Asia Pac. J. Math. 2024 **11**:27

EXPLORING STOCHASTIC FRACTIONAL DELAY INTEGRODIFFERENTIAL SYSTEM

M. LATHA MAHESWARI*, KARTHIK MUTHUSAMY

Department of Mathematics, PSG College of Arts & Science, India

*Corresponding author: lathamahespsg@gmail.com

Received Dec. 29, 2023

Abstract. This study examines the system of stochastic fractional delay integrodifferential equations with Gaussian noise. We employ the Picard-Lindelof successive approximation scheme for existence and uniqueness. Moreover, by means of Mittag-Leffler function, we establish the stability of the solution.

2020 Mathematics Subject Classification. 60H10; 34A12; 45J05; 34D20.

Key words and phrases. stochastic fractional system; existence & uniqueness; integrodifferential equation; stability.

1. Introduction

Fractional Calculus is an active and evolving area of research in various fields of science and engineering. It deals with derivatives and integrals of non-integer orders. Many researchers from all over the world have shown interest in the fractional calculus. Especially in the field of Biophysics, Econophysics, Fluid Dynamics and Quantum Mechanics and so on. For example, modeling the behaviour of biological systems with memory, such as neuronal firing patterns, studying financial markets and economic systems with long-term correlations and delayed feedback, understanding the behaviour of complex fluids with memory effects, like viscoelastic materials and describing quantum systems with non-local and memory-dependent operators have been studied with the applications of fractional calculus [4,8,9,11,15].

Over the last decade, stochastic fractional differential equations have surfaced as essential tools in mathematically modeling real-world phenomena. These type of equations have received a lot of attention in the field of mathematics, one can refer [1, 3, 5, 7, 14]. Several events are quantitatively represented in the scientific community using stochastic fractional delay integro-differential equations [2,10]. Integro-differential equations combine integral and differential terms. It is a delay differential

DOI: 10.28924/APJM/11-27

equation because the value of a variable at a given moment relies on its previous values, adding a temporal component to the equation [6,13,19,21].

Numerous applications have been addressed the existence, uniqueness, and stability of stochastic differential equations [16,17,20] as well as fractional differential equations [12,22]. In [20], Umamaheswari et al., discussed the investigation of a specific mathematical problem related to nonlinear stochastic fractional delay differential equations with Gaussian noise. The author explored the existence and uniqueness of solutions to this complex equation.

$$
\begin{aligned}
&{}^{C}D^{\alpha}x(t) = b(t, x(t), x(t-\delta)) + \sigma(t, x(t), x(t-\delta))\frac{dW(t)}{dt}, \quad t \in \mathcal{J} = [0, T]\\
&x(t) = \xi(t), \quad t \in [-\delta, 0]
\end{aligned}
$$

The author also had analyzed the nonlinear system, an equivalent nonlinear integral equation is introduced, which is deemed easier to work with. Her focus then shifted to establishing conditions that ensure the stability of the stochastic fractional differential equations when subjected to Gaussian noise. The Picard-Lindelof method, specifically the technique of successive approximation, was employed as a mathematical tool to derive the results.

Inspired by the aforementioned works, we considered stochastic fractional delay integrodifferential equations with gaussian noise of the form:

$$
\begin{aligned}
&{}^{C}\mathcal{D}^{\kappa}\mathfrak{h}(t) = b\big(t, \mathfrak{h}(t), \int_0^t f\big(t, \theta, \mathfrak{h}(t-\Delta)\big)d\theta\big) + b_1\big(t, \mathfrak{h}(t), \int_0^t g\big(t, \theta, \mathfrak{h}(t-\Delta)\big)d\theta\big)\frac{dW(t)}{dt},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in \mathcal{J} = [0, \mathfrak{v}],\\
&\mathfrak{h}(t) = \varepsilon(t), \qquad\qquad\qquad\qquad\qquad\qquad t \in [-\tau, 0].
\end{aligned} \tag{1}
$$

where $\kappa \in \left(\frac{1}{2}, 1\right),\ \mathcal{J} = [0, \mathfrak{v}],\ \Delta > 0,\ b, b_1, f, g$ are some suitable functions defined on a separable Hilbert space $\mathcal{D},\ \varepsilon$ is a $\mathcal{G}_0-$measurable $\mathcal{D}-$valued random variable and $W = \{W(t), 0 \leq t\}$ is $\mathcal{Q}-$wiener process on a complete probability space $(\Omega, \mathcal{G}, \mathcal{P})$.

This paper is organized as follows: Section 2 contains the basic notations, definitions, and lemmas. The main results from the existence and uniqueness are developed in Section 3. In Section 4, stability analysis is established.

2. Preliminaries

Consider the two real separable Hilbert spaces with their vector norms and inner products which are expressed by $(\mathcal{D}, ||.||_{\mathcal{D}}, \langle ., . \rangle_{\mathcal{D}})$ and $(\mathcal{K}, ||.||_{\mathcal{K}}, \langle ., . \rangle_{\mathcal{K}})$. Also, $\mathcal{L}(\mathcal{K}, \mathcal{D})$ the space of bounded linear operators from $\mathcal{K}$ into $\mathcal{D}$. We are working with the complete probability space $(\Omega, \mathcal{G}, \mathcal{P})$ that includes a normal filtration $\{\mathcal{G}_t\}_{t \in [0,\mathfrak{v}]}$. The Q-Wiener process on $(\Omega, \mathcal{G}, \mathcal{P})$ is defined by $W(t) = \{W(t), t \geq 0\}$, with the covariance operator $\mathcal{Q}$ satisfying $Tr(\mathcal{Q}) < \infty$. Assume that the existence of a complete orthonormal system $\{\gamma_m\} \in \mathcal{K}$, a bounded sequence of non-negative real numbers $\{\lambda_m\}_{m\in N}$ such that

$$\mathcal{Q}\gamma_m = \lambda_m \gamma_m, \quad \gamma_m \geq 0, \quad m \in N,$$

and a sequence of independent real-valued Brownian motion $\{\kappa_{1m}\}_{m\geq 1}$ such that

$$\langle W(t), \gamma \rangle_{\mathcal{K}} = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \langle \gamma_m, \gamma \rangle \kappa_{1m}(t), \quad \gamma \in \mathcal{K}, \quad t \in [0, \mathfrak{v}]$$

and $\mathcal{G}_t = \mathcal{G}_t^W$ is the σ - algebra induced by $\{W(\theta) : 0 < \theta \leq t\}$.

The space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathcal{K}$ into $\mathcal{D}$ is denoted as $\mathfrak{S}_2^0$, specifically represented as $\mathfrak{S}_2^0(Q^{\frac{1}{2}}\mathcal{K}, \mathcal{D})$. This set incorporates the inner product $\langle \bar{\Phi}, \bar{\phi} \rangle_{\mathfrak{S}_2^0} = Tr[\bar{\Phi} Q \bar{\phi}^*]$. Also, Banach space comprises all continuous functions from the interval $[0, \mathfrak{v}]$ to $\mathfrak{S}_2(\Omega, \mathcal{D})$ that satisfy certain specific conditions.

i.e., $\sup_{t\in[0,\mathfrak{v}]} \mathbb{E}||\mathfrak{h}(t)||^2 < \infty$ is determined as $\mathfrak{B}([0, \mathfrak{v}], \mathfrak{S}_2(\Omega, \mathcal{D}))$. Clearly, it is Banach space with the norm: $||\mathfrak{h}||_{\mathfrak{B}([0,\mathfrak{v}],\mathfrak{S}_2(\Omega,\mathcal{D}))} = \big(\sup_{t\in[0,\mathfrak{v}]} \mathbb{E}||\mathfrak{h}(t)||^2\big)^{\frac{1}{2}}$.

The purpose of this paper is to investigate the existence and uniqueness for the stochastic fractional delay integrodifferential equations with gaussian noise (1) whose the solution takes the form as below:

$$\begin{aligned} \mathfrak{h}(t) = \varepsilon(0) &+ \frac{1}{\Gamma(\kappa)} \int_0^t (t-\theta)^{\kappa-1} b\big(\theta, \mathfrak{h}(\theta), \int_0^\theta f\big(\theta, \tau, \mathfrak{h}(\tau - \Delta)\big) d\tau\big) d\theta \\ &+ \frac{1}{\Gamma(\kappa)} \int_0^t (t-\theta)^{\kappa-1} b_1\big(\theta, \mathfrak{h}(\theta), \int_0^\theta g\big(\theta, \tau, \mathfrak{h}(\tau - \Delta)\big) d\tau\big) dW(\theta) \end{aligned} \tag{2}$$

Definition 2.1. *The fractional integral of a function $\mathfrak{h} : [0, \infty) \to \mathfrak{R}$, with the order κ and zero as its lower limit, is denoted as:*

$$\mathcal{I}^\kappa \mathfrak{h}(t) = \frac{1}{\Gamma(\kappa)} \int_0^t \frac{\mathfrak{h}(\theta)}{(t-\theta)^{1-\kappa}} d\theta, \quad 0 < t, \quad \kappa \in \mathfrak{R}^+,$$

provided that the right hand side is defined pointwise over the interval $[0, \infty)$.

Definition 2.2. [18] *The Riemann-Liouville derivative of a function $\mathfrak{h} : [0, \infty) \to \mathfrak{R}$ with the order κ and zero as its lower limit, is denoted as:*

$${}^{L}\mathcal{D}^\kappa \mathfrak{h}(t) = \frac{1}{\Gamma(n-\kappa)} \frac{d^n}{dt^n} \int_0^t \frac{\mathfrak{h}(\theta)}{(t-\theta)^{1+\kappa-n}} d\theta, \quad 0 < t, \quad n-1 < \kappa < n.$$

Definition 2.3. [18] *The Caputo derivative of a function $\mathfrak{h} : [0, \infty) \to \mathfrak{R}$ with the fractional order κ and zero as its lower limit, is denoted as:*

$${}^{C}\mathcal{D}^\kappa \mathfrak{h}(t) = {}^{L}\mathcal{D}^\kappa \Big[\mathfrak{h}(t) - \sum_{j=0}^{n-1} \frac{t^j}{j!} \mathfrak{h}^j(0)\Big], \quad 0 < t, \quad n > \kappa > n-1.$$

Remark 2.1. (1) *If* $\mathfrak{h}(t) \in \mathfrak{B}^n[0,\infty)$, *then*

$$^{C}\mathcal{D}^{\kappa}\mathfrak{h}(t) = \frac{1}{\Gamma(n-\kappa)}\int\limits_0^t \frac{\mathfrak{h}^n(\theta)}{(t-\theta)^{n-\kappa-1}}d\theta = I^{n-\kappa}\mathfrak{h}^{(n)}(t), \quad 0 < t, \quad n > \kappa > n-1.$$

(2) *Assuming* $\mathfrak{h}$ *is an abstract function taking values in* $\mathcal{D}$, *the integrals in the definitions* (2.1) *and* (2.2) *are understood in the sense of Bochner.*

(3) $^{C}\mathcal{D}^{\kappa}(K) = 0$ (*where* K *is any constant*).

Definition 2.4. (*Mittag - Leffler function*) *The one - parameter and two - parameter Mittag-Leffler function is defined by*

$$E_{\kappa}(y) = \sum_{j=0}^{\infty}\frac{y^j}{\Gamma(\kappa j+1)}, \quad y \in \mathbb{C}, \quad Re(\kappa) > 0.$$

and

$$E_{\kappa,\kappa_1}(y) = \sum_{j=0}^{\infty}\frac{y^j}{\Gamma(\kappa j+\kappa_1)}, \quad y, \kappa_1 \in \mathbb{C}, \quad Re(\kappa) > 0.$$

Definition 2.5. (*Stochastic Process*)

A set containing the collection of random variables $\{\mathcal{Y}(t)|t \geq 0\}$ *is called as stochastic process.*

Definition 2.6. (*Chebyshev's Inequality*) *When* $\mathcal{Y}$ *is considered a random variable and* $1 \leq m < \infty$, *then*

$$\mathbb{P}(|\mathcal{Y}| \geq \lambda) \leq \frac{1}{\lambda^m}\mathbb{E}(|\mathcal{Y}|^m), \quad \textit{for all} \quad 0 < \lambda.$$

Lemma 2.1. (*Borel-Cantelli Lemma:*) *If* $\{S_m\} \subset \mathcal{G}$ *and* $\sum\limits_{m=1}^{\infty}\mathbb{P}(S_m) < \infty$, *then*

$$\mathbb{P}\left(\lim_{m\to\infty} sup\, S_m\right) = 0.$$

Lemma 2.2. *Consider the continuous functions* $\mathfrak{h}, \mathfrak{h}_1 : [0,\mathfrak{v}] \to [0,\infty)$ *and if* $\mathfrak{h}_1$ *is non-decreasing,* $\Delta \geq 0$ *and* $\kappa \geq 0 \ni$

$\mathfrak{h}(t) = \mathfrak{h}_1(t) + \Delta\int\limits_0^t (t-\theta)^{\kappa-1}x(\theta)d\theta, \quad t \in [0,\mathfrak{v}], \quad$ *next*

$\mathfrak{h}(t) = \mathfrak{h}_1(t) + \int\limits_0^t \big[\sum\limits_{n=1}^{\infty}\frac{(\Delta\Gamma(\kappa))^n}{\Gamma(n\kappa)}(t-\theta)^{n\Delta-1}\mathfrak{h}_1(\theta)\big]d\theta, \quad t \in [0,\mathfrak{v}],$

If $\mathfrak{h}_1(t) = c$, *a constant on the interval* $[0,\mathfrak{v}]$, *the inequality above is then simplified to:*

$\mathfrak{h}(t) \leq cE_{\kappa}\big(\Delta\Gamma(\kappa)t^{\kappa}\big), \quad t \in [0,\mathfrak{v}].$

The Mittag-Leffler function E_{κ} *in the instance above is obtained as*

$E_{\kappa}(z) = \sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(n\kappa+1)}, \quad z \in \mathbb{C}, \quad Re(\kappa) \geq 0.$

Lemma 2.3. *A measurable function* $f : [0,\mathfrak{v}] \to \mathcal{D}$ *is Bochner integrable, if* $||f||$ *is the Lebesgue integrable.*

Lemma 2.4. *Let the space* $\Theta = \{\Upsilon(\cdot,\cdot) : \Upsilon$ *is linearly bounded on* $[0,k] \times \Omega \ \ni \ \Upsilon(t)$ *is* $\mathcal{G}_t$ *-measurable* $\forall\, t \in [0,k]\} \ \in \mathrm{R}^n$. *If* $\Upsilon \in \Theta$ *with* $\int_0^k E|\Upsilon(t)|^2 dt < \infty$, *then*

$$\mathbb{E}\Big|\int_0^k \Upsilon(t,W)dW(t)\Big|^2 \leq Tr(\mathcal{Q})\int_0^k \mathbb{E}|\Upsilon(t)|^2 dt.$$

3. Main Result

Before proving the main result, let us assume the following hypotheses to demonstrate the existence and uniqueness of mild solutions for (1)

(H1) $b, b_1 : \mathcal{J} \times \mathcal{D} \times \mathcal{D} \to \mathfrak{S}_2^0$ fulfills:

(a) For all $(\Delta, \mathfrak{h}) \in \mathcal{D} \times \mathcal{D},\ b_1(\cdot, \Delta, \mathfrak{h}) : \mathcal{J} \to \mathfrak{S}_2^0$ is measurable.

(b) Arbitrary $\Delta_1, \Delta_2, \mathfrak{h}_1, \mathfrak{h}_2 \in \mathcal{D}$ fulfilling $\mathbb{E}||\Delta_1||^2,\ \mathbb{E}||\Delta_2||^2,\ \mathbb{E}||\mathfrak{h}_1||^2,\ \mathbb{E}||\mathfrak{h}_2||^2 \leq q$, $\exists\ \mathfrak{s}_b(q), \mathfrak{s}_{b_1}(q) > 0$, such that

(1) $\mathbb{E}||b(t,\Delta_1,\mathfrak{h}_1) - b(t,\Delta_2,\mathfrak{h}_2)||^2 \leq \mathfrak{s}_b(q)(\mathbb{E}||\Delta_1 - \Delta_2||^2 + \mathbb{E}||\gamma_1 - \gamma_2||^2)$,

(2) $\mathbb{E}||b_1(t,\Delta_1,\mathfrak{h}_1) - b_1(t,\Delta_2,\mathfrak{h}_2)||^2 \leq \mathfrak{s}_{b_1}(q)(\mathbb{E}||\Delta_1 - \Delta_2||^2 + \mathbb{E}||\gamma_1 - \gamma_2||^2)$, $\forall\, t \in [0,\mathfrak{v}]$.

(c) There exists $\mathfrak{r}_b, \mathfrak{r}_{b_1} \geq 0$, such that

(1) $\mathbb{E}||b(t,\Delta,\mathfrak{h})||^2 \leq \mathfrak{r}_b(1 + \mathbb{E}||\Delta||^2 + \mathbb{E}||\mathfrak{h}||^2),\ \forall\, \Delta, x \in \mathcal{D},\ t \in [0,\mathfrak{v}]$,

(2) $\mathbb{E}||b_1(t,\Delta,\mathfrak{h})||^2 \leq \mathfrak{r}_{b_1}(1 + \mathbb{E}||\Delta||^2 + \mathbb{E}||\mathfrak{h}||^2),\ \forall\, \Delta, x \in \mathcal{D},\ t \in [0,\mathfrak{v}]$.

(H2) $f,\ g : \mathfrak{J} \times \mathcal{D} \to \mathcal{D}$ fulfills:

(a) $f(t,\theta,\cdot) : \mathcal{D} \to \mathcal{D}$ is continuous $\forall\ (t,\theta) \in \mathfrak{J} = \{(t,\theta) \in \mathcal{D} \to \mathcal{D} | \theta \in [0,\mathfrak{v}]\}$, $g(t,\theta,\cdot) : \mathcal{D} \to \mathcal{D}$ is continuous $\forall\ (t,\theta) \in \mathfrak{J} = \{(t,\theta) \in \mathcal{D} \to \mathcal{D} | \theta \in [0,\mathfrak{v}]\}$.

(b) $f(\Delta,\theta,\cdot),\ g(\Delta,\theta,\cdot) : \mathcal{D} \to \mathcal{D}$ is continuous $\forall(\Delta,\theta) \in \mathfrak{J} = \{(\Delta,\theta) \in \mathcal{J} \times \mathcal{J},\ \theta \in [0,\mathfrak{v}]\}$.

(c) For arbitrary $(\Delta,\theta) \in \mathfrak{J}$, and $\Delta_1, \Delta_2 \in \mathcal{D}$ fulfilling $\mathbb{E}||\Delta_1||^2, \mathbb{E}||\Delta_2||^2 \leq q$, $\exists\ \ \mathfrak{s}_f(q),\ \mathfrak{s}_g(q) > 0$, such that

(1) $\mathbb{E}||f(t,\theta,\Delta_1) - f(t,\theta,\Delta_2)||^2 \leq \mathfrak{s}_f(q)(\mathbb{E}||\Delta_1 - \Delta_2||^2)$,

(2) $\mathbb{E}||g(t,\theta,\Delta_1) - g(t,\theta,\Delta_2)||^2 \leq \mathfrak{s}_g(q)(\mathbb{E}||\Delta_1 - \Delta_2||^2)$.

(d) There exists $\mathfrak{r}_f,\ \mathfrak{r}_g > 0$ such that

(1) $\mathbb{E}||f(t,\theta,\Delta)||^2 \leq \mathfrak{r}_f(1 + \mathbb{E}||\Delta||^2), \quad \forall \quad \Delta \in \mathcal{D}$,

(2) $\mathbb{E}||g(t,\theta,\Delta)||^2 \leq \mathfrak{r}_g(1 + \mathbb{E}||\Delta||^2), \quad \forall \quad \Delta \in \mathcal{D}$.

(H3): There exists $M > 0 \ \ni \ t \geq 0$,

(1) $||E_{\kappa,\kappa_1}(At^{\kappa})|| \leq Me^{-\rho t}$.

Theorem 3.1. (*Existence and Uniqueness*) *Let* $(t,\mathfrak{h})$ *belongs to the set* $\mathcal{J} \times \mathcal{D}$, *where* κ *is in the interval* $(\frac{1}{2}, 1)$, *and where* b *and* b_1 *belong to the set* $\mathcal{D}$. *Also, let* $W = \big\{W(t),\ t \geq 0\big\}$ *be a Q-Wiener process on a complete* *probability space $(\Omega, \mathcal{G}, \mathcal{P})$. Furthermore, assume that the conditions $(H1)$ and $(H2)$ are satisfied.*

Let us define a random variable, denoted as $\varepsilon(0)$, on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$. This random variable is independent of $\sigma-$ algebra $\mathcal{G}_\theta^t \subset \mathcal{G}$, which is generated by the collection $\{W(\theta),\ t \geq \theta \geq 0\}$, and such that $\mathbb{E}\big|\varepsilon(0)\big|^2 < \infty$. Then the initial value problem has a unique solution which is t-continuous with the property that $\mathfrak{h}(t,\omega)$ is adapted to the $\mathcal{G}_t^{\varepsilon_0}$ generated by ε_0 and $\{W(\theta)(\cdot), t \geq \theta\}$ and

$$\sup_{0\leq t\leq \mathfrak{v}} \mathbb{E}||\mathfrak{h}(t)||^2 < \infty. \tag{3}$$

Existence:

Proof. Let us prove the initial value problem (2.1) has a solution. Employing induction, let's define $\mathfrak{h}^0(t) = \varepsilon(0)$ and $\mathfrak{h}^m(t) = \mathfrak{h}^m(t,\omega)$ as follows.

$$\begin{aligned}\mathfrak{h}^{m+1}(t) = \varepsilon(0) &+ \frac{1}{\Gamma(\kappa)}\int_0^t (t-\theta)^{\kappa-1} b\Big(\theta, \mathfrak{h}^m(\theta), \int_0^\theta f\big(\theta,\tau,\mathfrak{h}^m(\tau-\Delta)\big)d\tau\Big)d\theta \\ &+ \frac{1}{\Gamma(\kappa)}\int_0^t (t-\theta)^{\kappa-1} b_1\Big(\theta, \mathfrak{h}^m(\theta), \int_0^\theta g\big(\theta,\tau,\mathfrak{h}^m(\tau-\Delta)\big)d\tau\Big)dW(\theta) \\ &\qquad\qquad for\ m = 0,1,2,\cdots.\end{aligned} \tag{4}$$

If, for a fixed value m (where $m \geq 0$), the approximation $\mathfrak{h}^m(t)$ is measurable with respect to the $\sigma-$ algebra $\mathcal{G}_t$ and exhibits continuity on the set $\mathcal{J}$, then it can be deduced from the conditions (H1) and (H2) that the integral in the equation (4) is well-defined. Consequently, the subsequent process denoted as $\mathfrak{h}^{m+1}$ becomes measurable with respect to the $\sigma-$ algebra $\mathcal{G}_t$ and maintains continuity on the set $\mathcal{J}$.

Since the initial value $\mathfrak{h}^0(t)$ is evidently measurable with respect to $\mathcal{G}_t$ and continuous on $\mathcal{J}$, we can establish by induction that this property holds true for every $\mathfrak{h}^m(t)$ with $m = 0,1,2,\cdots$.

As $\varepsilon(0)$ is $\mathcal{G}_t$ -measurable and $\mathbb{E}||\varepsilon(0)||^2$ is finite and hence

$$\sup_{0\leq t\leq v} \mathbb{E}||\varepsilon(0)||^2 < \infty.$$

Incorporating the Ito isometry, the hypotheses and the Cauchy-Schwartz inequality, we establish from (4) that

$$\begin{aligned}\mathbb{E}||\mathfrak{h}^{m+1}(t)||^2 \leq\ & 3\mathbb{E}||\varepsilon(0)||^2 + \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t ||b\Big(\theta,\mathfrak{h}^m(\theta), \int_0^\theta f\big(\theta,\tau,\mathfrak{h}^m(\tau-\Delta)\big)d\tau\Big)||^2 d\theta \\ &+ \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)} Tr(\mathcal{Q})\int_0^t ||b_1\Big(\theta,\mathfrak{h}^m(\theta), \int_0^\theta g\big(\theta,\tau,\mathfrak{h}^m(\tau-\Delta)\big)d\tau\Big)||^2 d\theta\end{aligned}$$

$$\mathbb{E}||\mathfrak{h}^{m+1}(t)||^2 \leq 3\mathbb{E}||\varepsilon(0)||^2 + \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int\limits_0^t \big[\mathfrak{r}_b\big(1+\mathbb{E}||\mathfrak{h}^m(\theta)||^2\big)$$
$$+ v\int\limits_0^\theta \mathfrak{r}_f\big(1+\mathbb{E}||\mathfrak{h}^m(\tau)||^2\big)d\tau\big]d\theta + \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}Tr(\mathcal{Q})$$
$$\times \int\limits_0^t \big[\mathfrak{r}_{b_1}\big(1+\mathbb{E}||\mathfrak{h}^m(\theta)||^2\big) + v\int\limits_0^\theta \mathfrak{r}_g\big(1+\mathbb{E}||\mathfrak{h}^m(\tau)||^2\big)d\tau\big]d\theta.$$

Hence,

$$\mathbb{E}||\mathfrak{h}^{m+1}(t)||^2 \leq 3\mathbb{E}||\varepsilon(0)|| + \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\big[\mathfrak{r}_b\big(1+\mathfrak{r}_f v^2\big)$$
$$+ Tr(\mathcal{Q})\mathfrak{r}_{b_1}\big(1+\mathfrak{r}_g v^2\big)\big]\int\limits_0^t \big[1+\mathbb{E}||\mathfrak{h}^m(\theta)||^2\big]d\theta, \qquad \text{for } m = 0,1,2,\cdots.$$

By means of induction, we get $\sup\limits_{0\leq t\leq \mathfrak{v}} \mathbb{E}||\mathfrak{h}^m(t)||^2 \leq K_0 < \infty$, for $m = 1,2,3,\cdots$, and K_0 is a positive constant.

Next we claim that

$$\mathbb{E}||\mathfrak{h}^{m+1}(t) - \mathfrak{h}^m(t)||^2 \leq AB^m\frac{(Ct)^{m+1}}{(m+1)!}, \qquad \text{for } m = 0,1,2,\cdots.$$

Depending on some constants A, B and C, where

$$A = \big[\mathfrak{r}_b(1+\mathfrak{r}_f v^2) + Tr(\mathcal{Q})\mathfrak{r}_{b_1}(1+\mathfrak{r}_g v^2)\big]\big[1+\mathbb{E}||\varepsilon_0||^2\big],$$
$$B = \mathfrak{s}_b(q)(1+\mathfrak{r}_f v^2) + Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)(1+\mathfrak{r}_g v^2), \text{ and } C = \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}.$$

Applying the Schwartz inequality, and Ito isometry along with the hypothesis (H1) and (H2) we get,

$$\mathbb{E}||\mathfrak{h}^{m+1}(t) - \mathfrak{h}^m(t)||^2 \leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int\limits_0^t \mathbb{E}\big[\big|\big|b\big(\theta,\mathfrak{h}^m(\theta),\int\limits_0^\theta f(\theta,\tau,\mathfrak{h}^m(\tau-\Delta))d\tau\big)$$
$$- b\big(\theta,\mathfrak{h}^{m-1}(\theta),\int\limits_0^\theta f(\theta,\tau,\mathfrak{h}^{m-1}(\tau-\Delta))d\tau\big)\big|\big|^2\big]d\theta$$
$$+ \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}Tr(\mathcal{Q})\int\limits_0^t \mathbb{E}\big[\big|\big|b_1\big(\theta,\mathfrak{h}^m(\theta),\int\limits_0^\theta g(\theta,\tau,\mathfrak{h}^m(\tau-\Delta))d\tau\big)$$
$$- b_1\big(\theta,\mathfrak{h}^{m-1}(\theta),\int\limits_0^\theta g(\theta,\tau,\mathfrak{h}^{m-1}(\tau-\Delta))d\tau\big)\big|\big|^2\big]d\theta,$$

$$\mathbb{E}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 \leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t \Big[\mathfrak{s}_b(q)\big(\mathbb{E}||\mathfrak{h}^{m}(\theta)-\mathfrak{h}^{m-1}(\theta)||^2$$
$$+v\int_0^\theta \mathfrak{s}_f(q)\mathbb{E}\big|\big|\mathfrak{h}^{m}(\tau)-\mathfrak{h}^{m-1}(\tau)\big|\big|^2 d\tau\big)\Big]d\theta$$
$$+\frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}Tr(\mathcal{Q})\int_0^t \Big[\mathfrak{s}_{b_1}(q)\big(\mathbb{E}||\mathfrak{h}^{m}(\theta)-\mathfrak{h}^{m-1}(\theta)||^2$$
$$+v\int_0^\theta \mathfrak{s}_g(q)\mathbb{E}\big|\big|\mathfrak{h}^{m}(\tau)-\mathfrak{h}^{m-1}(\tau)\big|\big|^2 d\tau\big)\Big]d\theta,$$

$$\mathbb{E}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 \leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\Big[\mathfrak{s}_b(q)\big(1+v^2\mathfrak{s}_f(q)\big)+Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)\big(1+v^2\mathfrak{s}_g(q)\big)\Big]$$
$$\times\int_0^t \mathbb{E}\big|\big|\mathfrak{h}^{m}(\theta)-\mathfrak{h}^{m-1}(\theta)\big|\big|^2 d\theta. \tag{5}$$

for $m=0$,

$$\mathbb{E}||\mathfrak{h}^{1}(t)-\mathfrak{h}^{0}(t)||^2 \leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t \mathbb{E}||b\big(\theta,\mathfrak{h}^0(\theta),\int_0^\theta f(\theta,\tau,\mathfrak{h}^0(\tau-\Delta))d\tau\big)d\theta||^2$$
$$+\frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t \mathbb{E}||b_1\big(\theta,\mathfrak{h}^0(\theta),\int_0^\theta g(\theta,\tau,\mathfrak{h}^0(\tau-\Delta))d\tau\big)dW(\theta)||^2$$

$$\mathbb{E}||\mathfrak{h}^{1}(t)-\mathfrak{h}^{0}(t)||^2 \leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t \mathfrak{r}_b\big(1+\mathbb{E}||\mathfrak{h}^0(\theta)||^2+\mathfrak{r}_f v^2\big[1+\mathbb{E}||\mathfrak{h}^0(\theta)||^2\big]\big)d\theta$$
$$+\frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}Tr(\mathcal{Q})\int_0^t \mathfrak{r}_{b_1}\big(1+\mathbb{E}||\mathfrak{h}^0(\theta)||^2+\mathfrak{r}_g v^2\big[1+\mathbb{E}||\mathfrak{h}^0(\theta)||^2\big]\big)d\theta$$
$$\leq \frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\big[\mathfrak{r}_b(1+\mathfrak{r}_f v^2)+Tr(\mathcal{Q})\mathfrak{r}_{b_1}(1+\mathfrak{r}_g v^2)\big](t)\big[1+\mathbb{E}||\varepsilon_0||^2\big].$$

Now for $m=1$,

$$\mathbb{E}||\mathfrak{h}^{2}(t)-\mathfrak{h}^{1}(t)||^2 \leq \left[\frac{2}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\right]^2\big(\mathfrak{s}_b(q)(1+\mathfrak{r}_f v^2)+Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)(1+\mathfrak{r}_g v^2)\big)$$
$$\times\int_0^t \mathbb{E}\big[||\mathfrak{h}^{1}(t)-\mathfrak{h}^{0}(t)||^2\big]d\theta$$
$$\leq C^2B\big(\mathfrak{r}_b(1+\mathfrak{r}_f v^2)+Tr(\mathcal{Q})\mathfrak{r}_{b_1}(1+\mathfrak{r}_g v^2)\big)\big[1+\mathbb{E}||\varepsilon_0||^2\big]\frac{t^2}{2!}.$$

Proceeding as before, we will have for $m=2$

$$\mathbb{E}||\mathfrak{h}^{3}(t)-\mathfrak{h}^{2}(t)||^2 \leq C^3B^2\big(\mathfrak{r}_b(1+\mathfrak{r}_f v^2)+Tr(\mathcal{Q})\mathfrak{r}_{b_1}(1+\mathfrak{r}_g v^2)\big)\big[1+\mathbb{E}||\varepsilon_0||^2\big]\frac{t^3}{3!}.$$

Thus, using the mathematical induction principle, we obtain

$$\mathbb{E}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 \le AB^m\frac{(Ct)^{m+1}}{(m+1)!}, \qquad \text{for } m=0,1,2,\cdots.$$

where A, B, and C are constants which depend on $\kappa, \upsilon, \mathfrak{s}_b, \mathfrak{s}_{b_1}, \mathfrak{r}_f, \mathfrak{r}_g$ and $\mathbb{E}||\varepsilon_0||$.

Note that

$$\begin{aligned}\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 \le 2\sup_{0\le t\le \mathfrak{v}}\int_0^t (t-\theta)^{\kappa-1}\Big[||b\big(\theta,\mathfrak{h}^m(\theta),\int_0^\theta f(\theta,\tau,\mathfrak{h}^m(\tau-\Delta))d\tau\big)\\
-b\big(\theta,\mathfrak{h}^{m-1}(\theta),\int_0^\theta f(\theta,\tau,\mathfrak{h}^{m-1}(\tau-\Delta))d\tau\big)||^2\Big]d\theta\\
+2\sup_{0\le t\le \mathfrak{v}}\int_0^t (t-\theta)^{\kappa-1}||b_1\big(\theta,\mathfrak{h}^m(\theta),\int_0^\theta g(\theta,\tau,\mathfrak{h}^m(\tau-\Delta))d\tau\big)\\
-b_1\big(\theta,\mathfrak{h}^{m-1}(\theta),\int_0^\theta g(\theta,\tau,\mathfrak{h}^{m-1}(\tau-\Delta))d\tau\big)||^2 dW(\theta).\end{aligned}$$

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2\Big] \le \frac{2}{\Gamma(\kappa)^2}\frac{\upsilon^{2\kappa-1}}{(2\kappa-1)}\big(\mathfrak{s}_b(q)(1+\mathfrak{r}_f\upsilon^2)+Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)(1+\mathfrak{r}_g\upsilon^2)\big)\\
\times\mathbb{E}\Big[\sup_{0\le t\le \mathfrak{v}}\int_0^t \big|\big|\mathfrak{h}^m(\theta)-\mathfrak{h}^{m-1}(\theta)\big|\big|^2 d\theta\Big].\end{aligned}$$

Using the submartingale theorem, it yields

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2\Big] &\le 4BC\int_0^t \mathbb{E}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 d\theta\\
&\le AB^m\frac{(Ct)^{m+1}}{(m+1)!},\end{aligned}$$

where A, B and C are constants depending on $\kappa, \upsilon, \mathfrak{s}_b, \mathfrak{s}_{b_1}, \mathfrak{s}_f, \mathfrak{s}_g, \mathfrak{r}_b, \mathfrak{r}_{b_1}, \mathfrak{r}_f$, and $\mathfrak{r}_g$.

With the help of Chebyshev's inequality, it yields

$$P\Big[\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 > \frac{1}{K^2}\Big] \le \frac{1}{\big(\frac{1}{K^2}\big)^2}\mathbb{E}\Big[\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2\Big].$$

Using the above two inequalities and summing up the results, we get

$$\sum_{k=0}^{\infty}\big(P\big(\sup_{0\le t\le \mathfrak{v}}||\mathfrak{h}^{m+1}(t)-\mathfrak{h}^{m}(t)||^2 > \frac{1}{K^2}\big)\big) \le \sum_{k=0}^{\infty}AB^m\frac{K^4(Ct)^{m+1}}{(m+1)!},$$

where the convergence of the *RHS* series is demonstrated by the ratio test.

Since the series on the left hand side converges, we may deduce from the Borel-Cantelli lemma that $\sup_{0\le t\le \mathfrak{v}} ||\mathfrak{h}^{m+1}(t) - \mathfrak{h}^{m}(t)||^2$ very likely converges to 0 (i.e., the successive approximations $\mathfrak{h}^m(t)$ converge uniformly and almost certainly on $\mathcal{J}$) at a limit $\mathfrak{h}(t)$ given by

$$\lim_{n\to\infty}\big(\mathfrak{h}^0(t)+\sum_{k=1}^{n}\big[\mathfrak{h}^m(t)-\mathfrak{h}^{m-1}(t)\big]\big)=\lim_{n\to\infty}\mathfrak{h}^n(t)=\mathfrak{h}(t)$$

Hence from equation (2),

$$\mathfrak{h}(t)=\varepsilon(0)+\frac{1}{(\Gamma(\kappa))}\int_0^t(t-\theta)^{\kappa-1}b\big(\theta,\mathfrak{h}(\theta),\int_0^\theta f\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\big)d\theta$$
$$+\frac{1}{(\Gamma(\kappa))}\int_0^t(t-\theta)^{\kappa-1}b_1\big(\theta,\mathfrak{h}(\theta),\int_0^\theta g\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\big)dW(\theta)\qquad\forall\, t\in\mathcal{J}.$$

Hence the existence of a solution is obtained. □

Uniqueness:

Proof. Let the two solutions be $\mathfrak{h}_1(t,\omega)$ and $\mathfrak{h}_2(t,\omega)$ with $(0,\varepsilon(0))=\varepsilon(0)(\omega)$ and $(0,\varphi(0))=\varphi(0)(\omega)$, $\omega\in\Omega$. Then as a result of Itô isometry and Schwartz inequality, we have

$$\begin{aligned}\mathbb{E}\big|\big|\mathfrak{h}_1(t)-\mathfrak{h}_2(t)\big|\big|^2\le\ & 3\mathbb{E}\big|\big|\varepsilon(0)-\varphi(0)\big|\big|^2\\
&+\frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\int_0^t\mathbb{E}\big(\big|\big|b\big(\theta,\mathfrak{h}_1(\theta),\int_0^\theta f(\theta,\tau,\mathfrak{h}_1(\tau-\Delta))d\tau\big)\\
&-b\big(\theta,\mathfrak{h}_2(\theta),\int_0^\theta f(\theta,\tau,\mathfrak{h}_2(\tau-\Delta))d\tau\big)\big|\big|^2\big)d\theta\\
&+\frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{2\kappa-1}Tr(\mathcal{Q})\int_0^t\mathbb{E}\big(\big|\big|b_1\big(\theta,\mathfrak{h}_1(\theta),\int_0^\theta g(\theta,\tau,\mathfrak{h}_1(\tau-\Delta))d\tau\big)\\
&-b_1\big(\theta,\mathfrak{h}_2(\theta),\int_0^\theta g(\theta,\tau,\mathfrak{h}_2(\tau-\Delta))d\tau\big)\big|\big|^2\big)d\theta,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\big|\big|\mathfrak{h}_1(t)-\mathfrak{h}_2(t)\big|\big|^2\le&\frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\big(\mathfrak{s}_b(q)\big(1+v^2\mathfrak{s}_f(q)\big)+Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)\big(1+v^2\mathfrak{s}_g(q)\big)\big)\\
&\times\int_0^t\mathbb{E}\big|\big|\mathfrak{h}_1(\theta)-\mathfrak{h}_2(\theta)\big|\big|^2d\theta.\end{aligned}$$

We state $h(t)=\mathbb{E}\big|\big|\mathfrak{h}_1(t)-\mathfrak{h}_2(t)\big|\big|^2$.

Subsequently the function h fulfills $h(t) \leq X + Y \int_0^t h(\theta)d\theta$, where $X = 3\mathbb{E}\big|\big|\varepsilon(0) - \varphi(0)\big|\big|^2$ and $Y = \frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{(2\kappa-1)}\big(\mathfrak{s}_b(q)\big(1+v^2\mathfrak{s}_f(q)\big) + Tr(\mathcal{Q})\mathfrak{s}_{b_1}(q)\big(1+v^2\mathfrak{s}_g(q)\big)\big)$.

By means of applying Gronwall inequality, we found that

$$h(t) \leq X exp(Yt).$$

Let $\varepsilon(0) = \varphi(0)$. Then X = 0 and hence h(t) = 0 $\forall\, t \geq 0$. i.e.,

$$\mathbb{E}\big|\big|\mathfrak{h}_1(t) - \mathfrak{h}_2(t)\big|\big|^2 = 0.$$

Therefore,

$$\int_0^2 \big|\big|\mathfrak{h}_1(t) - \mathfrak{h}_2(t)\big|\big|^2 dP = 0.$$

Hence $\mathfrak{h}_1(t) = \mathfrak{h}_2(t)$ a.s. $\forall\, t \in \mathcal{J}$. i.e.,

$$P\big\{||\mathfrak{h}_1(t,\omega) - \mathfrak{h}_2(t,\omega)|| = 0 \quad \forall\, t \in \mathcal{J}\big\} = 1.$$

Hence the uniqueness is proved for the given stochastic fractional delay integrodifferential equation. □

4. Stability Analysis

In this part, we examine the quadratic mean of a trivial solution's exponentially asymptotic stability. Think about the stochastic fractional nonlinear system described below.

$$\begin{aligned} {}^{C}D^{\kappa}\mathfrak{h}(t) &= b\big(t, \mathfrak{h}(t), \int_0^t f\big(t,\theta,\mathfrak{h}(\theta-\Delta)\big)\big)d\theta + b_1\big(t, \mathfrak{h}(t), \int_0^t g\big(t,\theta,\mathfrak{h}(\theta-\Delta)\big)\big)dW(\theta), \\ &\qquad t \in \mathcal{J} = [0, \mathfrak{v}], \\ \mathfrak{h}(t) &= \varepsilon(0), \quad t \in [-\Delta, 0], \end{aligned} \tag{6}$$

where $\kappa \in \big(\frac{1}{2}, 1\big)$, $b,\ b_1 \in \big(\mathcal{J}\times\mathcal{D}\times\mathcal{D},\ \mathcal{D}\big)$, and $W = \big\{W(t),\ t \geq 0\big\}$ is a Q-Wiener process on a complete probability space $(\Omega, \mathcal{G}, \mathcal{P})$. From this point onward, lets consider that $b(t,0,0) = b_1(t,0,0) \equiv 0$ for almost every t. Consequently, equation (6) possesses trivial solution.

Definition 4.1. *If there are constants K and r such that the trivial solution to equation (6) is exponentially stable in the quadratic mean, then*

$$E(||\mathfrak{h}(t)||^2) \leq K\mathbb{E}\big(||\varepsilon(0)||^2\big)exp(-rt),\ t \geq 0,$$

where $1 > \kappa > 0$, *and* $\kappa_1 = 1, 2$, *and* κ.

Lemma 4.1. *Supposing that the hypothesis (H3) is valid, for any stochastic process $F:[0,\infty)\to\mathcal{D}$ which is strongly measurable with $\int_0^r \mathbb{E}||F(t)||^2 d\theta<\infty,\ 0<\mathfrak{v}\leq\infty$, the below inequality is true on $(0,\mathfrak{v}]$,*

$$\mathbb{E}||\int_0^t E_{\kappa,\kappa_1}\big(A(t-\theta)^\kappa\big)F(\theta)d\theta||^2\leq\big(M^2/a\big)\int_0^t exp\big(-\rho(t-\theta)\big)\mathbb{E}||F(\theta)||^2d\theta,$$

where $\kappa\in(1/2,1)$ and $\kappa_1=1,2$ and κ.

Proof. Assume that the hypothesis (H3) holds; then $\exists$ a constant $\rho>0$ and $M>0\ni$ for $t\geq 0$,

$$||E_{\kappa,\kappa_1}\big(At^\kappa\big)||\leq Me^{-\rho t},\ \text{where } 1>\kappa>0 \text{ and } \kappa_1=1,2 \text{ and } \kappa.$$

Using Holder's inequality, we get $r\geq t>0$,

$$\begin{aligned}\mathbb{E}||\int_0^t E_{\kappa,\kappa_1}\big(A(t-\theta)^\kappa\big)F(\theta)d\theta||^2 &\leq \mathbb{E}\Big(\int_0^t Mexp\big(-(\rho/2)(t-\theta)\big)\\ &\quad\times exp\big(-(\rho/2)(t-\theta)\big)||F(\theta)||d\theta\Big)^2\\ &\leq \mathbb{E}\Big(\int_0^t Mexp\big(-(\rho/2)(t-\theta)\big)d\theta\Big)^2\\ &\quad\times\mathbb{E}\Big(\int_0^t exp\big(-(\rho/2)(t-\theta)\big)||F(\theta)||d\theta\Big)^2,\end{aligned}$$

$$\mathbb{E}||\int_0^t E_{\kappa,\kappa_1}\big(A(t-\theta)^\kappa\big)F(\theta)d\theta||^2\leq(M^2/\rho)\Big(\int_0^t exp\big(-\rho(t-\theta)\big)\mathbb{E}||F(\theta)||d\theta\Big)^2.$$

Thus, the lemma is established. □

Lemma 4.2. *Supposing that the hypothesis (H3) is valid, then for any B_t- adapted predictable process $\varphi:[0,\infty)\to\mathcal{D}$ with $\int_0^t\mathbb{E}||\varphi(\theta)||^2d\theta<\infty,\ 0\leq t$, the below inequality is true on $(0,\mathfrak{v}]$,*

$$\mathbb{E}||E_{\kappa,\kappa_1}\big(A(t-\theta)^\kappa\big)\varphi(\theta)dW(\theta)||^2\leq M^2\int_0^t exp\big(-\rho(t-\theta)\big)\mathbb{E}||\varphi(\theta)||^2d\theta,\quad \frac{1}{2}<\kappa<1,$$

where $\kappa_1=1,2$ and κ.

Theorem 4.1. *Assuming that the conditions of theorem (3.1) are met, the solution of the equation (6) is exponentially stable in the quadratic mean provided*

$$\Theta\big(\rho,\mathfrak{r}_b,\mathfrak{r}_{b_1},\mathfrak{r}_f,\mathfrak{r}_g,M\big)=\frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{2\kappa-1}M^2\big[\big(\mathfrak{r}_b/\rho\big)\big(1+\mathfrak{r}_fv^2\big)+Tr(\mathcal{Q})\mathfrak{r}_{b_1}\big(1+\mathfrak{r}_gv^2\big)\big].$$

Proof. The solution of the equation (6) takes the form:

$$\mathfrak{h}(t) = E_\kappa\big(At^\kappa\big)\varepsilon(0) + \frac{1}{(\Gamma(\kappa))}\int_0^t (t-\theta)^{\kappa-1}E_{\kappa,\kappa}\big[A(t-\theta)^\kappa\big]b\Big(\theta,\mathfrak{h}(\theta),\int_0^\theta f\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\Big)d\theta + \frac{1}{(\Gamma(\kappa))}\int_0^t (t-\theta)^{\kappa-1}E_{\kappa,\kappa}\big[A(t-\theta)^\kappa\big]b_1\Big(\theta,\mathfrak{h}(\theta),\int_0^\theta g\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\Big)dW(\theta).$$

By using Holder's inequality and lemmas (4.1) and (4.2),

$$\mathbb{E}||\mathfrak{h}(t)||^2 \leq 3M^2exp\big(-at\big)\mathbb{E}\big|\big|\varepsilon(0)\big|\big|^2 + \frac{3}{(\Gamma(\kappa))^2}\frac{M^2}{a}\frac{v^{2\kappa-1}}{2\kappa-1}\int_0^t exp\big[-a(t-\theta)^\kappa\big] \times \mathbb{E}||b\Big(\theta,\mathfrak{h}(\theta),\int_0^\theta f\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\Big)||^2d\theta + \frac{3}{(\Gamma(\kappa))^2}M^2\frac{v^{2\kappa-1}}{2\kappa-1}Tr(\mathcal{Q})\int_0^t exp\big[-a(t-\theta)^\kappa\big] \times \mathbb{E}||b_1\Big(\theta,\mathfrak{h}(\theta),\int_0^\theta g\big(\theta,\tau,\mathfrak{h}(\tau-\Delta)\big)d\tau\Big)||^2d\theta.$$

Using hypothesis (H3) and for $b(t,0,0) = b_1(t,0,0) \equiv 0$ a.e. t gives,

$$\begin{aligned} exp(\rho t)\mathbb{E}||\mathfrak{h}(t)||^2 \leq\ & 3M^2\mathbb{E}\big|\big|\varepsilon(0)\big|\big|^2 \\ & + \frac{3}{(\Gamma(\kappa))^2}\frac{M^2}{\rho}\frac{v^{2\kappa-1}}{2\kappa-1}\big[\mathfrak{r}_b\big(1+\mathfrak{r}_f v^2\big)\big]\int_0^t exp(\rho\theta)\mathbb{E}||\mathfrak{h}(\theta)||^2d\theta \\ & + \frac{3}{(\Gamma(\kappa))^2}M^2\frac{v^{2\kappa-1}}{2\kappa-1}Tr(\mathcal{Q})\big[\mathfrak{r}_{b_1}\big(1+\mathfrak{r}_g v^2\big)\big]\int_0^t exp(\rho\theta)\mathbb{E}||\mathfrak{h}(\theta)||^2d\theta \\ \leq\ & 3M^2\mathbb{E}||\varepsilon(0)||^2 + \frac{3}{(\Gamma(\kappa))^2}M^2\frac{v^{2\kappa-1}}{2\kappa-1} \\ & \times\big[(\mathfrak{r}_b/a)\big(1+\mathfrak{r}_f v^2\big) + Tr(\mathcal{Q})\mathfrak{r}_{b_1}\big(1+\mathfrak{r}_g v^2\big)\big]\int_0^t exp(\rho\theta)\mathbb{E}||\mathfrak{h}(\theta)||^2. \end{aligned}$$

With the help of Gronwall's inequality, the above inequality becomes,

$$exp(\rho t)\mathbb{E}||\mathfrak{h}(t)||^2 \leq 3M^2\mathbb{E}||\varepsilon(0)||^2exp\Big[\frac{3}{(\Gamma(\kappa))^2}\frac{v^{2\kappa-1}}{2\kappa-1}M^2\big[\big(1+\mathfrak{r}_f v^2\big)\frac{\mathfrak{r}_b}{\rho} + Tr(\mathcal{Q})\mathfrak{r}_{b_1}\big(1+\mathfrak{r}_g v^2\big)\big]t\Big].$$

As a result, $\mathbb{E}||\mathfrak{h}(t)||^2 \leq K\mathbb{E}||\varepsilon(0)||^2exp(-rt),\ t\geq 0$ where $r=\rho-\kappa_1$ and $K=3M^2$. □

5. Conclusion

This study delves into the analysis of stochatic fractional delay integrodifferential equations under the influence of Gaussian noise. Utilizing the Picard-Lindelof successive approximation scheme, the research establishes the existence and uniqueness of solutions. Additionally, the stability of these solutions is demonstrated through the application of the Mittag-Leffler function. These findings contribute valuable insights into the dynamics and behaviour of complex systems governed by stochastic fractional equations, furthering our understanding or their mathematical properties and potential applications.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R.P. Agarwal, S. Hristova, Boundary value problem for impulsive delay fractional differential equations with several generalized proportional Caputo fractional derivatives, Fractal Fract. 7 (2023), 396. https://doi.org/10.3390/fractalfract7050396.

[2] A. Anguraj, M.L. Maheswari, Existence of solutions for fractional impulsive neutral functional infinite delay integrodifferential equations with nonlocal conditions, J. Nonlinear Sci. Appl. 05 (2012), 271–280. https://doi.org/10.22436/jnsa.005.04.03.

[3] A. Anguraj, K. Ravikumar, J.J. Nieto, On stability of stochastic differential equations with random impulses driven by Poisson jumps, Stochastics. 93 (2020), 682–696. https://doi.org/10.1080/17442508.2020.1783264.

[4] K. Aslam, A.A. Zafar, N.A. Shah, B. Almutairi, MHD free convection flows for maxwell fluids over a porous plate via novel approach of Caputo fractional model, Symmetry. 15 (2023), 1731. https://doi.org/10.3390/sym15091731.

[5] K. Balachandran, K. Sumathy, J.K. Kim, Existence and stability of solutions of general stochastic integral equations, Nonlinear Funct. Anal. Appl. 12 (2007), 219–235.

[6] P. Balasubramaniam, P. Tamilalagan, The solvability and optimal controls for impulsive fractional stochastic integro-differential equations via resolvent operators, J. Optim. Theory Appl. 174 (2016), 139–155. https://doi.org/10.1007/s10957-016-0865-6.

[7] K. Diethelm, N.J. Ford, Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002), 229–248. https://doi.org/10.1006/jmaa.2000.7194.

[8] S. He, D. Vignesh, L. Rondoni, S. Banerjee, Chaos and firing patterns in a discrete fractional Hopfield neural network model, Nonlinear Dyn. 111 (2023), 21307–21332. https://doi.org/10.1007/s11071-023-08972-z.

[9] A. Fu, Optimal research in piano shape sound and sound propagation model based on nonlinear differential equations, Appl. Math. Nonlinear Sci. 8 (2022), 1135–1144. https://doi.org/10.2478/amns.2022.2.0100.

[10] M. Johnson, V. Vijayakumar, An Analysis on the Optimal Control for Fractional Stochastic Delay Integrodifferential Systems of Order $1 < \gamma < 2$, Fractal Fract. 7 (2023), 284. https://doi.org/10.3390/fractalfract7040284.

[11] R. Kazakevičius, A. Kononovicius, B. Kaulakys, V. Gontis, Understanding the nature of the long-range memory phenomenon in socioeconomic systems, Entropy. 23 (2021), 1125. https://doi.org/10.3390/e23091125.

[12] AA Kilbas, HM Srivastava, JJ Trujillo, Theory and applications of fractional differential equations, Elsevier, Amsterdam, (2006).

[13] S. Kumar, Mild solution and fractional optimal control of semilinear system with fixed delay, J. Optim. Theory Appl. 174 (2015), 108–121. https://doi.org/10.1007/s10957-015-0828-3.

[14] F. Mainardi, Fractional calculus: theory and applications, Mathematics. 6 (2018), 145. https://doi.org/10.3390/math6090145.

[15] S. Noor, B.M. Alotaibi, R. Shah, S.M.E. Ismaeel, S.A. El-Tantawy, On the solitary waves and nonlinear oscillations to the fractional Schrödinger-KdV equation in the framework of the Caputo operator, Symmetry. 15 (2023), 1616. https://doi.org/10.3390/sym15081616.

[16] B. Øksendal, Stochastic differential equations, Springer, Berlin, Heidelberg, 1995. https://doi.org/10.1007/978-3-662-03185-8.

[17] R. Sakthivel, R. Ganesh, S. Suganya, Approximate controllability of fractional neutral stochastic system with infinite delay, Rep. Math. Phys. 70 (2012), 291–311. https://doi.org/10.1016/s0034-4877(12)60047-0.

[18] P. Umamaheswari, K. Balachandran, N. Annapoorani, Existence of solutions of stochastic fractional integrodifferential equations, Discontin. Nonlinear. Complex. 7 (2018), 55–65. https://doi.org/10.5890/dnc.2018.03.005.

[19] P. Umamaheswari, K. Balachandran, N. Annapoorani, On the solution of stochastic fractional integrodifferential equations, Nonlinear Funct. Anal. Appl. 22 (2017), 335–354.

[20] P. Umamaheswari, K. Balachandran, N. Annapoorani, Existence and stability results for stochastic fractional delay differential equations with gaussian noise, Math. Model. Optim. Anal. Numer. Solutions. (2020), 399–412. https://doi.org/10.1007/978-981-15-0928-5_20.

[21] J. Wang, Y. Zhou, M. Medved, On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay, J. Optim. Theory Appl. 152 (2011), 31–50. https://doi.org/10.1007/s10957-011-9892-5.

[22] Y. Zhou, J. Wang, L. Zhang, Basic theory of fractional differential equations, 2nd ed. World Scientific, Singapore, 2016. https://doi.org/10.1142/10238.