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ISHIKAWA ITERATIVE PROCESS FOR TWO NONLINEAR MAPPING IN CAT(0) SPACES

KRITSANA SOKHUMA

Department of Mathematics, Faculty of Science and Technology, Phranakhon Rajabhat University, Bangkok 10220, Thailand
k_sokhuma@yahoo.co.th

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Abstract. In this paper, we construct an iteration scheme involving a hybrid pair of the asymptotically nonexpansive mapping t and the Suzuki generalized multi-valued nonexpansive mappings T of a complete CAT(0) spaces. In process, we remove a restricted condition (called end-point condition) in Akkasriworn and Sokhuma's results [12] and utilize the same to prove some convergence theorems. This results we obtain are analogs of CAT(0) spaces results of Sokhuma and Sokhuma [10].

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1. Introduction

Fixed point theory in a CAT(0) space was first studied by Kirk [18], [19]. He showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the existence problem of fixed point and the $\triangle-$ convergence problem of iterative sequences to a fixed point for nonexpansive mappings, asymptotically nonexpansive mappings in a CAT(0) space have been rapidly developed and many papers have appeared.

Let (X, d) be a geodesic metric space. We denote by $FB(K)$ the collection of all nonempty closed bounded subsets of X, $KC(K)$ the collection of all nonempty compact convex subsets of X . A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = \text{dist}(x, K) = \inf\{d(x, y) : y \in K\}.$$

We shall denote by $PB(K)$, the collection of all nonempty bounded proximinal subsets of K.

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Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\{\sup_{x\in A} \text{dist}(x,B), \sup_{y\in B} \text{dist}(y,A)\}, \quad A,B \in FB(X),$$

where $\text{dist}(x,B) = \inf\{d(x,y) : y \in B\}$ is the distance from the point x to the subset B.

A mapping $t : K \to K$ is said to be *nonexpansive* if

$$d(tx,ty) \leq d(x,y) \text{ for all } x,y \in K.$$

A point x is called a fixed point of t if $tx = x$.

A mapping $t : K \to K$ is called asymptotically nonexpansive if there is a sequence $\{k_n\}$ of positive numbers with the property $\lim_{n\to\infty} k_n = 1$ such that

$$d(t^n x, t^n y) \leq k_n d(x,y) \text{ for all } n \geq 1, x,y \in K.$$

A multi-valued mapping $T : K \to FB(K)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \leq d(x,y) \text{ for all } x,y \in K.$$

In 2010, Abkar and Eslamian [1] mentioned the Suzuki generalized multi-valued nonexpansive mapping as follows:

A multi-valued mapping $T : K \to FB(K)$ is said to be a *Suzuki generalized multi-valued nonexpansive mapping* if

$$\frac{1}{2}\text{dist}(x,Tx) \leq d(x,y) \Rightarrow H(Tx,Ty) \leq d(x,y) \text{ for all } x,y \in K.$$

Let $T : K \to PB(K)$ be a multi-valued mapping and define the mapping P_T for each x by

$$P_T(x) := \{y \in Tx : d(x,y) = \text{dist}(x,Tx)\}.$$

A point x is called a fixed point for a multi-valued mapping T if $x \in Tx$.

We say that $I - T$ is strongly demiclosed if for every sequence $\{x_n\}$ in K which converges to $x \in K$ and such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have $x \in T(x)$.

We note that for every continuous mapping $T : K \to 2^K$, $I - T$ is strongly demiclosed but the converse is not true. Notice also that if T satisfies condition (E), then $I - T$ is strongly demiclosed.

We use the notation $\text{Fix}(T)$ stands for the set of fixed points of a mapping T and $\text{Fix}(t) \cap \text{Fix}(T)$ stands for the set of common fixed points of t and T. Precisely, a point x is called a common fixed point of t and T if $x = tx \in Tx$.

In 2009, Laokul and Panyanak [15] defined the iterative and proved the $\triangle-$converges for nonexpansive mapping in CAT(0) spaces as follows:

Let C be a nonempty closed convex subset of a complete CAT(0) space and $t : C \to C$ be a nonexpansive mapping with $\text{Fix}(t) := \{x \in C : tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by

$x_1 \in C$,

$$y_n = \beta_n x_n \oplus (1-\beta_n)x_n,$$

$$x_{n+1} = \alpha_n t y_n \oplus (1-\alpha_n)x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ such that one of the following two conditions is satisfied:

(i) $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ for some a, b with $0 < a \le b < 1$,

(ii) $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \le b < 1$.

Then the sequence $\{x_n\}$ $\triangle-$converges to a fixed point of t.

In 2010, Sokhuma and Kaewkhao [9] proved the convergence theorem for a common fixed point in Banach spaces as follow:

Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in \text{Fix}(t) \cap \text{Fix}(T)$. Suppose $\{x_n\}$ is generated iterative by $x_1 \in E$,

$$y_n = (1-\beta_n)x_n + \beta_n z_n,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n,$$

for all $n \in \mathbb{N}$ where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \le \alpha_n, \beta_n \le b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2013, Sokhuma [8] proved the convergence theorem for a common fixed point in CAT(0) spaces as follow:

Let K be a nonempty compact convex subset of a complete CAT(0) space X, and $t : K \to K$ and $T : K \to FC(K)$ a single valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively, and $\text{Fix}(t) \cap \text{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \text{Fix}(t) \cap \text{Fix}(T)$. Let $\{x_n\}$ is generated iterative by $x_1 \in K$,

$$y_n = (1-\beta_n)x_n \oplus \beta_n z_n,$$

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t^n y_n,$$

for all $n \in \mathbb{N}$ where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \le \alpha_n, \beta_n \le b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2013, Laowang and Panyanak proved the convergence theorem for a common fixed point in CAT(0) spaces as follow:

Corollary 1.1. [20] *Let C be a nonempty bounded closed convex subset of a complete CAT(0) spaces X. Let $f : C \to C$ be a pointwise asymtotically nonexpansive mapping, and $g : C \to C$ a quasi-nonexpansive mapping,* *and let $T : C \to KC(C)$ be a multi-valued mapping satisfying conditions (E) and C_λ for some $\lambda \in (0,1)$. If f, g and T are pairwise commuting, then there exists a point $z \in C$ such that $z = f(z) = g(z) \in T(z)$.*

In 2015, Akkasriworn and Sokhuma [12] proved the convergence theorem for a common fixed point in a complete CAT(0) spaces as follow:

Theorem 1.2. *Let E be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : E \to E$ and $T : E \to FB(E)$ an asymptotically nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume that t and T are commuting and Fix$(t) \cap$ Fix$(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in$ Fix$(t) \cap$ Fix(T) and $\sum\limits_{n=1}^{\infty} (k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iterates defined by*

$$y_n = (1-\beta_n)x_n \oplus \beta_n z_n,$$

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t^n y_n,$$

for all $n \in \mathbb{N}$ where $z_n \in T(t^n x_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in [0,1]$. Then $\{x_n\}$ $\triangle-$converges to a common fixed point of t and T.

In 2016, Uddin and Imdad [7] introduce the following iteration scheme:

Let K be a nonempty closed, bounded and convex subset of Banach space X, let $t : K \to K$ be a single valued nonexpansive mapping and let $T : K \to CB(K)$ be a multi-valued nonexpansive mapping with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is a nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$y_n = \alpha_n z_n + (1-\alpha_n)x_n,$$

$$x_{n+1} = \beta_n t y_n + (1-\beta_n)x_n,$$

where $x_0 \in K,\ z_n \in P_T(x_n)$ and $0 < a \le \alpha_n,\ \beta_n \le b < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of t and T.

In 2022, Sokhuma and Sokhuma [10] proved the convergence theorem for two nonlinear mappings in CAT(0) spaces as follows:

Let K be a nonempty closed, bounded and convex subset of CAT(0) space X, let $t : K \to K$ and $T : K \to PB(K)$ be a Suzuki generalized nonexpansive single valued and multi-valued mapping, respectively with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is a nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$y_n = (1-\beta_n)x_n \oplus \beta_n z_n,$$

$$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t y_n,$$

for all $n \in \mathbb{N}$, where $z_n \in P_T(t x_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0,1)$.

The purpose of this paper is to study the iterative process, called the Ishikawa iteration method with respect to a pair of single valued asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping. We also establish the convergence theorem of a sequence from such process in a nonempty bounded closed convex subset of a complete CAT(0) space. We remove a restricted condition (called end-point condition) in Akkasriworn and Sokhuma's results [12].

Now, we introduce an iteration method modifying the above ones and call it the Ishikawa iteration method.

Definition 1.3. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ be a single valued asymptotically nonexpansive mapping, and $T : K \to PB(K)$ be a Suzuki generalized multi-valued nonexpansive mapping and*

$$P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}.$$

For fixed $x_1 \in K$. The sequence $\{x_n\}$ of the Ishikawa iteration is defined by

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n t^n y_n, \end{cases} \tag{1}$$

for all $n \in \mathbb{N}$ where $z_n \in P_T(t^n x_n)$ and $\{\alpha_n\}$, $\{\beta_n\} \in (0, 1)$.

2. Preliminaries

With a view to make, our presentation self contained, we collect some relevant basic definitions, results and iterative methods which will be used frequently in the text later.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, s] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(s) = y$, and $d(c(t), c(u)) = |t - u|$ for all $t, u \in [0, s]$. In particular, c is an isometry and $d(x, y) = s$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$, $d(x,y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if

$$y_0 = \frac{1}{2}y_1 \oplus \frac{1}{2}y_2,$$

then the CAT(0) inequality implies that

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{2}$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality [11].

The following results and methods deal with the concept of asymptotic centers. Let K be a nonempty closed convex subset of a CAT(0) space X and $\{x_n\}$ be a bounded sequence in X. For $x \in X$, define the asymptotic radius of $\{x_n\}$ at x as the number

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

Let

$$r \equiv r(K, \{x_n\}) := \inf \{r(x, \{x_n\}) : x \in K\}$$

and

$$A \equiv A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r and the set A are, respectively, called the asymptotic radius and asymptotic center of $\{x_n\}$ relative to K.

It is easy to know that if X is a complete CAT(0) spaces and K is a closed convex subset of X, then $A(K, \{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in CAT(0) space X is said to be $\triangle-$convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. A bounded sequence $\{x_n\}$ is said to be regular with respect to K if for every subsequence $\{x'_n\}$, we get

$$r(K, \{x_n\}) = r(K, \{x'_n\}).$$

We now give the definition of $\triangle-$convergence.

Definition 2.1.([19], [16]) *A sequence $\{x_n\}$ in a CAT(0) space X is said to $\triangle-$converge to $x \in X$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\triangle - lim_n x_n = x$ and call x the $\triangle-$limit of $\{x_n\}$.*

We now collect some elementary facts about CAT(0) spaces which will be used in the proofs of our main results. The following lemma can be found in ([19], [13], [14]).

Lemma 2.2.([19]) *Every bounded sequence in a complete $CAT(0)$ space has a $\triangle-$convergent subsequence.*

Lemma 2.3.([13]) *If K is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.4.([14]) *Let (X, d) be a $CAT(0)$ space.*

(i) For $x, y \in X$ and $u \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that

$$d(x, z) = ud(x, y) \;\; and \;\; d(y, z) = (1 - u)d(x, y). \tag{3}$$

We use the notation $(1 - u)x \oplus ty$ for the unique point z satisfying (3).

(ii) For $x, y, z \in X$ and $u \in [0, 1]$, we have

$$d((1 - u)x \oplus uy, z) \leq (1 - u)d(x, z) + ud(y, z).$$

We now collect some basic properties of the Suzuki generalized nonexpansive mapping. Although the proofs follow the idea of the proofs in [22]. The following two propositions are very easy to verify.

Proposition 2.5. *Let K be a nonempty subset of a $CAT(0)$ space X and $t : K \to K$ be a nonexpansive mapping. Then t is a Suzuki generalized nonexpansive mapping.*

Proposition 2.6. *Let K be a nonempty subset of a $CAT(0)$ space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping and has a fixed point. Then t is a quasi-nonexpansive mapping.*

Proposition 2.7. *Let K be a nonempty subset of a CAT(0) space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mapping. Then*

$$d(x, ty) \leq 3d(tx, x) + d(x, y)$$

holds for all x, y ∈ K.

The existence of fixed points for generalized Suzuki nonexpansive mappings in CAT(0) spaces was proved by Nanjaras et al. [2] as the following result.

Theorem 2.8. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X. Suppose $t : K \to K$ is a Suzuki generalized nonexpansive mappings. Then t has a fixed point in K.*

Lemma 2.9. *Let K be a closed and convex subset of a complete $CAT(0)$ space X and let $t : K \to X$ be a generalized Suzuki nonexpansive mappings. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = w$. Then $tw = w$.*

The existence of fixed points for asymptotically nonexpansive mappings in CAT(0) spaces was proved by Kirk [17] as the following result.

Theorem 2.10. *Let K be a nonempty bounded closed and convex subset of a complete $CAT(0)$ space X and let $t : K \to K$ be asymptotically nonexpansive. Then t has a fixed point.*

Theorem 2.11.([3]) *Let X be a complete $CAT(0)$ space and K be a nonempty bounded closed and convex subset of X and $t : K \to K$ be an asymptotically nonexpansive mapping. Then $I - t$ is demiclosed at 0.*

Corollary 2.12.([14]) *Let K be a closed and convex subset of a complete $CAT(0)$ space X and let $t : K \to X$ be an asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in K such that $\lim_{n\to\infty} d(tx_n, x_n) = 0$ and $\triangle - \lim_{n\to\infty} x_n = w$. Then $tw = w$.*

Lemma 2.13.([21]) *Let X be a complete $CAT(0)$ space and let $x \in X$. Suppose $\{\alpha_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r, \limsup_{n\to\infty} d(y_n, x) \leq r$, and $\lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n y_n, x) = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

The following fact is well-known.

Lemma 2.14. *Let X be a CAT(0) space and K be a nonempty compact convex subset of X and $\{x_n\}$ be the sequence in K. Then,*

$$dist(y, Ty) \leq d(y, x_n) + dist(x_n, Tx_n) + H(Tx_n, Ty)$$

where $y \in K$ and T be a multi-valued mapping from K in to $FB(K)$.

The important property can be found in [5].

Lemma 2.15. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 + b_n)a_n,$$

for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists. In particular, if there is a subsequence of $\{a_n\}$ which converges to 0 then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We first prove the following lemmas, which play very important roles in this section.

Lemma 3.1. *Let $T : K \to PB(K)$ be a multi-valued mapping and $P_T(x) = \{y \in Tx : d(x, y) = dist(x, Tx)\}$. Then the following are equivalent*

(1) $x \in Fix(T)$, that is $x \in Tx$;

(2) $P_T(x) = \{x\}$, that is $x = y$ for each $y \in P_T(x)$;

(3) $x \in Fix(P_T)$, that is $x \in P_T(x)$.

Further, $Fix(T) = Fix(P_T)$.

Proof. (1) implies (2). Since $x \in Tx$, then $d(x, Tx) = 0$. Therefore, for any $y \in P_T(x)$, $d(x, y) = \text{dist}(x, Tx) = 0$ and so $x = y$. That is, $P_T(x) = \{x\}$.

(2) implies (3). Since $P_T(x) = \{x\}$, then $x \in \text{Fix}(P_T)$ and we get $x \in P_T(x)$.

(3) implies (1). Since $x \in \text{Fix}(P_T)$, then $x \in P_T(x)$. Therefore, $d(x, x) = \text{dist}(x, Tx) = 0$ and so $x \in Tx$ by the closedness of Tx.

This implies that $\text{Fix}(T) = \text{Fix}(P_T)$. □

Lemma 3.2. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with $Fix(t) \cap Fix(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\lim_{n\to\infty} d(x_n, w)$ exists for all $w \in Fix(t) \cap Fix(T)$.*

Proof. Let $x_1 \in K$ and $w \in \text{Fix}(t) \cap \text{Fix}(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Now consider,

$$\begin{aligned}
d(x_{n+1}, w) &= d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(t^n y_n, t^n w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n d(y_n, w) \\
&= (1-\alpha_n)d(x_n, w) + \alpha_n k_n d((1-\beta_n)x_n \oplus \beta_n z_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n (1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n d(z_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n (1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n \text{dist}(z_n, P_T(w)) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n (1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n H(P_T(t^n x_n), P_T(w)) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n (1-\beta_n)d(x_n, w) + \alpha_n k_n \beta_n d(t^n x_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n (1-\beta_n)d(x_n, w) + \alpha_n \beta_n k_n^2 d(x_n, w) \\
&= [1 + \alpha_n(k_n - 1) + \alpha_n \beta_n k_n (k_n - 1)]d(x_n, w) \\
&= [1 + \alpha_n(1 + \beta_n k_n)(k_n - 1)]d(x_n, w).
\end{aligned}$$

By the convergence of k_n and $\alpha_n,\ \beta_n \in (0,1)$, then there exists some $M > 0$ such that

$$d(x_{n+1}, w) \leq [1 + M(k_n - 1)]d(x_n, w).$$

By condition $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and Lemma 2.15, we know that $\lim_{n\to\infty} d(x_n, w)$ exists. □

Lemma 3.3. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\lim_{n\to\infty} d(t^n y_n, x_n) = 0$.*

Proof. Let $x_1 \in K$ and $w \in \text{Fix}(t) \cap \text{Fix}(T)$, in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. From Lemma 3.2, we setting $\lim_{n\to\infty} d(x_n, w) = c$. Now consider,

$$\begin{aligned}
d(y_n, w) &= d((1-\beta_n)x_n \oplus \beta_n z_n, w) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(z_n, w) \\
&= (1-\beta_n)d(x_n, w) + \beta_n dist(z_n, P_T(w)) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n H(P_T(t^n x_n), P_T(w)) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n d(t^n x_n, w) \\
&\leq (1-\beta_n)d(x_n, w) + \beta_n k_n d(x_n, w).
\end{aligned}$$

We have

$$\begin{aligned}
d(t^n y_n, w) &\leq k_n d(y_n, w) \\
&\leq k_n[(1-\beta_n)d(x_n, w) + \beta_n k_n d(x_n, w)] \\
&= k_n(1-\beta_n)d(x_n, w) + \beta_n k_n^2 d(x_n, w) \\
&= (k_n - k_n\beta_n + \beta_n k_n^2)d(x_n, w) \\
&= [k_n + \beta_n k_n(k_n - 1)]d(x_n, w) \\
&\leq [1 + \beta_n k_n(k_n - 1)]d(x_n, w).
\end{aligned}$$

Then we have,

$$\limsup_{n\to\infty} d(t^n y_n, w) \leq \limsup_{n\to\infty} k_n d(y_n, w) \leq \limsup_{n\to\infty} [1 + \beta_n k_n(k_n - 1)]d(x_n, w).$$

By $k_n \to 1$ as $n \to \infty$ and $\alpha_n, \beta_n \in (0, 1)$, which implies that

$$\limsup_{n\to\infty} d(t^n y_n, w) \leq \limsup_{n\to\infty} d(y_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c. \tag{4}$$

Since, $c = \lim_{n\to\infty} d(x_{n+1}, w) = \lim_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w)$.
Then by condition of α_n and Lemma 2.13, we have $\lim_{n\to\infty} d(t^n y_n, x_n) = 0$. □

Lemma 3.4. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\lim_{n\to\infty} d(x_n, z_n) = 0$.*

Proof. Let $x_1 \in K$ and $w \in$ Fix$(t) \cap$ Fix(T), in view of Lemma 3.1 we have $w \in P_T(w) = \{w\}$. Consider,

$$\begin{aligned}
d(x_{n+1}, w) &= d((1-\alpha_n)x_n \oplus \alpha_n t^n y_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n d(t^n y_n, w) \\
&\leq (1-\alpha_n)d(x_n, w) + \alpha_n k_n d(y_n, w)
\end{aligned}$$

and hence

$$\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n} \leq k_n d(y_n, w) - d(x_n, w).$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w) \leq k_n d(y_n, w).$$

Thus,

$$\liminf_{n\to\infty} \left\{\left(\frac{d(x_{n+1}, w) - d(x_n, w)}{\alpha_n}\right) + d(x_n, w)\right\} \leq \liminf_{n\to\infty} k_n d(y_n, w).$$

It follows that

$$c \leq \liminf_{n\to\infty} d(y_n, w).$$

Since, from (4), $\limsup_{n\to\infty} d(y_n, w) \leq c$, we have

$$c = \lim_{n\to\infty} d(y_n, w) = \lim_{n\to\infty} d((1-\beta_n)x_n \oplus \beta_n z_n, w).$$

Recall that

$$d(z_n, w) = \text{dist}(z_n, P_T(w)) \leq H(P_T(t^n x_n), P_T(w)) \leq d(t^n x_n, w) \leq k_n d(x_n, w).$$

Hence we have

$$\limsup_{n\to\infty} d(z_n, w) \leq \limsup_{n\to\infty} k_n d(x_n, w) \leq \limsup_{n\to\infty} d(x_n, w) = c.$$

Thus, $\lim_{n\to\infty} d(x_n, z_n) = 0$. □

Lemma 3.5. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\lim_{n\to\infty} d(t^n x_n, x_n) = 0$.*

Proof. Consider,

$$\begin{aligned}
d(t^n x_n, x_n) &\leq d(t^n x_n, t^n y_n) + d(t^n y_n, x_n) \\
&\leq k_n d(x_n, y_n) + d(t^n y_n, x_n) \\
&= k_n d(x_n, (1-\beta_n)x_n \oplus \beta_n z_n) + d(t^n y_n, x_n) \\
&\leq k_n[(1-\beta_n)d(x_n, x_n) + \beta_n d(x_n, z_n)] + d(t^n y_n, x_n) \\
&= k_n \beta_n d(x_n, z_n) + d(t^n y_n, x_n).
\end{aligned}$$

Then, we have

$$\lim_{n\to\infty} d(t^n x_n, x_n) \leq \lim_{n\to\infty} k_n \beta_n d(z_n, x_n) + \lim_{n\to\infty} d(t^n y_n, x_n).$$

Hence, by Lemma 3.3 and Lemma 3.4, $\lim_{n\to\infty} d(t^n x_n, x_n) = 0$. □

Lemma 3.6. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates by (1). Then $\lim_{n\to\infty} d(t x_n, x_n) = 0$.*

Proof. Consider,

$$d(t x_n, x_n) = d(x_n, t x_n)$$

$$\begin{aligned}
&\leq d(x_n, t^n x_n) + d(t^n x_n, t x_n) \\
&\leq d(x_n, t^n x_n) + k_1[d(t^{n-1}x_n, t^{n-1}x_{n-1}) + d(t^{n-1}x_{n-1}, x_n)] \\
&\leq d(x_n, t^n x_n) + k_1 k_{n-1} d(x_n, x_{n-1}) + k_1 d(t^{n-1}x_{n-1}, x_n) \\
&\leq d(x_n, t^n x_n) + k_1 k_{n-1}\alpha_{n-1} d(t^{n-1}y_{n-1}, x_{n-1}) \\
&+ k_1(1-\alpha_{n-1}) d(x_{n-1}, t^{n-1}x_{n-1}) + k_1 k_{n-1}\alpha_{n-1} d(y_{n-1}, x_{n-1}) \\
&\leq d(x_n, t^n x_n) + k_1 k_{n-1}\alpha_{n-1} d(t^{n-1}y_{n-1}, x_{n-1}) \\
&+ k_1(1-\alpha_{n-1}) d(x_{n-1}, t^{n-1}x_{n-1}) + k_1 k_{n-1}\alpha_{n-1}\beta_{n-1} d(z_{n-1}, x_{n-1}).
\end{aligned}$$

It follows from Lemma 3.3, Lemma 3.4 and Lemma 3.5, we have $\lim_{n\to\infty} d(tx_n, x_n) = 0$. □

Theorem 3.7. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\{x_n\}$ $\triangle-$converges to y implies $y \in$ Fix$(t) \cap$ Fix(T).*

Proof. Since that $\{x_n\}$ $\triangle-$converges to y. From Lemma 3.6, we have

$$\lim_{n\to\infty} d(tx_n, x_n) = 0.$$

By Corollary 2.12, we have $y \in K$ and $ty = y$, that is $y \in \text{Fix}(t)$. From Lemma 2.14 we have

$$\begin{aligned}
\text{dist}(y, P_T(y)) &\leq d(y, x_n) + \text{dist}(x_n, P_T(x_n)) + H(P_T(x_n), P_T(y)) \\
&\leq d(y, x_n) + d(x_n, z_n) + d(x_n, y) \to 0 \ \text{ as } n \to \infty.
\end{aligned}$$

It follows that $y \in \text{Fix}(P_T)$, we get $y \in \text{Fix}(T)$. Therefore $y \in \text{Fix}(t) \cap \text{Fix}(T)$ as desired. □

Theorem 3.8. *Let K be a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ and $T : K \to PB(K)$ an asymptotically nonexpansive mapping and a Suzuki generalized multi-valued nonexpansive mapping, respectively, with Fix$(t) \cap$ Fix$(T) \neq \emptyset$ such that P_T is nonexpansive and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (1). Then $\{x_n\}$ $\triangle-$converges to a common fixed point of t and T.*

Proof. Since Lemma 3.6 guarantees that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(tx_n, x_n) = 0$. We now let $\omega_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset \text{Fix}(t) \cap \text{Fix}(T)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_{n\to\infty} v_n = v \in K$. Since

$\lim\limits_{n\to\infty} d(tv_n, v_n) = 0$, then $v \in \text{Fix}(t)$. Since,

$$\begin{aligned}
\text{dist}(v, P_T(v)) &\leq \text{dist}(v, P_T(v_n)) + H(P_T(v_n), P_T(v)) \\
&\leq d(v, z_n) + d(v_n, v) \\
&\leq d(v, v_n) + d(v_n, z_n) + d(v_n, v) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that $v \in \text{Fix}(P_T)$, we get $v \in \text{Fix}(T)$ by Lemma 3.1. Therefore, $v \in \text{Fix}(t) \cap \text{Fix}(T)$ as desired. We claim that $u = v$. Suppose not, since t is asymptotically nonexpansive mapping and $v \in \text{Fix}(t) \cap \text{Fix}(T)$, $\lim\limits_{n\to\infty} d(x_n, v)$ exists by Lemma 3.2. Then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\leq \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&\leq \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v)
\end{aligned}$$

a contradiction, and hence $u = v \in \text{Fix}(t) \cap \text{Fix}(T)$.

To show that $\{x_n\}$ $\triangle-$converges to a common fixed point, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.2 and Lemma 2.3 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim\limits_{n\to\infty} v_n = v \in K$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in \text{Fix}(t) \cap \text{Fix}(T)$.

We can complete the proof by showing that $x = v$. Suppose not, since $\lim\limits_{n\to\infty} d(x_n, v)$ exists, then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v)
\end{aligned}$$

a contradiction, and hence the conclusion follows. □

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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