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$D-$DIVISIBILITY OF ALMOST DISTRIBUTIVE LATTICES

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Abstract. The notion of divisibility is introduced in an Almost Distributive Lattice(ADL) with respect to a filter and it is proved that the set of all multipliers of an element is a filter. A congruence is defined on an ADL with respect to these multiplier filters and established a set of equivalent conditions for the corresponding quotient lattice to become a Boolean algebra. The concepts of $D-$prime elements and $D-$irreducible elements are introduced and characterized in terms of corresponding multiplier filters.

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1. Introduction

The concept of an Almost Distributive Lattice(ADL) was introduced by Swamy U.M., and Rao G.C., [9] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set of all principal ideals of an ADL forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. In [7], the notion of divisibility was introduced and investigated their properties. In [1], the concept of divisibility was introduced in distributive lattices with respect to dense set. The notion of $D-$filters in ADLs was introduced and studied their properties in [8] by Rafi, et.al. In this paper, the concept of divisibility is introduced with respect to dense elements

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in an ADL and obtained that the set of all multipliers of an element forms a filter. Later it is proved that the class of all these filters forms a complete Boolean algebra. The notions of $D-$prime elements and $D-$irreducible elements are also introduced in an ADL and established a relation between these elements and the corresponding filters formed by their multipliers. Finally, it is proved that every $D-$irreducible element is a $D-$prime element.

2. Preliminaries

In this section, we recall certain definitions and important results from [3] and [9], those will be required in the text of the paper.

Definition 2.1. [9] An algebra $R = (R, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions:

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$
(6) $0 \wedge a = 0$
(7) $a \vee 0 = a$, for all $a, b, c \in R$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

Theorem 2.3. [9] *If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in R$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$
(2) $a \vee b = b \Leftrightarrow a \wedge b = a$
(3) $\wedge$ *is associative in* R
(4) $a \wedge b \wedge c = b \wedge a \wedge c$
(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(6) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(7) $a \wedge (a \vee b) = a,\ (a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(8) $a \wedge a = a$ *and* $a \vee a = a$.

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL R a distributive lattice.

As usual, an element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$. The set of all maximal elements of R is denoted by $MaxR$.

As in distributive lattices [2,6], a non-empty subset I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$. Also, a non-empty subset F of R is said to be a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in R$.

The set $\mathfrak{I}(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in \mathfrak{I}(R)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal(filter) P of R is called a prime ideal(filter) if, for any $x, y \in R$, $x \wedge y \in P (x \vee y \in P) \Rightarrow x \in P$ or $y \in P$. A proper ideal(filter) M of R is said to be maximal if it is not properly contained in any proper ideal(filter) of R. It can be observed that every maximal ideal(filter) of R is a prime ideal(filter). Every proper ideal(filter) of R is contained in a maximal ideal(filter). For any subset S of R the smallest ideal containing S is given by $(S] := \{(\bigvee\limits_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in R$ and $n \in \mathbb{N}\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$ and such an ideal is called the principal ideal of R. Similarly, for any $S \subseteq R$, $[S) := \{x \vee (\bigwedge\limits_{i=1}^{n} s_i) \mid s_i \in S, x \in R$ and $n \in \mathbb{N}\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$ and such a filter is called the principal filter of R.

For any $a, b \in R$, it can be verified that $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$. Hence the set $(\mathfrak{I}^{PI}(R), \vee, \cap)$ of all principal ideals of R is a sublattice of the distributive lattice $(\mathfrak{I}(R), \vee, \cap)$ of all ideals of R. Also, we have that the set $(\mathfrak{F}(R), \vee, \cap)$ of all filters of R is a bounded distributive lattice.

Theorem 2.4. [5] *Let R be an ADL with maximal elements. Then P is a prime ideal of R if and only if $R \setminus P$ is a prime filter of R.*

Definition 2.5. [4] For any nonempty subset A of an ADL R, define $A^* = \{\, x \in R \mid a \wedge x = 0$ for all $a \in A\,\}$. Here A^* is called the annihilator of A in R.

For any $a \in R$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a. An element a of an ADL R is called dense element if $(a]^* = \{0\}$ and the set D of all dense elements in ADL is a filter if D is non-empty.

Definition 2.6. [8] Let D be a dense set of an ADL R with maximal elements. For any non-empty subset of S of R, define $S^D = \{a \in R \mid s \vee a \in D$, for all $s \in S\}$.

Lemma 2.7. [8] *Let S, T ba any non-empty subset of R. Then we have the following:*

(1) $R^D = D$;

(2) $D^D = R$;

(3) $D \subseteq S^D$;

(4) $(MaxR)^D = R$;

(5) $S \subseteq T \Rightarrow T^D \subseteq S^D$;

(6) $S \subseteq S^{DD}$;

(7) $S^{DDD} = S^D$;

(8) $S^D = R \Leftrightarrow S \subseteq D$;

(9) S^D *is a filter of* R *containing* D.

Theorem 2.8. [8] *Let S, T be any two filters of R. Then we have the following:*

(1) $S^D \cap S^{DD} = D$;

(2) $(S \vee T)^D = S^D \cap T^D$;

(3) $(S \cap T)^{DD} = S^{DD} \cap T^{DD}$.

For any $x \in R$, we simply represents $(\{x\})^D$ by $(x)^D$. Clearly, we have that $(0)^D = MaxR$ and $(m)^D = R$, where $m \in MaxR$.

Lemma 2.9. [8] *Let m be any maximal element of R. For any $x, y \in R$, we have the following:*

(1) $(0)^D = D$;

(2) $(m)^D = R$;

(3) $[x)^D = (x)^D$;

(4) $x \leq y \Rightarrow (x)^D \subseteq (y)^D$;

(5) $(x \wedge y)^D = (x)^D \cap (y)^D$;

(6) $(x \vee y)^{DD} = (x)^{DD} \cap (y)^{DD}$;

(7) $(x)^D = R \Leftrightarrow x \in D$;

(8) $(x \wedge y)^D = (y \wedge x)^D, (x \vee y)^D = (y \vee x)^D$;

(9) $(x)^D = (y)^D \Rightarrow (x \wedge a)^D = (y \wedge a)^D; (x \vee a)^D = (y \vee a)^D$, *for all* $a \in R$.

Definition 2.10 ([3]). An equivalence relation θ on an ADL R is called a congruence relation on R if $(a \wedge c, b \wedge \,), (a \vee c, b \vee \,) \in \theta$, for all $(a, b), (c, \,) \in \theta$.

Definition 2.11 ([3]). For any congruence relation θ on an ADL R and $a \in R$, we define $[a]_\theta = \{b \in R \mid (a, b) \in \theta\}$ and it is called the congruence class containing a.

Theorem 2.12 ([3]). *An equivalence relation θ on an ADL R is a congruence relation if and only if for any $(a, b) \in \theta,\ x \in R, (a \vee x, b \vee x), (x \vee a, x \vee b), (a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ are all in θ.*

3. $D-$Divisibility on ADLs

The notion of divisibility is introduced with respect to dense elements in an ADL and obtained that the set of all multipliers of an element forms a filter. Later it is proved that the class of all these filters forms a complete Boolean algebra. The notions of $D-$prime elements and $D-$irreducible elements are also introduced in an ADL and established a relation between these elements and the corresponding filters formed by their multipliers. Finally, it is proved that every $D-$irreducible element is a $D-$prime element.

Definition 3.1. An element x of an ADL R is said to be $D-$divisor of an element y of R if $(y)^D = (x \vee t)^D$, for some $t \in R$. In this case we write it as $(x|y)_D$.

Lemma 3.2. *For any $x, y, z \in R$, we have the following:*
(1) $(x|x)_D$;
(2) If $x \leq y$ then $(x|y)_D$;
(3) If $(x)^D = (y)^D$ then $(x|y)_D$ and $(y|x)_D$;
(4) If $(x|y)_D$ and $(y|z)_D$ then $(x|z)_D$;
(5) If $(x|y)_D$ then $(x|(y \vee t))_D$ for all $t \in R$;
(6) If $(x|y)_D$ then $((x \vee t)|(y \vee t))_D$ and $((x \wedge t)|(y \wedge t))_D$ for all $t \in R$.

Proof. (1) Since $(x)^D = (x \vee x)^D$, we get $(x|x)_D$.
(2) Assume $x \leq y$. Then $y = x \vee y$ and hence $(y)^D = (x \vee y)^D = (y \vee x)^D$. Therefore $(x|y)_D$.
(3) Assume $(x)^D = (y)^D$. Then $(x)^D = (y \vee y)^D$ and $(y)^D = (x \vee x)^D$. Therefore $(y|x)_D$ and $(x|y)_D$.
(4) Assume $(x|y)_D$ and $(y|z)_D$. Then $(y)^D = (x \vee s)^D$ for some $s \in R$ and $(z)^D = (y \vee t)^D$, for some $t \in R$. Now $(z)^{DD} = (y \vee t)^{DD} = (y)^{DD} \cap (t)^{DD} = (x \vee s)^{DD} \cap (t)^{DD} = (x)^{DD} \cap (s)^{DD} \cap (t)^{DD} = (x \vee s \vee t)^{DD}$. Hence $(z)^D = (x \vee s \vee t)^D$. Therefore $(x|z)_D$.
(5) Assume $(x|y)_D$. Then $(y)^D = (x \vee s)^D$, for some $s \in R$. Let $t \in R$. Now $(y \vee t)^{DD} = (y)^{DD} \cap (t)^{DD} = (x \vee s)^{DD} \cap t^{DD} = (x \vee s \vee t)^{DD}$. Therefore $(y \vee t)^D = (x \vee s \vee t)^D$ and hence $(x|y \vee t)_D$.
(6) Assume $(x|y)_D$. Then $(y)^D = (x \vee s)^D$, for some $s \in R$. Let $t \in R$. Now $(y \vee t)^{DD} = (y)^{DDD} \cap (t)^{DD} = (x \vee s)^{DD} \cap (t)^{DD} = (x \vee s \vee t)^{DD} = (x \vee t \vee s)^{DD}$. Therefore $(y \vee t)^D = (x \vee t \vee s)^D$ and hence $(x \vee t|y \vee t)_D$. Now $(y \wedge t)^D = (y)^D \cap (t)^D = (x \vee s)^D \cap (t)^D = ((x \vee s) \wedge t)^D = ((x \wedge t) \vee (s \wedge t))^D$. Therefore $((x \wedge t)|(y \wedge t))_D$. □

Definition 3.3. For any $x \in R$, define $(x)^\perp = \{t \in R \mid (x|t)_D\}$

Lemma 3.4. *For any $x \in R$, $(x)^\perp$ is a filter of R containing D.*

Proof. Since $(x|x)_D$, for all $x \in R$, we get $(x)^\perp \neq \phi$. Let $s, t \in (x)^\perp$. Then $(x|s)_D$ and $(x|t)_D$. That implies $(s)^D = (x \vee a)^D, (t)^D = (x \vee b)^D$, for some $a, b \in R$. Now $(s \wedge t)^D = (s)^D \cap (t)^D = (x \vee a)^D \cap (x \vee b)^D =$

$((x \vee a) \wedge (x \vee b))^D = (x \vee (a \wedge b))^D$. Therefore $(x|s \wedge t)_D$ and hence $s \wedge t \in (x)^\perp$. Let $s \in (x)^\perp$. Then $(x|s)_D$. That implies $(s)^D = (x \vee a)^D$, for some $a \in R$. Let $r \in R$. Now $(r \vee s)^{DD} = (r)^{DD} \cap (s)^{DD} = (r)^{DD} \cap (x \vee a)^{DD} = (r \vee x \vee a)^{DD} = (x \vee r \vee a)^{DD}$. Therefore $(r \vee s)^D = (x \vee r \vee a)^D$ and hence $(x|r \vee s)_D$. Which follows $r \vee s \in (x)^\perp$. Thus $(x)^\perp$ is a filter of R. Let $s \in D$. Then $(s)^D = R$. Let $t \in D$. Then $x \vee t \in D$, for all $x \in R$ and which leads to $(x \vee t)^D = R$. Therefore $(s)^D = (x \vee t)^D$. That implies $(x|s)^D$. Therefore $s \in (x)^\perp$. Hence $D \subseteq (x)^\perp$. □

Lemma 3.5. *Let $x, y, z \in R$. Then we have the following:*

(1) $x \in (x)^\perp$;

(2) $x \in (y)^\perp \Rightarrow (x)^\perp \subseteq (y)^\perp$;

(3) $x \leq y \Rightarrow (y)^\perp \subseteq (x)^\perp$;

(4) $(x)^D = (y)^D \Rightarrow (x)^\perp = (y)^\perp$;

(5) $(x)^\perp \cap (y)^\perp = (x \vee y)^\perp$;

(6) $(x \vee y)^\perp = (y \vee x)^\perp$ and $(x \wedge y)^\perp = (y \wedge x)^\perp$;

(7) $(x)^\perp = (y)^\perp \Rightarrow (x \vee t)^\perp = (y \vee t)^\perp$ and $(x \wedge t)^\perp = (y \wedge t)^\perp$, for all $t \in R$.

Proof. (1) Since $(x|x)_D$, we get $x \in (x)^\perp$.

(2) Let $x \in (y)^\perp$. Then $(y|x)_D$. Let $t \in (x)^\perp$. Then $(x|t)_D$. Since $(y|x)_D$, we get $(y|t)_D$ and hence $t \in (y)^\perp$. Therefore $(x)^\perp \subseteq (y)^\perp$.

(3) Assume $x \leq y$. Then $y = x \vee y$. Let $t \in (y)^\perp$. Then $(y|t)_D$. That implies $(t)^D = (y \vee s)^D$, for some $s \in R$. Therefore $(t)^D = (x \vee y \vee s)^D$ and hence $(x|t)_D$. Which follows $t \in (x)^\perp$. Thus $(y)^\perp \subseteq (x)^\perp$.

(4) Assume $(x)^D = (y)^D$. Let $t \in (x)^\perp$. Then $(x|t)_D$. That implies $(t)^D = (x \vee s)^D$, for some $s \in R$. Now $(t)^D = (x \vee s)^{DDD} = ((x)^{DD} \cap (s)^{DD})^D = ((y)^{DD} \cap (s)^{DD})^D = (y \vee s)^{DDD} = (y \vee s)^D$. Therefore $(y|t)_D$ and hence $t \in (y)^\perp$. Thus $(x)^\perp \subseteq (y)^\perp$. Similarly, we get $(y)^\perp \subseteq (x)^\perp$. Hence $(x)^\perp = (y)^\perp$.

(5) Clearly, we have $(x \vee y)^\perp \subseteq (x)^\perp \cap (y)^\perp$. Let $t \in (x)^\perp \cap (y)^\perp$. Then $(x|t)_D$ and $(y|t)_D$. That implies $(t)^D = (x \vee a)_D$, for some $a \in R, (t)^D = (y \vee b)^D$, for some $b \in R$. Now $(x \vee y \vee a \vee b)^{DD} = (x \vee a)^{DD} \cap (y \vee b)^{DD} = (t)^{DD} \cap (t)^{DD} = (t)^{DD}$. It follows $((x \vee y)|t)_D$. That implies $t \in (x \vee y)^\perp$. Therefore $(x)^\perp \cap (y)^\perp \subseteq (x \vee y)^\perp$. Hence $(x \vee y)^\perp = (x)^\perp \cap (y)^\perp$.

(6) and (7) are clear. □

Theorem 3.6. *Let $x \in R$. Then $(x)^D = D$ if and only if $(x)^\perp = R$.*

Proof. Assume $(x)^D = D$. Then $(x)^D = (0)^D$. That implies $(x)^\perp = (0)^\perp$. Since $x = x \vee 0$, for all $x \in R$, we get $(x)^D = (x \vee 0)^D$ and hence $(0|x)_D$ for all $x \in R$. Therefore $x \in (0)^\perp$, for all $x \in R$. Thus $(0)^\perp = R$. Since $(x)^\perp = (0)^\perp$, we get $(x)^\perp = R$. Conversely, assume that $(x)^\perp = R$. Then $0 \in (x)^\perp$. Then $(x|0)_D$. That implies $(0)^D = (x \vee t)^D$, for some $t \in R$. Now $(x)^D = (x \vee 0)^D = (x \vee 0)^{DDD} = ((x)^{DD} \cap (0)^{DD})^D = ((x)^{DD} \cap (x \vee t)^{DD})^D = (x \vee x \vee t)^{DDD} = (x \vee t)^D = (0)^D = D$. Since $(0)^D = D$, we get $(x)^D = D$. □

The set $\mathfrak{F}^{\perp}(R)$ denotes the set of all filters of the form $(x)^{\perp},\ x \in R$. In general $\mathfrak{F}^{\perp}(R)$ is not a sublattice of $\mathfrak{F}(R)$.

Example 3.7. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and define $\vee,\ \wedge$ on R as follows:

$\wedge$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	1	2	3	4	5	6	7
3	0	3	3	3	0	0	3	0
4	0	4	5	0	4	5	7	7
5	0	4	5	0	4	5	7	7
6	0	6	6	3	7	7	6	7
7	0	7	7	0	7	7	7	7

$\vee$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	1	2	3	1	2	6	6
4	4	1	1	1	4	4	1	4
5	5	2	2	2	5	5	2	5
6	6	1	2	6	1	2	6	6
7	7	1	2	6	4	5	6	7

Then $(R, \vee,\ \wedge)$ is an ADL. Clearly, we have that $D = \{1, 2, 6\}$ is the dense set of R. Also, we have that $(3)^{\perp} = \{1, 2, 3, 6\}$ and $(4)^{\perp} = \{1, 2, 4, 5, 6, 7\}$. Therefore $(3)^{\perp} \vee (4)^{\perp} = \{1, 2, 6\}$ and $(3 \wedge 4)^{\perp} = (0)^{\perp} = R$. Hence $(3)^{\perp} \vee (4)^{\perp} \neq (3 \wedge 4)^{\perp}$.

Theorem 3.8. *In an ADL R, $\mathfrak{F}^{\perp}(R)$ forms a complete distributive lattice on its own.*

Proof. Let $(x)^{\perp}, (y)^{\perp} \in \mathfrak{F}^{\perp}(R)$. Define $(x)^{\perp} \cap (y)^{\perp} = (x \vee y)^{\perp}$ and $(x)^{\perp} \sqcup (y)^{\perp} = (x \wedge y)^{\perp}$. Clearly we have that $(x)^{\perp} \cap (y)^{\perp} = (x \vee y)^{\perp}$. Therefore $(x \vee y)^{\perp}$ is the greatest lower bound of $(x)^{\perp}$ and $(y)^{\perp}$ in $\mathfrak{F}^{\perp}(R)$. Clearly, $(x)^{\perp} \subseteq (x \wedge y)^{\perp}$ and $(y)^{\perp} \subseteq (x \wedge y)^{\perp}$. Thus $(x \wedge y)^{\perp}$ is a upper bound of $(x)^{\perp}$ and $(y)^{\perp}$. Let $(z)^{\perp}$ be any upper bound of $(x)^{\perp}$ and $(y)^{\perp}$. Then $(x)^{\perp} \subseteq (z)^{\perp}$ and $(y)^{\perp} \subseteq (z)^{\perp}$. That implies $x, y \in (z)^{\perp}$ and hence $x \wedge y \in (z)^{\perp}$. Therefore $(x \wedge y)^{\perp} \subseteq (z)^{\perp}$. Thus $(x \wedge y)^{\perp}$ is the least upper bound of $(x)^{\perp}$ and $(y)^{\perp}$. That follows $(x)^{\perp} \sqcup (y)^{\perp} = (x \wedge y)^{\perp}$. Hence $\mathfrak{F}^{\perp}(R)$ is a lattice. It is easy to verify that $(\mathfrak{F}^{\perp}(R), \cap, \sqcup, R, \{m\})$ is a bounded complete distributive lattice. □

Definition 3.9. Define a relation θ on an ADL L as for any $x, y \in R$, $(x, y) \in \theta$ if and only if $(x)^{\perp} = (y)^{\perp}$. It is easy to verify that θ is a congruence relation on R.

Let θ be a congruence relation on R. Then proved easily that $R/\theta = \{[x]_{\theta} \mid x \in R\}$ is a quotient lattice by defining $[x]_{\theta} \wedge [y]_{\theta} = [x \wedge y]_{\theta}$ and $[x]_{\theta} \vee [y]_{\theta} = [x \vee y]_{\theta}$, for all $x, y \in R$.
Define $f : R \to R/\theta$ by $f(x) = [x]_{\theta}$, for all $x \in R$. Clearly f is a natural onto homomorphism.

Definition 3.10. An element x of an ADL R is said to be $D-$dense if $(x)^{D} = D$. The set of all $D-$dense elements denoted by $D'(R)$.

Lemma 3.11. *$D'(R)$ is an ideal of an ADL R.*

Proof. Clearly, $0 \in D'(R)$ and hence $D'(R) \neq \phi$. Let $a, b \in D'(R)$. Then $(a)^D = D = (b)^D$. Now $(a \vee b)^D = (a \vee b)^{DDD} = ((a)^{DD} \cap (b)^{DD})^D = (D^D \cap D^D)^D = R^D = D$. That implies $a \vee b \in D'(R)$. Let $a \in D'(R)$. Then $(a)^D = D$. Let $r \in R$. Clearly $D \subseteq (a \wedge r)^D \subseteq (a)^D = D$ and hence $(a \wedge r)^D = D$. Therefore $a \wedge r \in D'(R)$. Hence $D'(R)$ is an ideal of R. □

Theorem 3.12. *$D'(R)$ is the zero congruence class of R/θ and D is the unit congruence of R/θ.*

Proof. Let $x, y \in D'(R)$. Then $(x)^D = D = (y)^D$. That implies $(x)^\perp = (y)^\perp$ and hence $(x, y) \in \theta$. Thus $D'(R)$ is a congruence class of R/θ. Let $a \in R$. Thus we prove that $D'(R) \subseteq [a]_\theta$. Let $b \in D'(R)$. Thus $(b)^D = D$. Since $[b]_\theta \wedge [a]_\theta = [b \wedge a]_\theta$ and $b \in D'(R)$, we get $b \wedge a \in D'(R)$ and hence $(b \wedge a)^D = D$. Now $D'(R) \cap [a]_\theta = [b]_\theta \cap [a]_\theta = [b \wedge a]_\theta = D'(R)$. Therefore $D'(R) \subseteq [a]_\theta$, for all $a \in R$. Hence $D'(R)$ is the zero congruence of R/θ. Let $x, y \in D$. Then $(x)^D = R = (y)^D$ and hence $(x)^\perp = (y)^\perp$. Therefore D is a congruence class of R/θ. Let $x \in D$ and $a \in R$. Clearly, $[x]_\theta \vee [a]_\theta = [x \vee a]_\theta = D$, because of $x \vee a \in D$. Since $x \in D$, we get that $D \vee [a]_\theta = [x/]_\theta \vee [a]_\theta = [x \vee a]_\theta = D$. Therefore $[a]_\theta \subseteq D$, for all $a \in R$. Hence D is the unit congruence of R/θ. □

Theorem 3.13. *Let R be an ADL. Thus the following are equivalent:*
(1) $R|\theta$ is a Boolean algebra;
(2) $\mathfrak{F}^\perp(R)$ is Boolean algebra;
(3) For any $a \in R$, there exists $b \in R$ such that $a \wedge b \in D'(R)$ and $a \vee b \in D$.

Proof. $(1) \Rightarrow (2)$: Assume (1). Define $g : R|\theta \to \mathfrak{F}^\perp(R)$ by $g([a]_\theta) = (a)^\perp$, for all $[a]_\theta \in R/\theta$. Let $a, b \in R$ with $[a]_\theta = [b]_\theta$. Then $(a, b) \in \theta$. That implies $(a)^\perp = (b)^\perp$ and hence $g([a]_\theta) = g([b]_\theta)$. Hence g is well defined. Let $a, b \in R$ with $(a)^\perp = (b)^\perp$. Then $(a, b) \in \theta$. That implies $[a]_\theta = [b]_\theta$ and hence g is one - one. Let $a \in R$. Then $[a]_\theta \in R/\theta$. Since f is onto, there exists an element $b \in R$ such that $f(b) = [a]_\theta$ and hence $g([a]_\theta) = (a)^\perp$. Hence g is onto. Clearly, g is homomorphism and hence g is isomorphism. Therefore $\mathfrak{F}^\perp(R)$ is a Boolean algebra.
$(2) \Rightarrow (3)$: Assume (2). Let $a \in R$. Then $(a)^\perp \in \mathfrak{F}^\perp(R)$. By our assumption, There exist $(b)^\perp \in \mathfrak{F}^\perp(R)$ such that $(a \vee b)^\perp = (a)^\perp \cap (b)^\perp = (0)^\perp$ and $(a \wedge b)^\perp = (a)^\perp \sqcup (b)^\perp = R$. Hence $a \vee b \in D$ and $(a \wedge b)^D = D$. Therefore $a \vee b \in D$ and $a \wedge b \in D'(R)$.
$(3) \Rightarrow (1)$: Assume (3). Let $[a]_\theta \in R/\theta$, where $a \in R$. Then $a \wedge b \in D^{'}(R), a \vee b \in D$, for some $b \in R$. That implies $[a]_\theta \wedge [b]_\theta = [a \wedge b]_\theta = D'(R)$ and $[a]_\theta \vee [b]_\theta = [a \vee b]_\theta = D$. Hence R/θ is a Boolean algebra. □

Definition 3.14. A non-zero element x of an ADL R is said to be a $D-$prime element if for any $y, z \in R, (x|y \vee z)_D$ implies $(x|y)_D$ or $(x|z)_D$.

Theorem 3.15. *A non dense element x of an ADL R is $D-$prime if and only if $(x)^\perp$ is a prime filter of R.*

Proof. Let x be any non dense element of R. Assume x is $D-$prime. Let $a, b \in R$ with $a \vee b \in (x)^{\perp}$. Thus $(x|a \vee b)_D$. By our assumption, we get that $(x|a)_D$ or $(x|b)_D$. That implies $a \in (x)^{\perp}$ or $b \in (x)^{\perp}$. Hence $(x)^{\perp}$ is prime. Conversely, assume that $(x)^{\perp}$ is prime. Let $a, b \in R$ with $(x|a \vee b)_D$. Then $a \vee b \in (x)^{\perp}$. By our assumption, we get that $a \in (x)^{\perp}$ or $b \in (x)^{\perp}$. That implies $(x|a)_D$ or $(x|b)_D$. Hence x is $D-$prime. □

Definition 3.16. A non dense element x of an ADL R is said to be $D-$irreducible if for any $y, z \in R$, $(x)^D = (y \vee z)^D$ implies $(y)^D = D$ or $(z)^D = D$.

Lemma 3.17. *Every $D-$dense element of R is $D-$irreducible.*

Proof. Let x be any $D-$dense element of R. Let $a, b \in R$ with $(x)^D = (a \vee b)^D$. Then $(x)^{DD} = (a)^{DD} \cap (b)^{DD}$. That implies $D^D = (a)^{DD} \cap (b)^{DD}$ and hence $R = (a)^{DD} \cap (b)^{DD}$. Therefore $R \subseteq (a)^{DD}$ and $R \subseteq (b)^{DD}$. That implies $(a)^{DDD} = D$ and $(b)^{DDD} = D$, which gives $(a)^D = D$ and $(b)^D = D$. Thus x is $D-$irreducible. □

Theorem 3.18. *Let $x \in R$ with $(x)^D \neq D$. Thus the following are equivalent:*
(1) x is $D-$irreducible;
(2)(i) $(x)^{\perp}$ is maximal among all proper filters of the form $(y)^{\perp}$;
(ii) for any $a \in R, (x)^D = (x \vee a)^D \Rightarrow (a)^D = D$.

Proof. $(1) \Rightarrow (2)$: Assume (1). Suppose $(x)^{\perp} \subseteq (y)^{\perp} \neq R$ for some $y \in R$. Thus $x \in (y)^{\perp}$. That implies $(y|x)_D$. There exists $z \in R$ such that $(x)^D = (y \vee z)^D$. By (1), $(y)^D = D$ or $(z)^D = D$. Since $(y)^{\perp} \neq R$ we get $(y)^D \neq D$ and hence $(z)^D = D$. That implies $(z)^D = (0)^D$ and hence $(x)^D = (y)^D$. Therefore $(x)^{\perp} = (y)^{\perp}$. Thus $(x)^{\perp}$ is maximal among all filters of the form $(y)^{\perp}$. Suppose $(x)^D = (x \vee y)^D$, for some $y \in R$. By (1), we get $(x)^D = D$ or $(y)^D = D$. Since $(x)^D \neq D$, we get $(y)^D = D$.
$(2) \Rightarrow (1)$: Assume (2). Let $x \in R$ with $(x)^D \neq D$. Suppose $(x)^D = (y \vee z)^D$, for some $y, z \in R$. Then $(z|x)_D$. That implies $x \in (z)^{\perp}$ and hence $(x)^{\perp} \subseteq (z)^{\perp}$. By (2)(i), we get $(x)^{\perp} = (z)^{\perp}$ or $(z)^{\perp} = R$. Suppose $(x)^{\perp} = (z)^{\perp}$. Thus $z \in (x)^{\perp}$. That implies $(x|z)_D$. So That $(z)^D = (x \vee a)^D$, for some $a \in R$. That implies $(z \vee y)^D = (x \vee a \vee y)^D$. By 2(ii), we get $(a \vee y)^D = D$. Since $(y)^D \subseteq (a \vee y)^D$, we get $(y)^D = D$. Suppose $(z)^{\perp} = R$. Then $0 \in (z)^{\perp}$. That implies $(z/0)_D$. So that $(0)^D = (z \vee a)^D$, for some $a \in R$. That implies $D = (z \vee a)^D$. Therefore $(z)^D \subseteq (z \vee a)^D = D$ and hence $(z)^D = D$. Thus x is $D-$irreducible. □

Theorem 3.19. *Every $D-$irreducible element of an ADL R is $D-$prime.*

Proof. Let x be any $D-$irreducible element of R. Let $a, b \in R$ with $(x|a \vee b)_D$. Suppose $a \notin (x)^{\perp}$ and $b \notin (x)^{\perp}$. Clearly $(x)^{\perp} \subsetneq (x)^{\perp} \sqcup (a)^{\perp} \subseteq (x \wedge a)^{\perp}$ and $(x)^{\perp} \subsetneq (x)^{\perp} \sqcup (b)^{\perp} \subseteq (x \wedge b)^{\perp}$. By maximality of $(x)^{\perp}$, we get $(x \wedge a)^{\perp} = R$ and $(x \wedge b)^{\perp} = R$. That implies $R = (x \wedge a)^{\perp} \cap (x \wedge b)^{\perp}$. So that

$R = ((x)^{\perp} \sqcup (a)^{\perp}) \cap ((x)^{\perp} \sqcup (b)^{\perp}) = (x)^{\perp} \sqcup ((a)^{\perp} \cap (b)^{\perp}) = (x)^{\perp} \sqcup (a \wedge b)^{\perp}$. Since $a \wedge b \in (x)^{\perp}$, we get $(a \vee b)^{\perp} \subseteq (x)^{\perp}$ and hence $(x)^{\perp} = R$. we get a contradiction. Hence $a \in (x)^{\perp}$ or $b \in (x)^{\perp}$. Therefore $(x|a)_D$ or $(x|b)_D$. Thus x is $D-$prime. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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