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ESTIMATES ON SOLUTIONS OF SOME NEW MULTIDIMENSIONAL VOLTERRA-FREDHOLM TYPE INTEGRAL INEQUALITIES WITH DELAY AND APPLICATION

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Abstract. In this article, we provide some new multidimensional nonlinear integral inequalities of Volterra-Fredholm type with delay. By adopting novel analysis techniques, the upper bounds of the unknown functions are given. Our main results can be applied to the research of boundedness, and uniqueness of solutions of a class of Volterra-Fredholm type integral equations of several variables. An application is given to show the validity of our established inequalities.

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1. Introduction

The fundamental integral inequality of Gronwall [8] and its generalizations contribute significantly to the study of existence, uniqueness, boundedness, oscillation, stability, and other qualitative properties of solutions of differential and integral equations. Over the last decades, many authors have given several significant extensions and developments to the forms of classical integral inequalities [1–3,5,10]. Recently, many researchers have been interested in giving new integral inequalities with delay of the Volterra-Fredholm type [4,7,11,14].

Pachpatte [13] has discussed the following linear retarded integral inequality of Volterra-Fredholm type in two variables

$$u(x,y) \leq c + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\lambda(y_0)}^{\lambda(y)} a(x,y,s,t)u(s,t)dtds + \int_{\gamma(x_0)}^{\gamma(M)} \int_{\lambda(y_0)}^{\lambda(N)} b(x,y,s,t)u(s,t)dtds. \tag{1}$$

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In [12], Ma and Pečarić proved the nonlinear retarded Volterra-Fredholm integral inequality in two variables as follows

$$\begin{aligned} u(x,y) \quad \leq \quad & k + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\lambda(y_0)}^{\lambda(y)} \sigma_1(s,t) \left[f(s,t)\omega\left(u(s,t)\right) + \int_{\gamma(x_0)}^{s} \int_{\lambda(x_0)}^{t} \sigma_2(\tau,\xi)\omega\left(u(\tau,\xi)\right) d\xi d\tau \right] dtds \\ & + \int_{\gamma(x_0)}^{\gamma(M)} \int_{\lambda(y_0)}^{\lambda(N)} \sigma_1(s,t) \left[f(s,t)\omega\left(u(s,t)\right) + \int_{\gamma(x_0)}^{s} \int_{\lambda(x_0)}^{t} \sigma_2(\tau,\xi)\omega\left(u(\tau,\xi)\right) d\xi d\tau \right] dtds \quad (2) \end{aligned}$$

Kendre et al. in [9] established the following nonlinear integral inequality

$$u^p(t) \leq c(t) + \int_a^t f(t,s)u(s)ds + \int_a^b g(t,s)u^p(s)ds. \quad (3)$$

The following Volterra-Fredholm integral inequality with delay, proved by El-Deeb and Ahmed [6].

$$\omega^p(t) \leq c(t) + \int_0^{\gamma(t)} k_1(t,s)\omega(s)ds + \int_a^b k_2(t,s)\omega^p(s)ds. \quad (4)$$

In some problems, it is desirable to establish some new integral inequalities of the above type in more general cases, in order to achieve a diversity of desired goals. In this paper, motivated by the above results, we discuss a class of useful retarded integral inequalities of Volterra-Fredholm type in several variables. The upper bound estimation of the unknown function is given by some integral inequality techniques. Finally, we propose an application of our results to study the boundedness of solutions of multidimensional Volterra-Fredholm integral equations with delay.

2. Main results

Throughout this article, let the following notations: $J = \left[x^0, T\right] = J_1 \times J_2 \times ... \times J_n$ where $J_i = \left[x_i^0, T_i\right], i = 1, ..., n$, and $x^0 = \left(x_1^0, ..., x_n^0\right), T = (T_1, ..., T_n) \in \mathbb{R}^n, \Delta = \left\{(x,s) \in J^2 : x^0 \leq s \leq x \leq T\right\}$. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we write $x \leq y$ $(x < y)$ if and only if $x_i \leq y_i$ $(x_i < y_i), i = 1, ..., n$. We put $x = (x_1, x^1)$, where $x^1 = (x_2, ..., x_n)$, $(x^0)^1 = \left(x_2^0, ..., x_n^0\right)$, and

- $D_i = \frac{\partial}{\partial x_i}, i = 1, ..., n,$
- $dx^1 = dx_n ... dx_2,$
- $\int_{x^0}^{x} ...ds = \int_{x_1^0}^{x_1} ... \int_{x_n^0}^{x_n} ....ds_n...ds_1 = \int_{x_1^0}^{x_1} \int_{(x^0)^1}^{x^1} ....ds^1 ds_1,$
- $\int_{\alpha(x^0)}^{\alpha(x)} ...ds = \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} ...ds_n...ds_1.$

In what follows, we give some new generalizations of the multidimentional Volterra-Fredholm type integral inequalities.

Theorem 2.1. *Let $u \in C(J, \mathbb{R}_+)$, and $f, g, a \in C(\Delta, \mathbb{R}_+)$ be nondecreasing functions in x for each $s \in J, \gamma(x) = (\gamma_1(x_1), ..., \gamma_n(x_n)) \in C^1(J, J)$, where $\gamma_i \in C^1(J_i, J_i)$ be nondecreasing functions on J_i, with $\gamma_i(x_i) \leq x_i, i = 1, ..., n$. Let $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\omega(u) > 0$, for $u > 0$, and*

$$F(v) = \int_{v_0}^{v} \frac{ds}{\omega^{-1}(s)}, v \geq v_0 > 0, F(+\infty) = +\infty, \quad (5)$$

$$H_1(u) = u - \int_{\gamma(x^0)}^{\gamma(T)} g(x,s) F^{-1}\Big[F(u) + \varphi(s)\Big] ds - u_0, \tag{6}$$

where $\varphi(x) = \int_{\gamma(x^0)}^{\gamma(x)} a(x,s) \left[1 + \int_{\gamma(x^0)}^{s} f(s,\tau) d\tau\right] ds$, H_1 *is increasing for* $u \geq u_0$ *and* $H_1(u) = 0$ *has a solution* c *for* $u \geq u_0$, *then. If* $u(x)$ *satisfies*

$$\begin{aligned} \omega(u(x)) &\leq u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(x,s) \left[u(s) + \int_{\gamma(x^0)}^{s} f(s,\tau) u(\tau) d\tau\right] ds \\ &\quad + \int_{\gamma(x^0)}^{\gamma(T)} g(x,s) \omega(u(s)) ds, \end{aligned} \tag{7}$$

for $x \in J$, *where* $u_0 \geq 0$ *is a constant, then*

$$u(x) \leq \omega^{-1}\left(F^{-1}\left[F(c) + \int_{\gamma(x^0)}^{\gamma(x)} a(x,s) \left[1 + \int_{\gamma(x^0)}^{s} f(s,\tau) d\tau\right] ds\right]\right), \tag{8}$$

for $x \in J$, *where* F^{-1} *and* H_1^{-1} *are the inverse functions of* F *and* H_1, *respectively.*

Proof. Let $u_0 > 0$ and $X = (X_1, ..., X_n) \in J$ fixed, then for $x^0 \leq x \leq X \leq T$, let

$$\begin{aligned} z(x) &= u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(X,s) \left[u(s) + \int_{\gamma(x^0)}^{s} f(s,\tau) u(\tau) d\tau\right] ds \\ &\quad + \int_{\gamma(x^0)}^{\gamma(T)} g(X,s) \omega(u(s)) ds. \end{aligned}$$

It is obvious to see that $z(x)$ is a positive and nondecreasing function on J, so we have

$$u(x) \leq \omega^{-1}(z(x)), \tag{9}$$

and

$$D_1 ... D_n z(x) \leq \gamma'(x) a(X, \gamma(x)) \omega^{-1}(z(\gamma(x))) \left[1 + \int_{\gamma(x^0)}^{\gamma(x)} f(x,\tau) d\tau\right],$$

then

$$\frac{D_1 ... D_n z(x)}{\omega^{-1}(z(x))} \leq \gamma'(x) a(X, \gamma(x)) \left[1 + \int_{\gamma(x^0)}^{\gamma(x)} f(x,\tau) d\tau\right],$$

or

$$D_n \left(\frac{D_1 ... D_{n-1} z(x)}{\omega^{-1}(z(x))}\right) \leq \gamma'(x) a(X, \gamma(x)) \left[1 + \int_{\gamma(x^0)}^{\gamma(x)} f(x,\tau) d\tau\right]. \tag{10}$$

Fixing $x_1, ..., x_{n-1}$, setting $x_n = s_n$ then the integration of (10) with respect to s_n from x_n^0 to x_n, gives

$$\begin{aligned} \frac{D_1 ... D_{n-1} z(x)}{\omega^{-1}(z(x))} &\leq \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a(X, \gamma_1(x_1), ..., \gamma_{n-1}(x_{n-1}), s_n) \times \\ &\quad \left[1 + \int_{\gamma_1(x_1^0)}^{\gamma_1(x_1)} ... \int_{\gamma_n(x_n^0)}^{s_n} f(X, \tau_1, ..., \tau_n) d\tau_n\right] \gamma_1'(x_1) \times ... \times \gamma_{n-1}'(x_{n-1}) ds_n. \end{aligned}$$

Using the same method above, we obtain

$$\frac{D_1 z(x)}{\omega^{-1}(z(x))} \leq \int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)} ... \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a(X, \gamma_1(x_1), s_2, ..., s_n) \times$$

$$\left[1+\int_{\gamma_1(x_1^0)}^{\gamma_1(x_1)}\int_{\gamma_2(x_2^0)}^{s_2}...\int_{\gamma_n(x_n^0)}^{s_n} f(X,\tau_1,....,\tau_n)d\tau_n\right]\gamma_1'(x_1)ds^1. \tag{11}$$

Keeping $x^1=(x_2,...,x_n)$ fixed, replacing x_1 by s_1 then the integration of (11) with respect to s_1 from x_1^0 to x_1, gives

$$F\left(z(x)\right)\le F\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a\left(X,s\right)\left[1+\int_{\gamma(x^0)}^{s} f(s,\tau)d\tau\right]ds,$$

then we get

$$z(x)\le F^{-1}\left[F\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a\left(X,s\right)\left[1+\int_{\gamma(x^0)}^{s} f(s,\tau)d\tau\right]ds\right]. \tag{12}$$

From (9) and (12), we obtain

$$u(x)\le \omega^{-1}\left\{F^{-1}\left[F\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a\left(X,s\right)\left[1+\int_{\gamma(x^0)}^{s} f(s,\tau)d\tau\right]ds\right]\right\}. \tag{13}$$

In addition, we have

$$z(x_1^0,x^1)=u_0+\int_{\gamma(x^0)}^{\gamma(T)} g(X,s)\omega(u(s))ds. \tag{14}$$

Using (13) in (14), we obtain

$$z(x_1^0,x^1)\le u_0+\int_{\gamma(x^0)}^{\gamma(T)} g(X,s)F^{-1}\left[F\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(s)} a\left(X,\theta\right)\left[1+\int_{\gamma(x^0)}^{\theta} f(\theta,\tau)d\tau\right]d\theta\right]ds.$$

Since X is chosen arbitrarily, we have

$$z(x_1^0,x^1)-u_0-\int_{\gamma(x^0)}^{\gamma(T)} g(x,s)F^{-1}\Big[F\left(z(x_1^0,x^1)\right)+\varphi(s)\Big]ds\le 0. \tag{15}$$

From (15) and the definition of H_1, we get

$$H_1\left(z(x_1^0,x^1)\right)\le 0=H_1(c).$$

Since H_1 is increasing, then the above inequality gives

$$z(x_1^0,x^1)\le c. \tag{16}$$

From (16), (13), and the fact that X is chosen arbitrarily, we get the result (8).

For $u_0=0$, we repeat the same procedure above replacing u_0 by $\varepsilon>0$ and finally let $\varepsilon\to 0$. □

Remark 2.2. *For $\omega(u)=u^p, p\ge 1$, $\gamma(x)=x$ and $x^1=(x_2,...,x_n)$ fixed, Theorem 2.1 reduces to Theorem 2.5 in [9]. Also for $\omega(u)=u^p, p\ge 1$, $\gamma(x)=x$, $f=0$ and $x^1=(x_2,...,x_n)$ fixed, inequality (7) in Theorem 2.1 reduces to (3).*

Remark 2.3. *For $\omega(u)=u^p, p\ge 1$, $f=0$, and $x^1=(x_2,...,x_n)$ fixed, inequality (7) in Theorem 2.1 reduces to (4).*

Theorem 2.4. *Let $u(x), f(x), g(x), a(x) \in C(J, \mathbb{R}_+)$ and γ is defined as in Theorem 2.1. Let $\omega, \frac{\omega^{-1}(u)}{u} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing with $\omega(u) > 0$, for $u > 0$, and*

$$G(v) = \int_{v_0}^{v} \frac{u_0 \exp(s) ds}{\omega^{-1}(u_0 \exp(s))},\ v \geq v_0 > 0,\ G(+\infty) = +\infty. \tag{17}$$

If $u(x)$ satisfies

$$\omega\left(u(x)\right) \leq u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(s) \left[u(s) + \int_{\gamma(x^0)}^{s} f(\tau)u(\tau)d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)\omega(u(\tau))d\tau\right] ds, \tag{18}$$

for $x \in I$, where $u_0 \geq 0$ is a constant, then

$$u(x) \leq \omega^{-1}\left\{u_0 \exp\left(G^{-1}\left[G\big(B(x)\big) + \int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[1 + \int_{\gamma(x^0)}^{s} f(\tau)d\tau\right] ds\right]\right)\right\}, \tag{19}$$

for $x \in I$, where

$$B(x) = \int_{\gamma(x^0)}^{\gamma(x)} a(s) \left(\int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right) ds. \tag{20}$$

Proof. Let $u_0 > 0$, and

$$z(x) = u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(s) \left[u(s) + \int_{\gamma(x^0)}^{s} f(\tau)u(\tau)d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)\omega(u(\tau))d\tau\right] ds,$$

where $z(x)$ is a positive and nondecreasing function on J, so we have

$$\begin{cases} u(x) \leq \omega^{-1}\left(z(x)\right) \\ z(x_1^0, x^1) = u_0, \end{cases}$$

and

$$\begin{aligned} D_1...D_n z(x) &\leq \gamma'(x)a(\gamma(x)) \left[\omega^{-1}\left(z(\gamma(x))\right) + \int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\omega^{-1}\left(z(\tau)\right)d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)z(\tau)d\tau\right] \\ &\leq \gamma'(x)a(\gamma(x)) \left[\omega^{-1}\left(z(\gamma(x))\right) + \int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\omega^{-1}\left(z(\tau)\right)d\tau + z(\gamma(x))\int_{\gamma(x^0)}^{\alpha(T)} g(\tau)d\tau\right] \\ &\leq \gamma'(x)a(\gamma(x))z(\gamma(x)) \left[\frac{\omega^{-1}\left(z(\gamma(x))\right)}{z(\gamma(x))} + \int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\frac{\omega^{-1}\left(z(\tau)\right)}{z(\tau)}d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right], \end{aligned}$$

or

$$\frac{D_1...D_n z(x)}{z(x)} \leq \gamma'(x)a(\gamma(x)) \left[\frac{\omega^{-1}\left(z(\gamma(x))\right)}{z(\gamma(x))} + \int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right],$$

then

$$D_n\left(\frac{D_1...D_{n-1}z(x)}{z(x)}\right) \leq \gamma'(x)a(\gamma(x)) \left[\frac{\omega^{-1}\big(z(\gamma(x))\big)}{z(\gamma(x))} + \int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau + \int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right]. \tag{21}$$

Fixing $x_1, ..., x_{n-1}$, setting $x_n = s_n$ then the integration of (21) with respect to s_n from x_n^0 to x_n, gives

$$\frac{D_1...D_{n-1}z(x)}{z(x)} \leq \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a\left(\gamma_1(x_1), ..., \gamma_{n-1}(x_{n-1}), s_n\right) \left[\frac{\omega^{-1}\left(z(\gamma_1(x_1), ..., \gamma_{n-1}(x_{n-1}), s_n)\right)}{z(\gamma_1(x_1), ..., \gamma_{n-1}(x_{n-1}), s_n)}\right.$$

$$+\int_{\gamma(x_1^0)}^{\gamma_1(x_1)}...\int_{\gamma(x_n^0)}^{s_n} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau+\int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\Bigg]\times$$
$$\gamma_1'(x_1)\times...\times\gamma_{n-1}'(x_{n-1})ds_n.$$

After $(n-1)$ steps, we find

$$\begin{aligned}\frac{D_1z(x)}{z(x)} &\le \int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)}...\int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a\left(\gamma_1(x_1),s_2....,s_n\right)\left[\frac{\omega^{-1}\Big(z(\gamma_1(x_1),s_2....,s_n)\Big)}{z(\gamma_1(x_1),s_2....,s_n)}\right.\\ &\left.+\int_{\gamma_1(x_1^0)}^{\gamma_1(x_1)}\int_{\gamma_2(x_2^0)}^{s_2}...\int_{\gamma_n(x_n^0)}^{s_n} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau+\int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right]\gamma_1'(x_1)ds^1. \end{aligned}\tag{22}$$

Keeping $x^1=(x_2,...,x_n)$ fixed in (22), replacing x_1 by s_1, then the integration of (22) with respect to s_1 from x_1^0 to x_1, gives

$$\ln\left(\frac{z(x)}{u_0}\right)\le\int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[\frac{\omega^{-1}\big(z(s)\big)}{z(s)}+\int_{\gamma(x^0)}^{s} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau+\int_{\gamma(x^0)}^{\gamma(T)} g(\tau)d\tau\right]ds.$$

Using (20), and since $B(x)$ is nondecreasing so the last inequality can be can be restated as follows

$$\ln\left(\frac{z(x)}{u_0}\right)\le\int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[\frac{\omega^{-1}\big(z(s)\big)}{z(s)}+\int_{\gamma(x^0)}^{s} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau\right]ds+B(X),$$

for $x\le X\le T$. Define a function $v(x)$ on J by

$$v(x)=\int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[\frac{\omega^{-1}\big(z(s)\big)}{z(s)}+\int_{\gamma(x^0)}^{s} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau\right]ds+B(X),$$

then, $v(x)$ is positive and nondecreasing and

$$z(x)\le u_0\exp\big(v(x)\big),\tag{23}$$

$v(x_1^0,x^1)=B(X)$, and

$$\begin{aligned}D_1...D_nv(x) &= \gamma'(x)a(\gamma(x))\left[\frac{\omega^{-1}\big(z(\gamma(x))\big)}{z(\gamma(x))}+\int_{\gamma(x^0)}^{\gamma(x)} f(\tau)\frac{\omega^{-1}\big(z(\tau)\big)}{z(\tau)}d\tau\right]\\ &\le \gamma'(x)a(\gamma(x))\frac{\omega^{-1}\Big(u_0\exp\big(v(\gamma(x))\big)\Big)}{u_0\exp\big(v(\gamma(x))\big)}\left[1+\int_{\gamma(x^0)}^{\gamma(x)} f(\tau)d\tau\right],\end{aligned}$$

or

$$\frac{u_0\exp\left(v(x)\right)D_1...D_nv(x)}{\omega^{-1}\left(u_0\exp\left(v(x)\right)\right)}\le\gamma'(x)a(\gamma(x))\left[1+\int_{\gamma(x^0)}^{\alpha(x)} f(\tau)d\tau\right].$$

let's integrate the above inequality from x_i^0 to x_i $(i=2,..,n)$, after $(n-1)$ steps, we obtain

$$\frac{u_0\exp\big(v(x)\big)D_1v(x)}{\omega^{-1}\Big(u_0\exp\big(v(x)\big)\Big)}\le\int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)}...\int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a\left(\gamma_1(x_1),s_2....,s_n\right)\left[1+\int_{\gamma_1(x_1^0)}^{\gamma_1(x_1)}\int_{\gamma_2(x_2^0)}^{s_2}...\int_{\gamma_n(x_n^0)}^{s_n} f(\tau)d\tau\right].\tag{24}$$

Keeping $x^1 = (x_2, ..., x_n)$ fixed in (24), replacing x_1 by s_1 and then integrating with respect to s_1 from x_1^0 to x_1, we obtain

$$G(v(x)) \le G(B(X)) + \int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[1 + \int_{\gamma(x^0)}^{s_n} f(\tau)d\tau\right] ds,$$

therfore, we have

$$v(x) \le G^{-1}\left[G(B(X)) + \int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[1 + \int_{\gamma(x^0)}^{s_n} f(\tau)d\tau\right] ds\right].$$

Since X is chosen arbitrarily, and from (23) we have

$$z(x) \le u_0 \exp\left(G^{-1}\left[G(B(x)) + \int_{\gamma(x^0)}^{\gamma(x)} a(s)\left[1 + \int_{\gamma(x^0)}^{s_n} f(\tau)d\tau\right] ds\right]\right). \tag{25}$$

Since $u(x) \le \omega^{-1}(z(x))$, then from (25), yields the result (19).

For $u_0 = 0$, we repeat the same procedure above replacing u_0 by $\varepsilon > 0$ and finally let $\varepsilon \to 0$. □

Remark 2.5. *For $\omega(u) = u^p, p \ge 1, \gamma(x) = x$ and $x^1 = (x_2, ..., x_n)$ fixed, Theorem 2.4 reduces to Theorem 2.6 in [9].*

Theorem 2.6. *Let $u \in C(J, \mathbb{R}_+), a, b \in C(\Delta, \mathbb{R}_+)$ and a, b are nondecreasing in x for each $s \in J$, either γ is defined as in Theorem 2.1. Let $\omega, \frac{\omega(u)}{u} \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\omega(u) > 0$, for $u > 0$, and*

$$F_1(v) = \int_{v_0}^{v} \frac{ds}{\omega(s)}, F_2(v) = \int_{v_0}^{v} \frac{ds}{F_1^{-1}(s)}, v \ge v_0 > 0, F_1(+\infty) = F_2(+\infty) = +\infty, \tag{26}$$

$$H(u) = F_2\Big(F_1(2u - u_0)\Big) - F_2\left(F_1(u) + \int_{\gamma(x^0)}^{\gamma(T)} b(x, s)ds\right), \tag{27}$$

is increasing for $u \ge u_0$. If $u(x)$ satisfies

$$\begin{aligned} u(x) \quad \le \quad & u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(x, s)\omega(u(s))\left[u(s) + \int_{\gamma(x^0)}^{s} b(s, \tau)\omega(u(\tau))d\tau\right] ds \\ & + \int_{\gamma(x^0)}^{\gamma(T)} a(x, s)\omega(u(s))\left[u(s) + \int_{\gamma(x^0)}^{s} b(s, \tau)\omega(u(\tau))d\tau\right] ds, \end{aligned} \tag{28}$$

for $x \in I$, where $u_0 \ge 0$ is a constant, then

$$u(x) \le F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(H_1^{-1}\left(\int_{\gamma(x^0)}^{\gamma(T)} a(x, s)ds\right)\right) + \int_{\gamma(x^0)}^{\gamma(x)} b(x, s)ds\right) + \int_{\gamma(x^0)}^{\gamma(x)} a(x, s)ds\right)\right], \tag{29}$$

for $x \in I$, where F_1^{-1}, F_2^{-1} and H^{-1} are the inverse functions of F_1, F_2 and H, respectively.

Proof. Let $u_0 > 0$, $X = (X_1, ..., X_n) \in I$ fixed, and for $x^0 \le x \le X \le T$, we define a function $z(x)$ by

$$z(x) \quad = \quad u_0 + \int_{\gamma(x^0)}^{\gamma(x)} a(X, s)\omega(u(s))\left[u(s) + \int_{\gamma(x^0)}^{s} b(s, \tau)\omega(u(\tau))d\tau\right] ds$$

$$+\int_{\gamma(x^0)}^{\gamma(T)} a(X,s)\omega(u(s))\left[u(s)+\int_{\gamma(x^0)}^{s} b(s,\tau)\omega(u(\tau))d\tau\right]ds,$$

where $z(x)$ is positive and nondecreasing, then

$$u(x) \le z(x), \tag{30}$$

and

$$\begin{aligned} D_1...D_n z(x) &\le \gamma'(x)a(X,\gamma(x))\omega(z(\gamma(x))\left[z(\gamma(x))+\int_{\gamma(x^0)}^{\gamma(x)} b(\gamma(x),\tau)\omega(z(\tau))d\tau\right] \\ &\le \gamma'(x)a(X,\gamma(x))\omega(z(\gamma(x)))\left[z(x)+\int_{\gamma(x^0)}^{\gamma(x)} b(x,\tau)\omega(z(\tau))d\tau\right] \\ &\le \gamma'(x)a(X,\gamma(x))\omega(z(\gamma(x)))\left[z(x)+\int_{\gamma(x^0)}^{\gamma(x)} b(X,\tau)\omega(z(\tau))d\tau\right] \\ &\le \gamma'(x)a(X,\gamma(x))\omega(z(\gamma(x)))z_1(x), \end{aligned} \tag{31}$$

where

$$z_1(x) = z(x) + \int_{\gamma(x^0)}^{\gamma(x)} b(X,\tau)\omega(z(\tau))d\tau.$$

Hence,

$$z_1(x^0) = z(x^0), \qquad z(x) \le z_1(x). \tag{32}$$

Differentiating $z_1(x)$ and using (31) and (32) we get

$$D_1...D_n z_1(x) \le \gamma'(x)a(X,\gamma(x))\omega(z_1(\gamma(x)))z_1(x) + \gamma'(x)b(X,\gamma(x))\omega(z_1(\gamma(x))).$$

So

$$\frac{D_1...D_n z_1(x)}{z_1(x)} \le \gamma'(x)\left[a(X,\gamma(x))\omega(z_1(\gamma(x))) + b(X,\gamma(x))\frac{\omega(z_1(\gamma(x)))}{z_1(x)}\right],$$

then

$$\begin{aligned} D_n\left(\frac{D_1...D_{n-1}z_1(x)}{z_1(x)}\right) &\le \gamma_1'(x_1)\times ... \times \gamma_n'(x_n)\left[a(X,\gamma_1(x_1),...,\gamma_n(x_n))\omega(z_1(\gamma_1(x_1),...,\gamma_n(x_n)))\right. \\ &\quad \left. + b(X,\gamma_1(x_1),...,\gamma_n(x_n))\frac{\omega(z_1(\gamma_1(x_1),...,\gamma_n(x_n)))}{z_1(x_1,...,x_n)}\right]. \end{aligned} \tag{33}$$

Fixing $x_1, ..., x_{n-1}$, setting $x_n = s_n$ and the integration of (33) with respect to s_n from x_n^0 to x_n, gives

$$\begin{aligned} \frac{D_1...D_{n-1}z_1(x)}{z_1(x)} &\le \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} \left[a(X,\gamma_1(x_1),....,\gamma_{n-1}(x_{n-1}),s_n)\omega\left(z_1\left(\gamma_1(x_1),....,\gamma_{n-1}(x_{n-1}),s_n\right)\right)\right. \\ &\quad \left. + b(X,\gamma_1(x_1),....,\gamma_{n-1}(x_{n-1}),s_n)\frac{\omega(z_1(\gamma_1(x_1),....,\gamma_{n-1}(x_{n-1}),s_n))}{z_1(x_1,...,x_{n-1},s_n)}\right]\times \\ &\quad \gamma_1'(x_1)\times ... \times \gamma_{n-1}'(x_{n-1})ds_n. \end{aligned}$$

Using the same method above, we obtain (after $n-1$ steps)

$$\frac{D_1 z_1(x)}{z_1(x)} \le \int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)} ... \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} \left[a(X,\gamma_1(x_1),s_2,...,s_n)\omega\left(z_1\left(\gamma_1(x_1),s_2,...,s_n\right)\right)\right.$$

$$+ b(X,\gamma_1(x_1),s_2,...,s_n)\frac{\omega(z_1(\gamma_1(x_1),s_2,...,s_n))}{z_1(x_1,s_2,...,s_n)}\Bigg]\gamma_1'(x_1)ds^1. \tag{34}$$

Keeping $x^1=(x_2,...,x_n)$ fixed in (34), replacing x_1 by s_1 then the integration with respect to s_1 from x_1^0 to x_1, gives

$$\ln\left(z_1(x)\right)\le \ln\left(z_1(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)}\left[a(X,s)\omega\left(z_1\left(s\right)\right)+b(X,s)\frac{\omega(z_1(s))}{z_1(s)}\right]ds. \tag{35}$$

Define a positive and nondecreasing function $z_2(x)$ by the right-hand side of (35), then

$$z_2(x_1^0,x^1)=\ln\left(z_1(x_1^0,x^1)\right),\quad z_1(x)\le \exp\left(z_2(x)\right). \tag{36}$$

Differentiating $z_2(x)$ and using (36) we have

$$\begin{aligned} D_1...D_nz_2(x) &= \gamma'(x)\left[a(X,\gamma(x))\omega\left(z_1\left(\gamma(x)\right)\right)+b(X,\gamma(x))\frac{\omega(z_1(\gamma(x)))}{z_1(\gamma(x))}\right]\\ &\le \gamma'(x)\left[a(X,\gamma(x))\omega\Big(\exp\left(z_2(\gamma(x))\right)\Big)+b(X,\gamma(x))\frac{\omega\Big(\exp\left(z_2(\gamma(x))\right)\Big)}{\exp\left(z_2(\gamma(x))\right)}\right]\\ &\le \gamma'(x)\frac{\omega\Big(\exp(z_2(x))\Big)}{\exp\left(z_2(x)\right)}\Big[a(X,\gamma(x))\exp\left(z_2(\gamma(x))\right)+b(X,\gamma(x))\Big], \end{aligned} \tag{37}$$

since ω, $\frac{\omega(z)}{z}$ are nondecreasing functions. From (37), we have

$$\frac{\exp\left(z_2(x)\right)D_1...D_nz_2(x)}{\omega\left(\exp\left(z_2(x)\right)\right)}\le \gamma'(x)\Big[a(X,\gamma(x))\exp\left(z_2(\gamma(x))\right)+b(X,\gamma(x))\Big],$$

then

$$\begin{aligned} D_n\left(\frac{\exp\left(z_2(x)\right)D_1...D_{n-1}z_2(x)}{\omega\Big(\exp(z_2(x))big)}\right) &\le \Big[a(X,\gamma_1(x_1),...,\gamma_n(x_n))\exp\left(z_2(\gamma(x))\right)+b(X,\gamma_1(x_1),...,\gamma_n(x_n))\Big]\\ &\quad \gamma_1'(x_1)\times...\times\gamma_n'(x_n), \end{aligned}$$

Fixing $x_1,...,x_{n-1}$, setting $x_n=s_n$ then the integration of the above inequality from x_n^0 to x_n, gives

$$\begin{aligned} \frac{\exp\left(z_2(x)\right)D_1...D_{n-1}z_2(x)}{\omega\left(\exp(z_2(x))\right)} &\le \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)}\Big[a(X,\gamma_1(x_1),...,\gamma_{n-1}(x_{n-1}),s_n)\exp z_2(x)+\\ &\quad b\left(X,\gamma_1(x_1),...,\gamma_{n-1}(x_{n-1}),s_n\right)\Big]\gamma_1'(x_1)\times...\times\gamma_{n-1}'(x_{n-1})ds_n. \end{aligned}$$

After $n-1$ steps, we get

$$\begin{aligned} \frac{\exp z_2(x)D_1z_2(x)}{\omega\left(\exp z_2(x)\right)} &\le \int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)}...\int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)}\Big[a(X,\gamma_1(x_1),s_2,....,s_n)\exp z_2(\gamma(x))+b(X,\gamma_1(x_1),s_2,....,s_n)\Big]\\ &\quad \times\gamma_1'(x_1)ds^1. \end{aligned}$$

Keeping $x^1=(x_2,...,x_n)$ fixed in the above inequality, replacing x_1 by s_1, integrating with respect to s_1 from x_1^0 to x_1 and from (26), we get

$$F_1\left(\exp z_2(x)\right) \le F_1\left(\exp z_2(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)}b(X,s)ds+\int_{\gamma(x^0)}^{\gamma(x)}a(X,s)\exp z_2(s)ds$$

$$\leq \quad F_1\left(\exp z_2(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(X_1)} b(X,s)ds+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)\exp z_2(s)ds, \qquad (38)$$

for $x^0 \leq x \leq X_1 \leq T$, where X_1 is arbitrary.

Define a positive and nondecreasing function $z_3(x)$ by the right-hand side of (38). Then

$$z_3(x_1^0,x^1)=F_1\Big(\exp\left(z_2(x_1^0,x^1)\right)\Big)+\int_{\gamma(x^0)}^{\gamma(X_1)} b(X,s)ds,\ \exp\left(z_2(x)\right)\leq F_1^{-1}\left(z_3(x)\right). \qquad (39)$$

We know that differentiating $z_3(x)$ and using (39), we deduce that

$$\begin{aligned} D_1...D_n z_3(x) &= \gamma'(x)a(X,\gamma(x))\exp\left(z_2(\gamma(x))\right) \\ &\leq \gamma'(x)a(X,\gamma(x))\exp\left(z_2(x)\right) \\ &\leq \gamma'(x)a(X,\gamma(x))F_1^{-1}\left(z_3(x)\right), \end{aligned}$$

then

$$\frac{D_1...D_n z_3(x)}{F_1^{-1}\left\{z_3(x)\right\}} \leq \gamma'(x)a(X,\gamma(x)),$$

and

$$D_n\left(\frac{D_1...D_{n-1}z_3(x)}{F_1^{-1}\left\{z_3(x)\right\}}\right) \leq \gamma_1'(x_1)\times...\times\gamma_n'(x_n)a(X,\gamma_1(x_1),...,\gamma_n(x_n)).$$

The integration of the above inequality from x_n^0 to x_n, gives

$$\frac{D_1...D_{n-1}z_3(x)}{F_1^{-1}\left\{z_3(x)\right\}} \leq \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a(X,\gamma_1(x_1),...,\gamma_{n-1}(x_{n-1}),s_n)\gamma_1'(x_1)\times...\times\gamma_{n-1}'(x_{n-1})ds_n.$$

After $(n-1)$ steps, we obtain

$$\frac{D_1 z_3(x)}{F_1^{-1}\left\{z_3(x)\right\}} \leq \int_{\gamma_2(x_2^0)}^{\gamma_2(x_2)} ... \int_{\gamma_n(x_n^0)}^{\gamma_n(x_n)} a(X,\gamma_1(x_1),s_2,....,s_n)\gamma_1'(x_1)ds^1.$$

Keeping $x^1=(x_2,...,x_n)$ fixed in the above inequality, replacing x_1 by s_1, integrating with respect to s_1 from x_1^0 to x_1, and using (26), we obtain

$$F_2\left(z_3(x)\right) \leq F_2\left(z_3(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)ds,$$

then

$$z_3(x) \leq F_2^{-1}\left(F_2\left(z_3(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)ds\right). \qquad (40)$$

Then from (30), (32), (36), (39) and (40) we have

$$\begin{aligned} u(x) &\leq F_1^{-1}\left[F_2^{-1}\left(F_2\left(z_3(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)ds\right)\right] \\ &= F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(\exp\left(z_2(x_1^0,x^1)\right)\right)+\int_{\gamma(x^0)}^{\gamma(X_1)} b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)ds\right)\right] \\ &= F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(z_1(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(X_1)} b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(x)} a(X,s)ds\right)\right] \end{aligned}$$

$$= \quad F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(X_1)}b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(x)}a(X,s)ds\right)\right].$$

Since $X_1 \in J$ is arbitrary, so we get

$$u(x) \leq F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(x)}b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(x)}a(X,s)ds\right)\right]. \tag{41}$$

Since

$$2z(x_1^0,x^1)-u_0 = u_0 + 2\int_{\gamma(x^0)}^{\gamma(T)}a(X,s)\omega(u(s))\left[u(s)+\int_{\gamma(x^0)}^{s}b(s,\tau)\omega(u(\tau))d\tau\right]ds = z(T),$$

and from (41), we have

$$2z(x_1^0,x^1)-u_0 = z(T) \leq F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(T)}b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(T)}a(X,s)ds\right)\right].$$

Hence

$$F_1\left(2z(x_1^0,x^1)-u_0\right) \leq F_2^{-1}\left(F_2\left(F_1\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(T)}b(X,s)ds\right)+\int_{\gamma(x^0)}^{\gamma(T)}a(X,s)ds\right),$$

so

$$F_2\left(F_1\left(2z(x_1^0,x^1)-u_0\right)\right)-F_2\left(F_1\left(z(x_1^0,x^1)\right)+\int_{\gamma(x^0)}^{\gamma(T)}b(X,s)ds\right) \leq \int_{\gamma(x^0)}^{\gamma(T)}a(X,s)ds,$$

and

$$H\left(z(x_1^0,x^1)\right) \leq \int_{\gamma(x^0)}^{\gamma(T)}a(X,s)ds.$$

Since H is increasing, we have

$$z(x_1^0,x^1) \leq H^{-1}\left(\int_{\gamma(x^0)}^{\gamma(T)}a(X,s)ds\right) \tag{42}$$

Since $X \in J$ is chosen arbitrary, so substituting (42) into (41), we get the estimate (29).

For $u_0 = 0$, we repeat the same procedure above replacing u_0 by $\varepsilon > 0$ and finally let $\varepsilon \to 0$. □

Remark 2.7. *For $b = 0$, $\omega(u) = 11$ and $x^2 = (x_3, ..., x_n)$ fixed, inequality (28) in Theorem 2.6 reduces to inequality (1).*

Remark 2.8. *When the known function $\sigma_1(s,t)$ in (2) is replaced by $a(x,s)\omega(u(s))$, and for $x^2 = (x_3, ..., x_n)$ fixed, the bound for $u(x)$ in (28) reduces to (2).*

3. Application

This section suggests an application of our results to study the boundedness of the solutions of certain multidimensional Volterra-Fredholm integral equations with delay of the form

$$\begin{aligned}\chi(x) &= \chi_0 + \int_{x^0}^{x} A\left[x, s, \chi\left(s-\lambda(s)\right), \int_{x^0}^{s} B\left(s,\tau,\chi\left(\tau-\lambda(\tau)\right)\right)d\tau\right]ds \\ &\quad + \int_{x^0}^{T} A\left[x, s, \chi\left(s-\lambda(s)\right), \int_{x^0}^{s} B\left(s,\tau,\chi\left(\tau-\lambda(\tau)\right)\right)d\tau\right]ds, \end{aligned} \tag{43}$$

where $\chi \in C(J,\mathbb{R})$, $A \in C(\Delta \times \mathbb{R}^2,\mathbb{R}), B \in C(\Delta \times \mathbb{R},\mathbb{R}), J = \left[x^0, T\right] \subset \mathbb{R}^n$, $\Delta = \left\{(x,s) \in J^2 : x^0 \le s \le x \le T\right\} \subset \mathbb{R}^n$ and $\lambda \in C^1(J,J)$ is nondecreasing on J such that $\lambda(x) = \left(\lambda_1(x_1), ..., \lambda_n(x_n)\right), x_i - \lambda_i(x_i) \ge 0, \lambda_i'(x_i) < 1$, and $\lambda_i(x_i^0) = 0$, for $i = 1, ..., n, x = (x_1, ..., x_n), x^0 = \left(x_1^0, ..., x_n^0\right) \in \mathbb{R}^n$.

In the following, we give the estimate of the unknown function χ in the multidimensional Volterra-Fredholm integral equation (43).

Theorem 3.1. *Suppose that the functions A, B in (43) satisfy the conditions*

$$|A(x,s,z,y)| \le a(x,s)\omega\left(|z|\right)\left[\,|z| + |y|\,\right], \tag{44}$$

$$|B(s,\tau,z)| \le b(s,\tau)\omega\left(|z|\right), \tag{45}$$

where a, b are as in Theorem 2.6.
Let $M = M_1 \times ... \times M_n$, where

$$M_i = \max_{x_i \in I_i} \frac{1}{1-\beta_i'(x_i)}, i = 1, ..., n \tag{46}$$

and $\gamma(x) = x - \lambda(x) \in C^1(J,J)$ is increasing on J, and $\gamma(x) \le x$. Assume that the function

$$H^*(u) = F_2\big(F_1\left(2u - |\chi_0|\right)\big) - F_2\left(F_1(u) + \int_{\gamma(x^0)}^{\gamma(T)} M\, b(x,\gamma^{-1}(s))ds\right), \tag{47}$$

is increasing for $u \ge |\chi_0|$. If χ is a solution of (43) on J, then

$$\begin{aligned}|\chi(x)| &\le F_1^{-1}\left[F_2^{-1}\left(F_2\left(F_1\left(H^{*-1}\left(\int_{\gamma(x^0)}^{\gamma(T)} M\, a(x,\gamma^{-1}(s))ds\right)\right) + \int_{\gamma(x^0)}^{\gamma(x)} M\, b(x,\gamma^{-1}(s))ds\right)\right.\right. \\ &\quad \left.\left. + \int_{\gamma(x^0)}^{\gamma(x)} M\, a(x,\gamma^{-1}(s))ds\right)\right], \end{aligned} \tag{48}$$

where F_1, F_2, F_1^{-1} and F_2^{-1} are as in theorem 2.6.

Proof. From the conditions (44), (45), and the equation (43), we can obtain the inequality

$$|\chi(x)| \le |\chi_0| + \int_{x^0}^{x} a(x,s)\omega\big(|\chi(s-\lambda(s))|\big)\left[|\chi(s-\lambda(s))| + \int_{x^0}^{s} b(s,\tau)\omega\left(|\chi\left(\tau-\lambda(\tau)\right)|\right)d\tau\right]ds$$

$$+\int_{x^0}^{T} a(x,s)\omega\big(|\chi(s-\lambda(s))|\big)\left[|\chi(s-\lambda(s))|+\int_{x^0}^{s} b(s,\tau)\omega\left(|\chi\left(\tau-\lambda(\tau)\right)|\right)d\tau\right]ds,$$

using the change of variables $\gamma(x)=x-\lambda(x)$, and (46), then the last inequality can be restated as follows

$$\begin{aligned}|\chi(x)| &\le |\chi_0|+\int_{x^0}^{x} a(x,s)\omega\big(|\chi(\gamma(s))|\big)\left[|\chi(\gamma(s))|+\int_{x^0}^{s} b(s,\tau)\omega\big(|\chi\left(\gamma(\tau)\right)|\big)d\tau\right]ds\\ &\quad+\int_{x^0}^{T} a(x,s)\omega\big(|\chi(\gamma(s))|\big)\left[|\chi(\gamma(s))|+\int_{x^0}^{s} b(s,\tau)\omega\big(|\chi\left(\gamma(\tau)\right)|\big)d\tau\right]ds\\ &\le |\chi_0|+\int_{\gamma(x^0)}^{\gamma(x)} M\,a\left(x,\gamma^{-1}(s)\right)\omega\big(|\chi(s)|\big)\left[|\chi(s)|+\int_{x^0}^{s} M\,b(s,\gamma^{-1}(\tau))\omega\big(|\chi\left(\tau\right)|\big)d\tau\right]ds\\ &\quad+\int_{\gamma(x^0)}^{\gamma(T)} M\,a\left(x,\gamma^{-1}(s)\right)\omega\big(|\chi(s)|\big)\left[|\chi(s)|+\int_{x^0}^{s} M\,b(s,\gamma^{-1}(\tau))\omega\big(|\chi\left(\tau\right)|\big)d\tau\right]ds, \qquad (49)\end{aligned}$$

for $x\in J$. Now, we can obtain the bound on the solution $\chi(x)$ given in (48) by applying Theorem 2.6 to (49). □

Conclusion

In this paper, we established some new multidimensional retarded integral inequalities of Volterra-Fredholm type in Theorem 2.1, Theorem 2.4, Theorem 2.6, which generalize some results given in [6], [9], [12], [13]. Using novel analysis techniques, the bounds of the unknown functions are given explicitly. These results can be used in the analysis of the qualitative properties to solutions of Volterra-Fredholm integral equations in n independent variables. An application of our results is given to study the boundedness of the solutions of some multidimensional Volterra-Fredholm integral equations with delay.

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this paper.

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