ALGEBRAS OF PRIMITIVE FUNCTIONS

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Abstract. In this paper a Banach space X is considered such that $A(\overline{U}) \subset X \subset Hol(U)$, where $A(\overline{U}) = \{f \in \mathcal{H}(\overline{U}) : \sum_{n=0}^{\infty} |a_n| < \infty,$ if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $U\}$. We shall define a norm in the space $\mathscr{A}_X(\overline{U})$ of all primitive functions to $f \in X$ such that $\mathscr{A}_X(\overline{U})$ is a Banach algebra with respect to the pointwise multiplication. The maximal ideals in $\mathscr{A}_X(\overline{U})$ are studied, too.

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1. Introduction

Denote by $Hol(\Sigma^0)$ the set of all analytic functions on Σ^0. Consider $Hol(U)$ with the topology of locally uniform convergence. Let X be a Banach space such that $A(\overline{U}) \subset X \subset Hol(U)$ and the both embeddings $A(\overline{U}) \hookrightarrow X \hookrightarrow Hol(U)$ are continuous. The pointwise multiplication of functions is a continuous operation both on $A(\overline{U})$ and $Hol(U)$. We shall suppose $f \cdot g \in X$ for $f \in A(\overline{U})$, $g \in X$ and

$$\|f \cdot g\|_X \leq \|f\|_{A(\overline{U})} \|g\|_X.$$

It follows, by the way, the pointwise multiplication "$\cdot$" : $A(\overline{U}) \times X \to X$ is continuous. We shall construct a Banach algebra $\mathscr{A}_X(\overline{U})$ taking X as an initial point of the process. Firstly,

$$\mathscr{A}_X(\overline{U}) = \{F \in Hol(U) : \ F' \in X\}$$

is a linear space. Suppose further $\mathscr{A}_X(\overline{U}) \subset A(\overline{U})$. Then for $F, G \in \mathscr{A}_X(U)$ we have $(F \cdot G)' = F'G + FG' \in X$ as $\{F'G; FG'\} \subset X$. So $\mathscr{A}_X(\overline{U})$ is an algebra. Putting

$$\|F\| = \|F\|_{A(\overline{U})} + \|F'\|_X \tag{1}$$

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for $F \in \mathscr{A}_X(\overline{U})$ we invert $\mathscr{A}_X(\overline{U})$ to a normed algebra:

$$\|F \cdot G\| = \|F \cdot G\|_{A(\overline{U})} + \|F'G + FG'\|_X \leq \|F\|_{A(\overline{U})}\|G\|_{A(\overline{U})} + \|G\|_{A(\overline{U})}\|F'\|_X + \|F\|_{A(\overline{U})}\|G'\|_X$$
$$\leq (\|F\|_{A(\overline{U})} + \|F'\|_X)(\|G\|_{A(\overline{U})} + \|G'\|_X) = \|F\| \cdot \|G\|$$

for any $F, G \in \mathscr{A}_X(\overline{U})$. The algebra $\mathscr{A}_X(\overline{U})$ is complete as it follows from completeness of X and $A(\overline{U})$.

Let $\alpha \in \overline{U}$. Then we can write $\mathscr{A}_X(\overline{U}) = \mathbb{C} \oplus I_\alpha$, where

$$I_\alpha = \{f \in \mathscr{A}_X(\overline{U});\ f(\alpha) = 0\}$$

is an ideal in $\mathscr{A}_X(\overline{U})$ for all $\alpha \in \overline{U}$. We are interested if no other maximal ideals in $\mathscr{A}_X(\overline{U})$ exist (see e.g. Proposition 2). We have $I_\alpha \cong X,\ \alpha \in \overline{U}$, where $g \mapsto g'$ is the isometry. So $\mathscr{A}_X(\overline{U})$ with the norm

$$\|F\|_\alpha = |F(\alpha)| + \|F'\|_X,\ \alpha \in \overline{U},$$

is a Banach space. Using the open mapping theorem (cf. [1], 5.10, p.116) we obtain the norms $\|\cdot\|$ and $\|\cdot\|_\alpha$ are equivalent on $\mathscr{A}_X(\overline{U})$ for all $\alpha \in \overline{U}$. Note however, that the condition

$$\|F \cdot G\|_\alpha \leq \|F\|_\alpha \|G\|_\alpha,\ F, G \in \mathscr{A}_X(\overline{U}),$$

need not be satisfied for $\alpha \in \overline{U}$. So $\mathscr{A}_X(\overline{U})$ with the norm $\|\cdot\|_\alpha,\ \alpha \in \overline{U}$, may be a Banach space only! An example will be mentioned later for a concrete X (see Remark 2).

We showed that the space $\mathscr{A}_X(\overline{U})$ is a Banach algebra with the norm (1). In the next sections we will investigate particular cases of X and consider maximal ideals on them. Some proofs had to be omitted due to space constraints. The detailed proofs will be given in our forthcoming publications.

2. The Cases $X = \mathcal{H}(\overline{U})$ and $X = H^\rho(U),\ 1 \leq \rho \leq \infty$

Let $f \in Hol(U),\ 1 \leq p < \infty$. Put

$$\|f\|_p = \sup_{r \in (0,1)} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}.$$

Then we obtain using Jensen's inequality that

$$\|f\|_{p_1} \leq \|f\|_{p_2} \leq \|f\|_\infty = \sup_{z \in U} |f(z)|$$

holds for all $f \in Hol(U),\ 1 \leq p_1 < p_2 < \infty$. We define

$$H^p(U) = \{f \in Hol(U);\ \|f\|_p < \infty\},\ 1 \leq p \leq \infty.$$

The linear spaces $H^p(U),\ 1 \leq p \leq \infty$, are Banach spaces (cf. [1], [2] e.g.).

It follows $\mathcal{H}(\overline{U}) \subset H^\infty(U) \subset H^{p_2}(U) \subset H^{p_1}(U) \subset Hol(U),\ 1 \leq p_1 < p_2 < \infty,$ every embedding given by any of these inclusion is continuous and the topology on a smaller space is always finer (with the only exception: $\mathcal{H}(\overline{U})$ is a subspace of $H^\infty(U)$). We can verify our assumptions e.g. for $X=\mathcal{H}(\overline{U})$ or $H^p(U)$, $1\le p\le\infty$. Using Hardy's theorem (cf. [2], p. 71) and the first consequence on p. 91, [2], we obtain $F\in A(\overline{U})$ if $F'\in H^1(U)$. So $\mathscr{A}_X(\overline{U})\subset A(\overline{U})\subset\mathcal{H}(\overline{U})$ for all considered X. As $\|f\|_{\mathcal{H}(\overline{U})}\le\|f\|_{A(\overline{U})}$, it remains to prove the inequality $\|f\cdot g\|_X\le\|f\|_{\mathcal{H}(\overline{U})}\|g\|_X$ for all $f\in\mathcal{H}(\overline{U})$ and $g\in X$. It is clear for $X=\mathcal{H}(\overline{U})$ or $X=H^\infty(U)$. Let $p\in\langle 1;+\infty)$, $f\in\mathcal{H}(\overline{U})$, $g\in H^p(U)$. Then g is defined almost everywhere on ∂U by its angle trace. Then

$$\begin{aligned}\|f\cdot g\|_{H^p(U)}&=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{it})g(e^{it})|^p dt\right)^{\frac{1}{p}}\\&\le\sup_{t\in\langle-\pi;\pi\rangle}|f(e^{it})|\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(e^{it})|^p dt\right)^{\frac{1}{p}}\\&=\|f\|_{\mathcal{H}(\overline{U})}\|g\|_{H^p(U)}.\end{aligned}$$

We have proved the next theorem.

Theorem 1. *The algebra* $\mathscr{A}_X(\overline{U})=\{F\in Hol(U);\ F'\in X\}$ *is a Banach algebra with respect to the pointwise multiplication of functions and the norm*

$$\|F\|=\|F\|_{A(\overline{U})}+\|F'\|_X,$$

if $X=\mathcal{H}(\overline{U})$ *or* $X=H^p(U)$, $1\le p\le\infty$.

For the sake of brevity we shall denote $\mathscr{A}_{\mathcal{H}(\overline{U})}(\overline{U})=\mathscr{A}^{(\infty)}(\overline{U})$, $\mathscr{A}_{H^p(U)}(\overline{U})=\mathscr{A}^p(\overline{U})$, $1\le p\le\infty$. Currently, the questions of holomorphic continuation of functions belonging to the class H^p are very relevant [5], [6], [7] , [8], [9], [10], [11].

Remark 1. *The whole construction of* $\mathscr{A}_X(\overline{U})$ *can be made using* $\mathcal{H}(\overline{U})$ *in the place of* $\mathscr{A}(\overline{U})$. *So* $\mathscr{A}_X(\overline{U})$ *is Banach algebra with respect to the norm* $\|F\|^\sim==\|F\|_{\mathcal{H}(\overline{U})}+\|F'\|_X$, $F\in\mathscr{A}_X(\overline{U})$, *for* $X=\mathcal{H}(\overline{U})$ *or* $X=H^p(U)$, $1\le p\le\infty$. *Since* $\|F\|^\sim\le\|F\|$, *the norms* $\|\cdot\|^\sim$ *and* $\|\cdot\|$ *are equivalent on* $\mathscr{A}_X(\overline{U})$ *due to the open mapping theorem. Let* $F\in A(\overline{U})$, $F(z)=\sum\limits_{n=0}^{\infty}a_nz^n$. *Then* $\|F\|_{A(\overline{U})}=\|F\|_{\mathcal{H}(\overline{U})}$ *if and only if there are exist* φ_0, $\varphi\in\mathbb{R}$ *such that* $\varphi_0+n\varphi\in\arg a_n$ *for all* $n\in\mathbb{N}\cup\{0\}$, *where* $\arg 0=\mathbb{R}$. *It follows a polynomial* F, $\deg F=2$, *exists such that* $\|F\|_{\mathcal{H}(\overline{U})}<\|F\|_{A(\overline{U})}$. *So the norms* $\|\cdot\|^\sim$ *and* $\|\cdot\|$ *are not equal on* $\mathscr{A}_X\left(\overline{U}\right)$.

We prefer the construction with $A(\overline{U})$ which yields a sequence $\mathscr{A}_X(\overline{U})$ different from this one described in Theorem 2.

Remark 2. *Let* $F(z)=z-\alpha$ *for* $\alpha\in\overline{U}$. *Then* $F,F^2\in\mathscr{A}_{\mathcal{H}(\overline{U})}(\overline{U})$, $F(\alpha)=F^2(\alpha)=0$,

$$F'(z)=1,\ (F^2)'(z)=2(z-\alpha),\ \|F'\|_{\mathcal{H}(\overline{U})}=1,\ \|(F^2)'\|_{\mathcal{H}(\overline{U})}=2(1+|\alpha|)>1=\|F'\|^2_{\mathcal{H}(\overline{U})}.$$

Note $\|G\|_{\mathcal{H}(\overline{U})} = \|G\|_{A(\overline{U})}$, *if* G *is polynomial,* $\deg G \leq 1$. *So,* $\|F^2\|_\alpha > \|F\|_\alpha^2$ *if the norm* $\|\cdot\|_\alpha$ *is used in* $\mathscr{A}_{A(\overline{U})}(\overline{U})$. *The space* $\mathscr{A}_{A(\overline{U})}(\overline{U})$ *with the norm* $\|\cdot\|_\alpha$ *is a Banach space only (see the construction* $\mathscr{A}_X(\overline{U})$ *for* $X = l_p(U)$, $1 \leq p < \infty$), *exactly like* $\mathscr{A}_{\mathcal{H}(\overline{U})}(\overline{U})$.

Since $f \in \mathscr{A}^1(\overline{U})$ if and only if $f(e^{it}) \in AC(\langle -\pi, \pi\rangle)$ and $f \in \mathscr{A}^\infty(\overline{U})$ if and only if $f(e^{it}) \in Lip(\langle -\pi, \pi\rangle)$, we shall occasionally use the notation $AC(\overline{U}) = \mathscr{A}^1(\overline{U})$, $Lip(\overline{U}) = \mathscr{A}^\infty(\overline{U})$.

Using the inequality $\|f\|_{p_1} \leq \|f\|_{p_2}$, $1 \leq p_1 < p_2 \leq \infty$, $f \in H^{p_2}(U)$ we obtain every embedding $\mathscr{A}^{(\infty)}(\overline{U}) \hookrightarrow \mathscr{A}^{p_2}(\overline{U}) \hookrightarrow \mathscr{A}^{p_1}(\overline{U})$, $1 \leq p_1 < p_2 < \infty$ is continuous and the topology on a smaller space is finer with the obvious exception: $\mathscr{A}^{(\infty)}(\overline{U})$ is a subspace of $\mathscr{A}^\infty(\overline{U})$. It is clear the embedding $\mathscr{A}^1(\overline{U}) = AC(\overline{U}) \hookrightarrow A(\overline{U})$ is continuous.

Let

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{2^k}, \ |z| \leq 1.$$

Then $f \in A(\overline{U}) \setminus AC(\overline{U})$. Denote

$$s_n(z) = \sum_{k=0}^{n-1} \frac{1}{2^k} z^{2^k}, \ n \in \mathbb{N}.$$

Then $s_n \to f$ in $A(\overline{U})$, but $\{s_n\}_{n=1}^\infty$ is not fundamental in $AC(\overline{U})$. It follows the topology of $AC(\overline{U})$ is a refinement of the topology of $A(\overline{U})$. So we have obtained a "long" sequence of continuous embeddings of Banach spaces

$$\mathscr{A}^{(\infty)}(\overline{U}) \hookrightarrow \mathscr{A}^{p_2}(\overline{U}) \hookrightarrow \mathscr{A}^{p_1}(\overline{U}) \hookrightarrow A(\overline{U}) \hookrightarrow$$

$$\mathcal{H}(\overline{U}) \hookrightarrow H^{p_2}(U) \hookrightarrow H^{p_1}(U), 1 \leq p_1 < p_2 \leq \infty,$$

where only the topologies on $\mathscr{A}^{(\infty)}(\overline{U})$, $\mathscr{A}^\infty(\overline{U})$ and $\mathcal{H}(\overline{U})$, $H^\infty(U)$ are the same.

Proposition 1. *Every maximal ideal in* $AC(\overline{U})$ *(resp. in* $AC(\partial U)$) *is of the form*

$$I_\alpha = \{f \in AC(\overline{U}) : \ f(\alpha) = 0\}, \ \alpha \in \overline{U} \ (resp. \ G(I_\alpha) = \{f|_{\partial U} : \ f \in I_\alpha\}).$$

Proof. It is enough to prove that for every homomorphism $F : AC(\overline{U}) \to \mathbb{C}$ some $\alpha \in \overline{U}$ exists such that $F(x) = x(\alpha)$ for all $x \in AC(\overline{U})$. Since the set of all polynomials is dense in $AC(U)$, we can repeat the proof of this fact from [1], p.400.

Remark. The norms $\|\cdot\|^\sim$ and $\|\cdot\|$ equivalent on $AC(\overline{U})$ and so complex homomorphisms and maximal ideals are the same in the both cases. However, it is easy to prove straightforward that $\sigma_n \to f$ in $A(\overline{U})$ for every $f \in A(\overline{U})$. It follows $\sigma_n \to f$ in $\mathscr{A}^{(\infty)}(\overline{U})$ for every $f \in \mathscr{A}^{(\infty)}(\overline{U}) \subset A(\overline{U})$. So the next proposition is true.

Proposition 2. *Every maximal ideal in* $\mathscr{A}^{(\infty)}(\overline{U})$ *(resp. in* $\mathscr{A}^{(\infty)}(\partial U)$), *is of the form*

$$I_\alpha = \{f \in \mathscr{A}^{(\infty)}(\overline{U}); \ f(\alpha) = 0\}, \ (resp. \ G(I_\alpha) = \{f|_\partial U; \ f \in I_\alpha\}),$$

where $\alpha \in \overline{U}$.

Similarly, using [1], p.388, Exercise 25, we can prove the following proposition.

Proposition 3. Let $p \in \langle 1; +\infty)$. Every maximal ideal in $\mathscr{A}^p(\overline{U})$ (resp. in $\mathscr{A}^p(\partial U)$) is also form

$$I_\alpha = \{f \in \mathscr{A}^p(\overline{U});\ f(\alpha) = 0\}$$

(resp. $G(I_\alpha) = \{f|_{\partial U};\ f \in I_\alpha\}$), where $\alpha \in \overline{U}$.

Problem 1. Are the ideals

$$I_\alpha = \{f \in \mathscr{A}^\infty(\overline{U});\ f(\alpha) = 0\},\ \alpha \in \overline{U}$$

the only maximal ideals in $\mathscr{A}^\infty(\overline{U})$?

It is clear that the mapping $T : AC(\overline{U}) \to AC(\overline{U})$ defined by $Tf = e^{i\beta} f \circ \tau$, where $\beta \in \mathbb{R}$ and τ is a conformal mapping U onto itself, is an isometric isomorphism $AC(\overline{U})$ with the norm $\|\cdot\|^\sim$ onto itself. Probably there are no other isometric isomorphisms $AC(\overline{U})$ with this norm onto itself. This assertion can be proved by method of [3], p.202-211 if

$$\|(Tf)'\|_{H^1(U)} = \|f'\|_{H^1(U)}$$

for all $f \in AC(\overline{U})$, i.e. if $\widetilde{T} : H^1(U) \to H^1(U)$ is isometric isomorphism where $\widetilde{T}f = (TF)',\ F' = f$. Maybe it can be proved taking $F \in I_{z_0} \subset AC(\overline{U})$ for an appropriate $z_0 \in \overline{U}$. We don't know whether $T : \mathscr{A}_X(\overline{U}) \to \mathscr{A}_X(\overline{U})$ as an isometrical isomorphism in all remaining cases: for $X = \mathcal{H}(\overline{U})$ or $X = H^p(U),\ 1 < p \leq \infty$ if $\|\cdot\|^\sim$ is taken as the norm in $\mathscr{A}_X(\overline{U})$ and generally for all X of Theorem 2, if $\mathscr{A}_X(\overline{U})$ is considered with the norm $\|\cdot\|$.

3. The Case $X = l_\rho,\ 1 \leq \rho < \infty$

Denote

$$l^p = \{\{a_n\}_0^\infty;\ \|\{a_n\}\|_p = \left(\sum |a_n|^p\right)^{\frac{1}{p}} < \infty\}$$

for $1 \leq p < \infty$,

$$l_\infty = \{\{a_n\}_0^\infty;\ \|\{a_n\}\|_\infty = \sup_{n\in\mathbb{N}_0} |a_n| < \infty\}.$$

Then l_p is a Banach space, $1 \leq p \leq \infty$, $l_{p_1} \subsetneqq l_{p_2}$, the embedding $l_{p_1} \hookrightarrow l_{p_2}$ is continuous and the topology is finer for the smaller space if
$1 \leq p_1 < p_2 \leq \infty$. We shall consider these spaces as spaces of functions from $Hol(U)$:

$$l_p(U) = \{f; f(z) = \sum_{n=0}^{\infty} a_n z^n,\ \|f\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{\frac{1}{p}} < \infty\} \subset Hol(U),\ 1 \leq p \leq \infty.$$

Put $X = l_p(U),\ 1 \leq p < \infty$. Then $\mathscr{A}_X(\overline{U}) \subset A(U)$. It is true for $p = 1$. For $1 < p < \infty$ let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ f \in l_p(U).$$

Then

$$F(z)=\sum_{n=1}^{\infty}\frac{a_{n-1}}{n}z^n,\ z\in U,$$

and

$$\sum_{n=1}^{\infty}\frac{|a_{n-1}|}{n}\le\left(\sum_{n=1}^{\infty}|a_{n-1}|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{1}{n^q}\right)^{\frac{1}{q}}=\|f\|_p\|\{\frac{1}{n^q}\}\|_q<\infty,\ \frac{1}{p}+\frac{1}{q}=1,$$

by Hölder's inequality.

Remark 3. *Note that for*

$$F_0(z)=-\ln(1-z)=\sum_{n=1}^{\infty}\frac{z^n}{n},\ |z|<1,$$

we have

$$F_0'(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}=f_0(z)$$

and so $f_0\in l_\infty(U)$, *but* F_0 *is not defined in* $\overline{U}$. *Moreover, we cannot define the norm in* $\mathscr{A}_{l_\infty(U)}(U)$ *by*

$$\|F\|=\sup_{|z|<1}|F(z)|+\|F'\|_{l_\infty(U)}$$

as F_0 *is not bounded in* U. *It holds* $F_0\in H^1(U)$, only.

Theorem 2. *The algebra* $\mathscr{A}_X(\overline{U})=\{F\in Hol(U);\ F'\in X\}$ *is a Banach algebra with respect to the pointwise multiplication of functions and the norm*

$$\|F\|=\|F\|_{A(\overline{U})}+\|F'\|_X,$$

if $X=l_p(U),\ 1\le p<\infty$.

For the sake of brevity we shall denote $\mathscr{A}_{l_p(U)}(\overline{U})=\mathscr{A}_p(\overline{U}),\ 1\le p<\infty$.

Lemma 1. Let $f\in l_p(U),\ 1\le p<\infty$,

$$f(z)=\sum_{k=0}^{\infty}a_kz^k,\ z\in\overline{U}.$$

Then $s_n\to f$ in $l_p(U)$, where $s_n(z)=\sum\limits_{k=0}^{n}a_kz^k$. The proof is trivial.

Proposition Every maximal ideal in $\mathscr{A}_p(\overline{U})$ (resp. in $\mathscr{A}_p(\partial U)$),
$1\le p<\infty$, is of the form

$$I_\alpha=\{f\in\mathscr{A}_p(\overline{U});\ f(\alpha)=0\},\ (\text{resp. } G(I_\alpha)=\{f|_{\partial}U;\ f\in I_\alpha\}),$$

where $\alpha\in\overline{U}$.

Proof. Using Lemma 1 we can repeat the proof of Proposition 2.

Remark 4. a) We obtain a "long" sequence of continuous embedding also for $l_p(U)$:

$$\mathscr{A}_{p_1}(\overline{U})\hookrightarrow\mathscr{A}_{p_2}(\overline{U})\hookrightarrow A(\overline{U})=l_1(U)\hookrightarrow l_{p_1}(U)\hookrightarrow l_{p_1}(U),\ 1\le p_1<p_2<\infty.$$

The corresponding inclusions are strict, the topologies on (strictly) smaller spaces are finer.

b) Note that $p\mapsto l_p(U)$ is increasing, $p\mapsto H^p(U)$ is decreasing, both for $p\in\langle 1;+\infty)$, in the sense of inclusion. We have $l_2(U) = H^2(U)$ (see [1], 17.10(a), p.371). So $\mathscr{A}_2(\overline{U}) = \mathscr{A}^2(\overline{U})$ and we can describe maximal ideals in $\mathscr{A}^2(\overline{U})$ by any of Propositions 2, 3.

c) We have $l_p(U) \subset H^p(U),\ p \in \langle 1; 2),\ l_p(U) \supset H^p(U),\ p \in \langle 2; +\infty)$. Inclusions are strict, topologies on smaller spaces are finer. The same holds for the algebra $\mathscr{A}_p(\overline{U}), \mathscr{A}^p(\overline{U}) : \mathscr{A}_p(\overline{U}) \subset \mathscr{A}^p(\overline{U}), p \in \langle 1; 2)$, $\mathscr{A}_p(\overline{U}) \supset \mathscr{A}^p(\overline{U}),\ p \in \langle 2; +\infty)$, with the same relation between topologies.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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