying arbitrary algebras into subclasses via strong hyperidentites, a generalized full hypersubstitution sending each n-ary operation symbol to each element of the set $W^{GT(\bar{n},Y)}_{\tau_n}(X)$ is given and its binary associative operation is defined. Finally, we determine a necessary conditions for which every identity in $clone_{GT(\bar{n},Y)}(\tau_n)$ to be generalized full hyperidentity in a variety of algebras.

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1. Introduction

Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as *letters*, to X as an *alphabet*, and also refer to the set $X_n = \{x_1, \ldots, x_n\}$ as an *n-element alphabet*. Let $(f_i)_{i \in I}$ be an indexed set which is disjoint from X. Each f_i is called *n_i-ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its

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arity. The function τ, on the values of τ written as $(n_i)_{i\in I}$ is called a *type*. Recall that an n-*ary term* of type τ is defined inductively as follows :

(i) The variables $x_1, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

By $W_\tau(X_n)$ we mean the smallest set which contains $x_1, \ldots, x_n$ and it is closed under finite application of (ii). It is clear that every n-ary term is also m-ary term for all $m \geq n$. The set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is set of all terms of type τ over the alphabet X. This set can be used as the universe of an algebra of type τ. For every $i \in I$, an n_i-ary operation $\overline{f_i}$ on $W_\tau(X)$ is defined by $\overline{f_i} : W_\tau(X)^{n_i} \longrightarrow W_\tau(X)$ with $(t_1, \ldots, t_{n_i}) \longmapsto f_i(t_1, \ldots, t_{n_i})$. The algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\overline{f_i})_{i\in I})$ is called the *absolutely free algebra of type* τ *over the set* X. There are a number of detailed researches on terms (see, [8]).

In [15], the complexity of terms was studied. Actually, the depth of a term is defined. Let $t \in W_\tau(X)$, the depth of a term t, denoted by $depth(t)$, is defined as follows:

(i) if $t = x \in X$, then $depth(t) := 0$,

(ii) if $t = f_i(t_1, \ldots, t_{n_i})$ where $t_1, \ldots, t_{n_i} \in W_\tau(X)$, then
$depth(t) := max\{depth(t_j) \mid 1 \leq j \leq n_i\} + 1.$

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we need to define the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps. For any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_n) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_n) := x_j$,

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_n) := f_i(S^n_m(s_1, t_1, \ldots, t_m), \ldots, S^n_m(s_{n_i}, t_1, \ldots, t_n)).$

As a consequence, the algebra $(W_\tau(X), S^m)$ is a Menger algebra of rank $m + 1$. More information about Menger algebras, see [6,7]. A relationship between Menger algebras and power set of terms was given in [20].

Applying the operation S^m, the generalized hypersubstitution σ can be extended to a mapping

$$\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$$

by the following steps:

(i) $\hat{\sigma}[x] := x \in X$; and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

We defined a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. In [11], it is proved that for arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma_1} \circ \sigma_2)\hat{} = \hat{\sigma_1} \circ \hat{\sigma_2}$.

It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where σ_{id} is its identity element. The monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

Let $\tau_n = (n, n, \ldots, n)$ be a type consisting of the same values equal to n, i.e. $\tau_n = (n_i)$ with $n_i = n$ for all $i \in I$. The concept of full terms is used in [4] to study the depth of terms and full hypersubstitutions, and solid varieties. The composed full terms are derived by operation symbols and terms in which all input variables occur. Thus the resulting subterms in each step of composition, content whole set of the input variables, which can be only permuted.

In 2013, Phuapong and Leeratanavalee [13] inductively defined generalized full terms of type τ_n, based on the permutations, as follows:

(i) if $s : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation, then $f_i(x_{s(1)}, \ldots, x_{s(n)})$ is a generalized full term,

(ii) if $j_1, j_2, \ldots, j_n$ are natural numbers and greater than n_i, then $f_i(x_{s'(j_1)}, \ldots, x_{s'(j_n)})$ is a generalized full term where s' is a permutation on $\{j_1, j_2, \ldots, j_n\}$,

(iii) if $t_1, \ldots, t_n$ are generalized full terms of type τ, then $f_i(t_1, \ldots, t_n)$ is a generalized full term of type τ.

Let $W^{GF}_{(n)}(X)$ be the set of all generalized full terms and let P_n be the set of all permutations on $\{1, \ldots, n\}$.

2. Generalized $T(\bar{n}, Y)$-full terms

The first aim of our main results is to propose the new concept of a specific term, based on full transformation mappings and the original notions of terms. For this, we recall the concept of the full transformations.

Let X be a nonempty set and let $T(X)$ denote the semigroup of the full transformations from X into itself under composition of mappings and let Y be a nonempty subset of X. Then $T(X, Y)$ was introduced by Symons [17] to be the set of all transformations from X to Y called the *full transformation semigroup with restricted range*, that means

$$T(X, Y) := \{\alpha \in T(X) \mid X\alpha \subseteq Y\}.$$

Clearly, $T(X,Y)$ is a subsemigroup of $T(X)$ and if $X = Y$ then $T(X,Y) = T(X)$. For more information about $T(X,Y)$, we refer to [16].

Let τ_n be a type and let $(f_i)_{i\in I}$ be an indexed set of operation symbols of type τ. The *full transformation semigroup* T_n consists of the set of all maps $\alpha : \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$ and the usual composition of mappings. Indeed, T_n is a monoid and identity map 1_n acts as its identity. Let $\bar{n} := \{1,2,\ldots,n\}$. For a fixed nonempty subset Y of $\bar{n}$, it is well-known that the set

$$T(\bar{n},Y) := \{\alpha \in T_n \mid \operatorname{Im}\alpha \subseteq Y\} \cup \{1_n\}$$

is a submonoid of T_n. Let $\{j_1,\ldots,j_n\} \subseteq \mathbb{N}$ where $j_1,\ldots,j_n$ are all distinct with $\{j_1,\ldots,j_n\} \cap \{1,\ldots,n\} = \emptyset$. By $T_{\{j_1,\ldots,j_n\}}$ we denote the set of all full transformations on $\{j_1,\ldots,j_n\}$ and by $1_{\{j_1,\ldots,j_n\}}$ we mean an identity mapping on $\{j_1,\ldots,j_n\}$. For, a fixed nonempty subset Y' of $\{j_1,\ldots,j_n\}$, we set

$$T(\{j_1,\ldots,j_n\},Y') := \{\alpha' \in T_{\{j_1,\ldots,j_n\}} \mid \operatorname{Im}\alpha' \subseteq Y'\} \cup \{1_{\{j_1,\ldots,j_n\}}\}.$$

is a submonoid of $T_{\{j_1,\ldots,j_n\}}$. Now we introduce the definition of generalized $T(\bar{n},Y)$-full term of type τ_n.

Definition 2.1. Let f_i be an n-ary operation symbol and $\alpha \in T(\bar{n},Y)$. A generalized $T(\bar{n},Y)$-full term of type τ_n is defined in the following way:

(i) if $\alpha \in T(\bar{n},Y)$, then $f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ is a generalized $T(\bar{n},Y)$-full term of type τ_n;

(ii) if $\alpha' \in T(\{j_1,\ldots,j_n\},Y')$, then $f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})$ is a generalized $T(\bar{n},Y)$-full term of type τ_n;

(iii) if $t_1,\ldots,t_n$ are generalized $T(\bar{n},Y)$-terms of type τ_n, then $f_i\,(t_1,\ldots,t_n)$ is a generalized $T(\bar{n},Y)$-full term of type τ_n.

Let $W_{\tau_n}^{GT(\bar{n},Y)}(X)$ be the set of all generalized $T(\bar{n},Y)$-full terms of type τ_n.

The set of all generalized $T(\bar{n},Y)$-full terms of type τ_n is closed under the superposition S^n can be proved in the following theorem.

Theorem 2.2. *Let $t,s_1,s_2,\ldots,s_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$. Then $S^n(t,s_1,\ldots,s_n)$ is also a generalized $T(\bar{n},Y)$-full terms of type τ_n.*

Proof. We give a proof by induction on the depth of a generalized $T(\bar{n},Y)$-full term t. If $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in T(\bar{n},Y)$, and $Depth(t) = 1$, then

$$\begin{aligned} S^n(t,s_1,\ldots,s_n) &= S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),s_1,\ldots,s_n) \\ &= f_i(s_{\alpha(1)},\ldots,s_{\alpha(n)}), \end{aligned}$$

which is a generalized $T(\bar{n}, Y)$-full term.

If $t = f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$, and $Depth(t) = 1$, then

$$\begin{aligned} S^n(t, s_1, \ldots, s_n) &= S^n(f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}), s_1, \ldots, s_n) \\ &= f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}), \end{aligned}$$

which is a generalized $T(\bar{n}, Y)$-full term.

If $t = f_i(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$ and assume that $S^n(r_k, s_1, \ldots, s_n)$ is a generalized $T(\bar{n}, Y)$-full terms for all $1 \leq k \leq n$ and $max_{1\leq k\leq n} Depth(r_k) = m$, then $Depth(t) = m + 1$ and we have

$$\begin{aligned} S^n(t, s_1, \ldots, s_n) &= S^n(f_i(r_1, \ldots, r_n), s_1, \ldots, s_n) \\ &= f_i(S^n(r_1, s_1, \ldots, s_n), \ldots, S^n(r_n, s_1, \ldots, s_n)). \end{aligned}$$

By Definition 2.1, $S^n(t, s_1, \ldots, s_n)$ is a generalized $T(\bar{n}, Y)$-full term.

□

Now we consider the algebra

$$clone_{GT(\bar{n},Y)}(\tau_n) := \left(W_{\tau_n}^{GT(\bar{n},Y)}(X), S^n\right)$$

which is called *the clone of all generalized $T(\bar{n}, Y)$-full terms of type τ_n*. The Theorem 2.3, presented below, shows that the algebra $\left(W_{\tau_n}^{GT(\bar{n},Y)}(X), S^n\right)$ satisfies the superassociative law (SASS):

$$S^n(X_0, S^n(Y_1, Z_1, \ldots, Z_n), \ldots, S^n(Y_n, Z_1, \ldots, Z_n)) \approx S^n(S^n(X_0, Y_1, \ldots, Y_n), Z_1, \ldots, Z_n) \quad (1)$$

where S^n is an $(n + 1)$-ary operation symbol and X_0, Y_j, Z_j are variables for all $1 \leq j \leq n$.

Next, we shall show that the superassociative law is satisfied in the clone of all generalized $T(\bar{n}, Y)$-full terms.

Theorem 2.3. *The algebra $clone_{GT(\bar{n},Y)}(\tau_n)$ satisfies the superassociative law (SASS).*

Proof. We give a proof by induction on the depth of a generalized $T(\bar{n}, Y)$-full term t which is substituted X_0 from (1). If we substitute X_0 from (1) by a generalized $T(\bar{n}, Y)$-full term $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in T(\bar{n}, Y)$ and $Depth(t) = 1$, then we have

$$\begin{aligned} &S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)) \\ &\quad= f_i(S^n(x_{\alpha(1)}, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n)), \ldots, \\ &\qquad\quad S^n(x_{\alpha(n)}, S^n(t_1, s_1, \ldots, s_n), \ldots, S^n(t_n, s_1, \ldots, s_n))) \\ &\quad= f_i(S^n(t_{\alpha(1)}, s_1, \ldots, s_n), \ldots, S^n(t_{\alpha(n)}, s_1, \ldots, s_n)) \\ &\quad= S^n(f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)}), s_1, \ldots, s_n) \end{aligned}$$

$$= \quad S^n(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n),s_1,\ldots,s_n).$$

If we substitute X_0 from (1) by a generalized $T(\bar{n},Y)$-full term $t = f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$ and $Depth(t) = 1$, then we have

$$\begin{aligned}
&S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n))\\
&= f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})\\
&= S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),s_1,\ldots,s_n)\\
&= S^n(S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),t_1,\ldots,t_n),s_1,\ldots,s_n).
\end{aligned}$$

If we substitute X_0 from (1) by a generalized $T(\bar{n},Y)$-full term $t = f_i(r_1,\ldots,r_n)$ where $r_1,\ldots,r_n \in W_{\tau_n}^{T(\bar{n},Y)}(X_n)$ and assume that

$$S^n(r_k,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n)) = S^n\left(S^n(r_k,t_1,\ldots,t_n),s_1,\ldots,s_n\right)$$

for all $1 \le k \le n$, and $max_{1\le k\le n}Depth(r_k) = m$, then $Depth(t) = m+1$ and we have

$$\begin{aligned}
&S^n(f_i(r_1,\ldots,r_n),S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n))\\
&= f_i(S^n(r_1,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n)),\ldots,\\
&\quad S^n(r_n,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n)))\\
&= f_i\left(S^n\left(S^n(r_1,t_1,\ldots,t_n),s_1,\ldots,s_n\right),\ldots,\left(S^n(r_n,t_1,\ldots,t_n),s_1,\ldots,s_n\right)\right)\\
&= S^n(f_i(S^n(r_1,t_1,\ldots,t_n),\ldots,S^{n_i}(r_n,t_1,\ldots,t_n)),s_1,\ldots,s_n)\\
&= S^n(S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n),s_1,\ldots,s_n).
\end{aligned}$$

□

An algebra $\mathcal{M} := (M,S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if $\mathcal{M}$ satisfies the condition (SASS) [1]. It follows immediately from Theorem 2.3 that $clone_{GT(\bar{n},Y)}(\tau_n)$ is a Menger algebra of rank n. For basics and some advanced developments of Menger algebras can be found in the works of W.A. Dudek and V.S. Trokhimenko, for example, see [5].

It is clear that $clone_{GT(\bar{n},Y)}(\tau_n)$ is generated by

$$F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)} := \left\{f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right) \mid i\in I,\alpha\in T(\bar{n},Y)\right\}\cup$$

$$\{f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}) \mid j_1,\ldots,j_n > n, \alpha' \in T_{\{j_1,\ldots,j_n\}}\}.$$

Let $V^{GT(\bar{n},Y)}$ be the variety of type $\tau = (n+1)$ generated by the superassociative law (SASS). Let $\mathcal{F}_{V^{GT(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ be the free algebra with respect to $V^{GT(\bar{n},Y)}$, freely generated by an alphabet $\{Y_l \mid l\in J\}$ where $J = \{(i,\alpha) \mid i\in I\,,\alpha\in T(\bar{n},Y)\}$. The operation of $\mathcal{F}_{V^{GT(\bar{n},Y)}}(\{Y_l \mid l\in J\})$ is denoted by $\tilde{S}^n$. Next, we are going to prove that the clone of all generalized $T(\bar{n},Y)$-full terms is a free algebra with respect to the variety $V^{GT(\bar{n},Y)}$.

Theorem 2.4. *The algebra $clone_{GT(\bar{n},Y)}(\tau_n)$ is isomorphic to $\mathcal{F}_{V^{GT(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ and therefore it is free with respect to the variety $V^{GT(\bar{n},Y)}$, and freely generated by the set*
$\{f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}) \mid i \in I, \alpha \in T(\bar{n},Y)\} \cup \{f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}) \mid j_1,\ldots,j_n > n, \alpha' \in T_{\{j_1,\ldots,j_n\}}\}$.

Proof. We define the mapping $\varphi : W_{\tau_n}^{T(\bar{n},Y)}(X_n) \longrightarrow \mathcal{F}_{V^{T(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ inductively as follows:

(i) $\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}) = y_{(i,\alpha)}$;
(ii) $\varphi(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}) = y_{(i,\alpha')}$;
(iii) $\varphi(f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})) = \tilde{S}^n(y_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n))$.

Since φ maps the generating system of $clone_{GT(\bar{n},Y)}(\tau_n)$ onto the generating system of $\mathcal{F}_{V^{GT(\bar{n},Y)}}(\{Y_l \mid l \in J\})$, it is surjective. We will prove the homomorphism property

$$\varphi(S^n(t_0,t_1,\ldots,t_n)) = \tilde{S}^n(\varphi(t_0),\varphi(t_1),\ldots,\varphi(t_n))$$

by induction on the depth of a generalized $T(\bar{n},Y)$-full term t_0.
If $t_0 = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in T(\bar{n},Y)$ and $Depth(t) = 1$, then we have

$$\begin{aligned}
&\varphi(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n)) \\
&\quad = \varphi(f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})) \\
&\quad = \tilde{S}^n(y_{(i,\alpha)},\varphi(t_1),\ldots,\varphi(t_n)) \\
&\quad = \tilde{S}^n(\varphi(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

If $t_0 = f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$ and $Depth(t) = 1$, then we have

$$\begin{aligned}
&\varphi(S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n)) \\
&\quad = \varphi(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})) \\
&\quad = y_{(i,\alpha')} \\
&\quad = \tilde{S}^n(y_{(i,\alpha')},\varphi(t_1),\ldots,\varphi(t_n)) \\
&\quad = \tilde{S}^n(\varphi(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

If $t_0 = f_i(r_1,\ldots,r_n)$ and assume that

$$\varphi(S^n(r_k,t_1,\ldots,t_n)) = \tilde{S}^n(\varphi(r_k),\varphi(t_1),\ldots,\varphi(t_n))$$

for all $1 \le k \le n$ and $max_{1\le k\le n}Depth(r_k) = m$, then $Depth(t) = m+1$ and we have

$$\begin{aligned}
&\varphi(S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n)) \\
&\quad = \varphi(f_i(S^n(r_1,t_1,\ldots,t_n),\ldots,S^n(r_n,t_1,\ldots,t_n))) \\
&\quad = \tilde{S}^n(y_{(i,1_n)},\varphi(S^n(r_1,t_1,\ldots,t_n)),\ldots,\varphi(S^n(r_n,t_1,\ldots,t_n))) \\
&\quad = \tilde{S}^n(y_{(i,1_n)},\tilde{S}^n(\varphi(r_1),\varphi(t_1),\ldots,\varphi(t_n)),\ldots, \\
&\qquad \tilde{S}^n(\varphi(r_n),\varphi(t_1),\ldots,\varphi(t_n)))
\end{aligned}$$

$$= \tilde{S}^n(\tilde{S}(y_{(i,1_n)}, \varphi(r_1), \ldots, \varphi(r_n)), \varphi(t_1), \ldots, \varphi(t_n))$$
$$= \tilde{S}^n(\varphi(f_i(r_1, \ldots, r_n)), \varphi(t_1), \ldots, \varphi(t_n)).$$

Thus φ is a homomorphism. The mapping φ is clearly bijective since the set $\{y_{(i,\alpha)} \mid i \in I,\ \alpha \in T(\bar{n}, Y)\}$ is free independent. Therefore we have

$$y_{(i,\alpha)} = y_{(j,\beta)} \Longrightarrow (i, \alpha) = (j, \beta) \Longrightarrow i = j,\ \alpha = \beta$$

and

$$y_{(i,\alpha')} = y_{(j,\beta')} \Longrightarrow (i, \alpha') = (j, \beta') \Longrightarrow i = j,\ \alpha' = \beta'.$$

So $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = f_j(x_{\beta(1)}, \ldots, x_{\beta(n)})$ and $f_i(x_{\alpha'(1)}, \ldots, x_{\alpha'(n)}) = f_j(x_{\beta'(1)}, \ldots, x_{\beta'(n)})$. Thus φ is a bijection between the generating sets of $clone_{GT(\bar{n},Y)}(\tau_n)$ and $\mathcal{F}_{V^{GT(\bar{n},Y)}}(\{Y_l \mid l \in J\})$ and therefore φ is an isomorphism. □

3. Generalized $T(\bar{n}, Y)$-full hypersubstitutions

In this section, we will generalize the concept of $T(\bar{n}, Y)$- full hypersubstitution. For any generalized $T(\bar{n}, Y)$-full term t we need the generalized $T(\bar{n}, Y)$-full term t_β derived from t by replacement a variable $x_{\alpha(j)}$ in t by a variable $x_{\beta(\alpha(j))}$ for a mapping $\beta \in T(\bar{n}, Y)$. This can be defined as follows.

Let $t, t_1, \ldots, t_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X_n)$ and $\alpha, \beta \in T(\bar{n}, Y)$. Then we define the generalized $T(\bar{n}, Y)$-full term t_β in the following steps:

(i) If $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then $t_\beta := f_i(x_{\beta(\alpha(1))}, \ldots, x_{\beta(\alpha(n))})$.
(ii) If $t = f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$,
then $t_\beta := f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})$.
(iii) If $t = f_i(t_1, \ldots, t_n)$, then $t_\beta := f_i((t_1)_\beta, \ldots, (t_n)_\beta)$.

It is clear that t_β is a generalized $T(\bar{n}, Y)$-full term for any generalized $T(\bar{n}, Y)$-full term t and for any $\alpha \in T(\bar{n}, Y)$.

Now, we call a mapping

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$$

a generalized $T(\bar{n}, Y)$-full hypersubstitution of type τ_n. By $Hyp_G^{T(\bar{n},Y)}(\tau_n)$ we denote the set of all generalized $T(\bar{n}, Y)$-hypersubstitutions of type τ_n.

Then any generalized $T(\bar{n}, Y)$-full hypersubstitution

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$$

of type τ_n can be extended to a mapping

$$\hat{\sigma} : W_{\tau_n}^{GT(\bar{n},Y)}(X) \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$$

as follows :

(i) $\hat{\sigma}[f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})] := (\sigma(f_i))_\alpha$,

(ii) $\hat{\sigma}[f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})] := S^n\left(\sigma(f_i),x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}\right)$,

(iii) $\hat{\sigma}[f_i(t_1,\ldots,t_n)] := S^n\left(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}]\right)$.

We define a binary operation $\circ_G^{T(\bar{n},Y)}$ on $Hyp_G^{T(\bar{n},Y)}(\tau_n)$ by $\sigma_1 \circ_G^{T(\bar{n},Y)} \sigma_2 := \hat{\sigma_1}\circ\sigma_2$ where $\circ$ denotes the usual composition of mappings. Together with the hypersubstitution σ_{id} defined by $\sigma_{id}(f) := f_i(x_1,\ldots,x_n)$, one has a monoid $\left(Hyp_G^{T(\bar{n},Y)}(\tau_n);\circ_G^{T(\bar{n},Y)},\sigma_{id}\right)$. The following lemma shows the property of a term t_α and the extension $\hat{\sigma}$.

Lemma 3.1. *Let $t,t_1,\ldots,t_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$. Then*

$$S^n(t,\hat{\sigma}[t_{\beta(1)}],\ldots,\hat{\sigma}[t_{\beta(n)}]) = S^n(t_\beta,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$$

for all $\beta \in T(\bar{n},Y)$.

Proof. We give a proof by induction on the depth of a generalized $T(\bar{n},Y)$-full term t.
If $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha\in T(\bar{n},Y)$ and $Depth(t)=1$, then
$S^n(t,\hat{\sigma}[t_{\beta(1)}],\ldots,\hat{\sigma}[t_{\beta(n)}])$

$$\begin{aligned} &= S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),\hat{\sigma}[t_{\beta(1)}],\ldots,\hat{\sigma}[t_{\beta(n)}]) \\ &= f_i(\hat{\sigma}[t_{\beta(\alpha(1))}],\ldots,\hat{\sigma}[t_{\beta(\alpha(n))}]), \end{aligned}$$

$S^n(t_\beta,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$

$$\begin{aligned} &= S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})_\beta,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n] \\ &= S^n(f_i(x_{\beta(\alpha(1))},\ldots,x_{\beta(\alpha(n))}),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n] \\ &= f_i(\hat{\sigma}[t_{\beta(\alpha(1))}],\ldots,\hat{\sigma}[t_{\beta(\alpha(n))}]). \end{aligned}$$

If $t = f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})$ where $\alpha'\in T_{\{j_1,\ldots,j_n\}}$, and $Depth(t)=1$,
then $S^n(t,\hat{\sigma}[t_{\beta(1)}],\ldots,\hat{\sigma}[t_{\beta(n)}])$

$$\begin{aligned} &= S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),\hat{\sigma}[t_{\beta(1)}],\ldots,\hat{\sigma}[t_{\beta(n)}]) \\ &= f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}), \end{aligned}$$

$S^n(t_\beta,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n])$

$$\begin{aligned} &= S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})_\beta,\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n] \\ &= S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_n] \\ &= f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}). \end{aligned}$$

If $t = f_i(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$ and assume that

$$S^n(s_k, \hat{\sigma}[t_{\beta(1)}], \ldots, \hat{\sigma}[t_{\beta(n)}]) = S^n((s_k)_\beta, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $1 \leq k \leq n$ and $max_{1\leq k\leq n} Depth(s_k) = m$, then $Depth(t) = m + 1$ and we have $S^n(t, \hat{\sigma}[t_{\beta(1)}], \ldots, \hat{\sigma}[t_{\beta(n)}])$

$$\begin{aligned}
&= S^n(f_i(s_1, \ldots, s_n), \hat{\sigma}[t_{\beta(1)}], \ldots, \hat{\sigma}[t_{\beta(n)}]) \\
&= f_i(S^n(s_1, \hat{\sigma}[t_{\beta(1)}], \ldots, \hat{\sigma}[t_{\beta(n)}]), \ldots, S^n(s_n, \hat{\sigma}[t_{\beta(1)}], \ldots, \hat{\sigma}[t_{\beta(n)}])) \\
&= f_i(S^n((s_1)_\beta, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]), \ldots, S^n((s_n)_\beta, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) \\
&= S^n(f_i((s_1)_\beta, \ldots, (s_n)_\beta), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(f_i(s_1, \ldots, s_n)_\beta, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(t_\beta, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

□

Using Lemma 3.1 we show that the extension $\hat{\sigma}$ of each generalized $T(\bar{n}, Y)$-full hypersubstitution σ preserves the operation S^n on the set $W_{\tau_n}^{GT(\bar{n},Y)}(X)$.

Theorem 3.2. *For $\sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$, the extension*

$$\hat{\sigma} : W_{\tau_n}^{GT(\bar{n},Y)}(X) \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$$

is an endomorphism on the algebra $clone_{GT(\bar{n},Y)}(\tau_n)$.

Proof. It is clear that $\hat{\sigma} : W_{\tau_n}^{GT(\bar{n},Y)}(X) \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$. Let $t_0, t_1, \ldots, t_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$. We will show by induction on the depth of t_0 that

$$\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).$$

If $t_0 = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ where $\alpha \in T(\bar{n}, Y)$ and $Depth(t_0) = 1$, then $\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)]$

$$\begin{aligned}
&= \hat{\sigma}[S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})] \\
&= S^n(\sigma(f_i), \hat{\sigma}[t_{\alpha(1)}], \ldots, \hat{\sigma}[t_{\alpha(n)}]) \\
&= S^n(\sigma(f_i)_\alpha, \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

If $t_0 = f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$ and $Depth(t_0) = 1$, then
$\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)]$

$$\begin{aligned}
&= \hat{\sigma}[S^n(f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})] \\
&= S^n\left(\sigma(f_i), x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}\right) \\
&= S^n(\sigma(f_i), \hat{\sigma}[S^n(x_{\alpha'(j_1)}, t_1, \ldots, t_n)], \ldots, \hat{\sigma}[S^n(x_{\alpha'(j_n)}, t_1, \ldots, t_n)]) \\
&= S^n(S^n(\sigma(f), x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)})], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

If $t_0 = f_i(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$ and we assume that

$$\hat{\sigma}[S^n(r_k, t_1, \ldots, t_n)] = S^n(\hat{\sigma}[r_k], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$$

for all $1 \leq k \leq n$ and $max_{1\leq k\leq n} Depth(r_k) = m$, then $Depth(t) = m+1$ and we have $\hat{\sigma}[S^n(t_0, t_1, \ldots, t_n)]$

$$\begin{aligned}
&= \hat{\sigma}[S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)] \\
&= \hat{\sigma}[f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^n(r_n, t_1, \ldots, t_n))] \\
&= S^n(\sigma(f_i), \hat{\sigma}[S^n(r_1, t_1, \ldots, t_n)], \ldots, \hat{\sigma}[S^n(r_n, t_1, \ldots, t_n)]) \\
&= S^n(\sigma(f_i), S^n(\hat{\sigma}[r_1], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]), \ldots, S^n(\hat{\sigma}[r_n], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])) \\
&= S^n(S^n(\sigma(f_i), \hat{\sigma}[r_1], \ldots, \hat{\sigma}[r_n]), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) \\
&= S^n(\hat{\sigma}[t_0], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).
\end{aligned}$$

□

We complete this section by studying the connection between generalized $T(\bar{n}, Y)$-full terms and the extension of a mapping which maps fundamental term to any generalized $T(\bar{n}, Y)$-full terms.

As mentioned, the algebra $clone_{GT(\bar{n},Y)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)} := \left\{f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right) \mid i \in I, \alpha \in T(\bar{n}, Y)\right\} \cup$$
$$\{f_i(x_{\alpha'(j_1)}, \ldots, x_{\alpha'(j_n)}) \mid j_1, \ldots, j_n > n, \alpha' \in T_{\{j_1,\ldots,j_n\}}\}.$$

Thus, any mapping

$$\eta : F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)} \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$$

called *generalized $T(\bar{n}, Y)$-full clone substitution*, can be uniquely extended to endomorphism

$$\bar{\eta} : W_{\tau_n}^{GT(\bar{n},Y)}(X) \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X).$$

Let $Subst_{GT(\bar{n},Y)}(\tau_n)$ be the set of all generalized $T(\bar{n},Y)$-full clone substitutions. On the set $Subst_{GT(\bar{n},Y)}(\tau_n)$, a binary operation $\odot$ can be defined by

$$\eta_1 \odot \eta_2 := \bar{\eta_1} \circ \eta_2$$

where $\circ$ denotes the usual composition of mappings. Furthermore, the identity mapping with respect to $\odot$ is denoted by $id_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}$.

Then clearly, $\left(Subst_{GT(\bar{n},Y)}(\tau); \odot, id_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}\right)$ forms a monoid.

Consider $\sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$ and by Theorem 3.2, $\hat{\sigma} : W_{\tau_n}^{GT(\bar{n},Y)}(X) \longrightarrow W_{\tau_n}^{GT(\bar{n},Y)}(X)$ is an endomorphism. Since $F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}$ generates
$clone_{GT(\bar{n},Y)}(\tau_n)$, we have $\hat{\sigma}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}$ is an generalized $T(\bar{n},Y)$-full clone substitution with

$$\overline{\hat{\sigma}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}} = \hat{\sigma}.$$

Define a mapping $\psi : Hyp_G^{T(\bar{n},Y)}(\tau_n) \longrightarrow Subst_{GT(\bar{n},Y)}(\tau_n)$ by

$$\psi(\sigma) = \hat{\sigma}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}.$$

We have ψ is a homomorphism. In fact: Let $\sigma_1, \sigma_2 \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$. Then

$$\begin{aligned}
\psi(\sigma_1 \circ_G^{T(\bar{n},Y)} \sigma_2) &= \left(\sigma_1 \circ_G^{T(\bar{n},Y)} \sigma_2\right)\hat{}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}} \\
&= (\hat{\sigma_1} \circ \hat{\sigma_2})\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}} \\
&= \overline{\hat{\sigma_1}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}}} \circ \hat{\sigma_2}\big|_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}} \\
&= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\
&= \psi(\sigma_1) \odot \psi(\sigma_2).
\end{aligned}$$

Clearly, ψ is an injection. Hence we have the following corollary.

Corollary 3.3. *The monoid* $\left(Hyp_G^{T(\bar{n},Y)}(\tau_n); \circ_G^{T(\bar{n},Y)}, \sigma_{id}\right)$ *can be embedded into* $(Subst_{GT(\bar{n},Y)}(\tau_n); \odot, id_{F_{W_{\tau_n}^{GT(\bar{n},Y)}(X)}})$.

4. Generalized $T(\bar{n},Y)$-full hyperidentities

In this section we examine the relationship between a variety V of type τ_n and the identity in the $clone_{GT(\bar{n},Y)}(\tau_n)$.

Let V be a variety of type τ_n and let IdV be the set of all identities of V. Let $Id^{GT(\bar{n},Y)}V$ be the set of all $s \approx t$ of V such that s and t are both generalized $T(\bar{n},Y)$-full term of type τ_n; that is

$$Id^{GT(\bar{n},Y)}V := \left(W_{\tau_n}^{GT(\bar{n},Y)}(X)\right)^2 \cap IdV.$$

It is well-known that IdV is a congruence on the free algebra $\mathcal{F}_\tau(X)$. However, in general this is not true for $Id^{GT(\bar{n},Y)}V$. The following theorem shows that $Id^{GT(\bar{n},Y)}V$ is a congruence on $clone_{GT(\bar{n},Y)}(\tau_n)$.

Theorem 4.1. *Let V be a variety of type τ_n. Then $Id^{GT(\bar{n},Y)}V$ is a congruence on the algebra $clone_{GT(\bar{n},Y)}(\tau_n)$.*

Proof. We will prove that if $t \approx r$, $t_k \approx r_k \in Id^{GT(\bar{n},Y)}V, k = 1,2,\ldots,n$, then $S^n(t,t_1,\ldots,t_n) \approx S^n(r,r_1,\ldots,r_n) \in Id^{GT(\bar{n},Y)}V$. Firstly, we give a proof by induction on the depth of a term $t \in W_{\tau_n}^{GT(\bar{n},Y)}(X)$ that for every $i \in I$ if $t_k \approx r_k \in Id^{GT(\bar{n},Y)}V, k = 1,2,\ldots,n$, then $S^n(t,t_1,\ldots,t_n) \approx S^n(t,r_1,\ldots,r_n) \in Id^{GT(\bar{n},Y)}V$.

If $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ where $\alpha \in T(\bar{n},Y)$ and $Depth(t) = 1$, then

$$\begin{aligned} S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n) &= f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)}) \\ &\approx f_i(r_{\alpha(1)},\ldots,r_{\alpha(n)}) \\ &= S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),r_1,\ldots,r_n) \\ &\in Id^{GT(\bar{n},Y)}V, \end{aligned}$$

where IdV is compatible with the operation $\overline{f_i}$ of the absolutely free algebra $\mathcal{F}_\tau(X)$ and by the definition of generalized $T(\bar{n},Y)$-full terms.

If $t = f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)})$ where $\alpha' \in T_{\{j_1,\ldots,j_n\}}$ and $Depth(t) = 1$, then

$$\begin{aligned} S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),t_1,\ldots,t_n) &= f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}) \\ &= S^n(f_i(x_{\alpha'(j_1)},\ldots,x_{\alpha'(j_n)}),r_1,\ldots,r_n) \\ &\in Id^{GT(\bar{n},Y)}V. \end{aligned}$$

If $t = f_i(l_1,\ldots,l_n) \in W_{\tau_n}^{T(\bar{n},Y)}(X)$ and assume that

$$S^n(l_k,t_1,...,t_n) \approx S^n(l_k,r_1,...,r_n) \in Id^{GT(\bar{n},Y)}V.$$

for all $1 \leq k \leq n$ and $max_{1\leq k\leq n}Depth(r_k) = m$, then $Depth(t) = m+1$ and we obtain

$$\begin{aligned} S^n(f_i(l_1,\ldots,l_n),t_1,\ldots,t_n) &= f_i(S^n(l_1,t_1,\ldots,t_n),\ldots,S^n(l_n,t_1,\ldots,t_n)) \\ &\approx f_i(S^n(l_1,r_1,\ldots,r_n),\ldots,S^{n_i}(l_n,r_1,\ldots,r_n)) \\ &= S^n(f_i(l_1,\ldots,l_n),r_1,\ldots,r_n) \in Id^{GT(\bar{n},Y)}V. \end{aligned}$$

This means

$$S^n(t,t_1,\ldots,t_n) \approx S^n(t,r_1,\ldots,r_n) \in Id^{GT(\bar{n},Y)}V.$$

This is a consequence of the fact that IdV is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X)$. Assume now that $t \approx r, t_k \approx r_k \in Id^{GT(\bar{n},Y)}V$. Then

$$S^n(t,t_1,\ldots,t_n) \approx S^n(r,t_1,\ldots,t_n) \approx S^n(r,r_1,\ldots,r_n) \in Id^{GT(\bar{n},Y)}V.$$

□

By using the concepts of generalized $T(\bar{n},Y)$-full hypersubstitution as we presented in Section 3, we shall define generalized $T(\bar{n},Y)$-full hyperidentities in a variety of type τ_n.

Let V be a variety of type τ_n. An identity $s \approx t \in Id^{GT(\bar{n},Y)}V$ is called a *generalized* $T(\bar{n},Y)$*-full hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all $\sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$. Moreover, the variety V is called *generalized* $T(\bar{n},Y)$*-full solid* if the following holds:

$$\forall s \approx t \in Id^{GT(\bar{n},Y)}V, \forall \sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n), \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV.$$

Next, we characterizes the generalized $T(\bar{n},Y)$-full solid variety as the following theorem.

Theorem 4.2. *Let V be a variety of type τ_n. If $Id^{GT(\bar{n},Y)}V$ is a fully invariant congruence on $clone_{GT(\bar{n},Y)}(\tau_n)$, then V is a generalized $T(\bar{n},Y)$-full solid.*

Proof. Assume that $Id^{GT(\bar{n},Y)}V$ is a fully invariant congruence on $clone_{GT(\bar{n},Y)}(\tau_n)$. Let $s \approx t \in Id^{GT(\bar{n},Y)}V$ and $\sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$. By Theorem 3.2, $\hat{\sigma}$ is an endomorphism of $clone_{GT(\bar{n},Y)}(\tau_n)$. Hence $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{GT(\bar{n},Y)}V$ which shows that V is generalized $T(\bar{n},Y)$-full solid. □

For a variety V of type τ_n, $Id^{GT(\bar{n},Y)}V$ is a congruence on $clone_{GT(\bar{n},Y)}(\tau_n)$ by using the Theorem 4.1. We can form the quotient algebra

$$clone_{GT(\bar{n},Y)}(V) := clone_{GT(\bar{n},Y)}(\tau_n)/Id^{GT(\bar{n},Y)}V.$$

This quotient algebra belongs to the class of a Menger algebra of rank n. Note that, we have a natural homomorphism

$$nat_{Id^{GT(\bar{n},Y)}V} : clone_{GT(\bar{n},Y)}(\tau_n) \longrightarrow clone_{GT(\bar{n},Y)}(V)$$

such that

$$nat_{Id^{GT(\bar{n},Y)}V}(t) = [t]_{Id^{GT(\bar{n},Y)}V}.$$

Finally, we prove the following connection between generalized $T(\bar{n},Y)$-full hyperidentities of a variety V and clone identities.

Theorem 4.3. *Let V be a variety of type τ_n. If $s \approx t \in Id^{GT(\bar{n},Y)}V$ is an identity in $clone_{GT(\bar{n},Y)}(V)$, then $s \approx t$ is a generalized $T(\bar{n},Y)$-full hyperidentity of V.*

Proof. Assume that $s \approx t \in Id^{GT(\bar{n},Y)}V$ is an identity in $clone_{GT(\bar{n},Y)}(V)$. Let $\sigma \in Hyp_G^{T(\bar{n},Y)}(\tau_n)$. Then $\hat{\sigma} : clone_{GT(\bar{n},Y)}(\tau_n) \longrightarrow clone_{GT(\bar{n},Y)}(\tau_n)$ is an endomorphism by Theorem 3.2. Thus

$$nat_{Id^{GT(\bar{n},Y)}V} \circ \hat{\sigma} : clone_{GT(\bar{n},Y)}(\tau_n) \longrightarrow clone_{GT(\bar{n},Y)}(V)$$

is a homomorphism. By assumption,

$$\left(nat_{Id^{GT(\bar{n},Y)}V} \circ \hat{\sigma}\right)(s) = \left(nat_{Id^{GT(\bar{n},Y)}V} \circ \hat{\sigma}\right)(t).$$

That is

$$nat_{Id^{GT(\bar{n},Y)}V}(\hat{\sigma}[s]) = nat_{Id^{GT(\bar{n},Y)}V}(\hat{\sigma}[t]).$$

Thus

$$[\hat{\sigma}[s]]_{Id^{GT(\bar{n},Y)}V} = [\hat{\sigma}[t]]_{Id^{GT(\bar{n},Y)}V},$$

and hence

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id^{GT(\bar{n},Y)}V.$$

Therefore, $s \approx t$ is a generalized $T(\bar{n}, Y)$-full hyperidentity of V. □

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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