Asia Pac. J. Math. 2024 **11**:28

ON REDUCTION THEOREMS FOR QTAG-MODULES

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Received Sep. 23, 2023

Abstract. For any $QTAG$-module M, we consider pairs $\{S_1, S_2\}$ of submodules of M such that S_2 is maximal with respect to the property $S_1 \cap S_2 = \{0\}$, and in some special cases we settle the question for certain kind of submodules, thus S_2 is not h-pure submodule of M. In addition, some interesting properties regarding center of h-purity are obtained. Moreover, some characterizations of maximality of submodules are investigated.

2020 Mathematics Subject Classification. 06F25; 13C05; 16K20.

Key words and phrases. QTAG-modules; N-high submodules; h-purity; center of h-purity.

1. Introduction and Fundamentals

As it is well-known, module theory can only be processed by generalizing the theory of abelian groups that provide novel viewpoints of various structures for torsion abelain groups. The theory of torsion abelian groups is significant as it generates the natural problems in $QTAG$-module theory. The notion of $QTAG$ (torsion abelian group like) module is one of the most important tool in module theory. Its importance lies behind the fact that this module can be applied in order to generalized torsion abelian group accurately. Significant work on $QTAG$-module was produced by many authors, concentrating in establishing when torsion abelian groups are actually $QTAG$-modules. In 1976, Singh [18] began his investigations into the torsion abelian groups or TAG-modules over an arbitrary (associative, unitary) ring R, defined by satisfying the following two conditions.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d\,(U_1/N) \leq d\,(U_2/\phi(N))$ holds.

DOI: 10.28924/APJM/11-28

It was shown that, for almost all applications, one of these conditions was not needed; ignoring this nearly superfluous condition, the slightly more general concept of a $QTAG$ module was initiated by the same author in [19]. Since then, many forms of this notion have been defined and studied by many authors. For instance, in a subsequent series of articles, the authors [14] have explored the same kind of developments for nice bases of $QTAG$-modules, and extended some results analogous to abelian groups from [1,3]. Likewise, in [7] the author also have investigate the criteria of exposing finitely generated submodules of $QTAG$-modules and obtained the results for the $QTAG$-module by generalizing [2, Problem 1-2]. The present work, then, translates a few of the ideas of the abelian p-groups from [16] over to the theory of modules with the above condition (i) only. It is fairly to note that many results in $QTAG$-modules are valid to the earlier results of TAG-modules [17].

We begin by reviewing some terminology. Rings considered here are with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules; our notations and terminology are standard and may be found in the texts [4,5]. A module M over a ring R is called uniserial if it has a unique decomposition series of finite length. A module M is called uniform if intersection of any two of its non-zero submodules is non-zero. An element x in M is called uniform if xR is a non-zero uniform (hence uniserial) module. For any module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element x of M, its exponent $e(x)$ is defined to be equal to the decomposition length $d(xR)$. For any $0 \neq x \in M$, $H_M(x)$ (the height of x in M) is defined by $H_M(x) = \sup\{d(yR/xR) : y \in M, x \in yR$ and y uniform $\}$. For $k \geq 0$, $H_k(M) = \{x \in M \mid H_M(x) \geq k\}$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k.

Next, we review the following concepts. The set of modules $\{H_k(M)\}_{k=0,1,\ldots,\infty}$ forms a base for the neighbourhood system of zero. This gives rise to a topology known as h-topology. The closure of a submodule $S \subset M$ is defined as $\overline{S} = \cap_{k=0}^{\infty} (S + H_k(M))$ and it is closed with respect to h-topology if $\overline{S} = S$. The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$. For any $k \geq 0, Soc^k(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{k+1}(M)/Soc^k(M) = Soc\left(M/Soc^k(M)\right)$.

Moreover, we add some basic definitions as well from [11]. The module M is called h-divisible if $H_1(M) = M$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. A submodule S of M is h-pure in M if $S \cap H_k(M) = H_k(S)$, for every integer $k \geq 0$. A submodule S of M is h-neat in M if $S \cap H_1(M) = H_1(S)$.

2. Main Results

The concept of high submodules of $QTAG$-modules was introduced by Khan in [10]: A submodule S_1 of M is high if it is maximal with respect to having zero intersection with $H_\omega(M)$. Mehdi [13] and, subsequently, Hasan [6] have also consider and give further results concerning these high submodules: A submodule S_2 of M maximal with respect to disjointness from S_1 is called a S_1-high submodule of M, or S_1-high in M. This kind of S_1-high submodules has been widely investigated in [8] and discussed its properties. Among the properties of S_1-high submodule is their h-purity in M. One of first questions, namely, for which submodules S_1 it is clear that all S_1-high submodules are h-pure, was recently unanswered. This question has been investigated in [9] to characterizing submodules S_1 of M such that all S_1-high submodules are ℓ-imbedded, in particular, h-pure in M. Our global aim here is to establish in this direction some new characterizations of such high submodules, and thereby to examine some assertions in the light of h-purity as well as maximality of these high submodules.

So, we are giving our first main characterization.

Theorem 2.1. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that S_2 is S_1-high in M. Then either for $t \geq 0$ the equality $S_2 \cap H_t(M) = H_t(S_2)$ hold or $\exists$ uniform elements $u \in S_2, v \in Soc(S_1)$ such that $H_M(u) = H_M(v) < H_M(v-u)$.*

Proof. Assume that $S_2 \cap H_t(M) \neq H_t(S_2)$. Then there exist equations $tc = x$ for some $c \in M, x \in S_2$ and t is an integer. But observing that these equations have no solution for $c \in S_2$. Among all such equations, let $ty = z$ be one for which t is least positive integer for some $y \in M$, $z \in S_2$. By the h-neatness of S_2 and $t > 1$, we have $d(yR/aR) = t$ for some $a \in S_2$, so that $H_1((y' - a)R) = 0$ where $d(yR/y'R) = t - 1$. Thus, $y' - a \in Soc(M)$ where $d(yR/y'R) = t - 1$. Since $Soc(M) = Soc(S_2) \oplus Soc(S_1)$, we get that $y' - a = u + v$, for some $u \in Soc(S_2), v \in Soc(S_1)$.

Let us assume now that $H_M(v) \geq t - 1$. then $v = w'$ for some $w \in M$ and $d(wR/w'R) = t - 1$, which in tern, yields that $a + u = H_{t-1}((y - w)R) = b'$ for some $b \in S_2$ and $d(bR/b'R) = t - 1$. This gives that $y' = b' + v$ with $d(yR/y'R) = d(bR/b'R) = t - 1$ or, $y' = b' = z$ with $d(yR/y'R) = d(bR/b'R) = t$, which is contrary to the choice of t. Therefore, $H_M(v) < t - 1$ and $y' - (u + a) = v$. If $u = -(u + a)$, we have

$$H_M(u) = H_M(v) < H_M(v - u),$$

and the result follows. □

Now, we proceed by proving

Corollary 2.1. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that S_2 is S_1-high in M. If either $S_2 \subseteq H_1(M)$ or $Soc^t(S_1) = H_1(Soc^t(S_1))$ for some t, then $S_2 \cap H_t(M) = H_t(S_2)$.*

Proof. By h-neatness of S_2 and $S_2 \subseteq H_1(M)$, we have $S_2 = H_1(S_2)$. Thus, for each $u \in S_2$ and $v \in Soc^t(S_1)$, either $H_M(u) = \infty$ or, $H_M(v) = \infty$. We are done. □

The following, then, reformulates Theorem 2.1.

Proposition 2.1. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that $\overline{M} = \overline{S_1}$ and let S_2 be S_1-high in M. If S_2 is not h-pure in M, then there exist uniform elements $u \in S_2, v \in Soc\,(S_1)$ such that $H_M(u) = 0 = H_M(v) < H_M(v - u)$.*

Proof. In virtue of Theorem 2.1, $(S_2 + y) \cap S_1 \neq 0$ for some $y \in M$. Then there exist nonzero uniform elements of S_1 of the form $x + ty$ and t a positive integer. Let $t_1 > 0$ such that $u + t_1 y \in S_1, u + t_1 y \neq 0$ for some $u \in S_2$.

Since $v = u + y'$ where $d\,(yR/y'R) = t_2$ for some $t_2 < t$, we have $v \in Soc\,(S_1)$ and $H_M(u) = H_M(v) < H_M(v - u) = t_2$. Now, $H_{t-1}\,(y') = z$ where $d\,(yR/y'R) = 1$ for some $z \in S_2$, so that by choice of z there exists $w \in S_2$ such that $H_{t-1}(wR) = z = H_t(yR)$. Hence $H_{t-1}\,((y' - w)\,R) = 0$ where $d\,(yR/y'R) = 1$ and $t - 1 > 0$. By using equality $\overline{M} = \overline{S_1}$, we obtain that $y' - w \in S_1$ where $d\,(yR/y'R) = 1$. It is plainly observe that $y' - w \neq 0$ where $d\,(yR/y'R) = 1$, so that $t_2 = 1$. This gives that $H_M(u) = H_M(v) = 0$, as wanted. □

Analogous to center of purity in abelian group [15], it was given the following notion in [12].

Definition 2.1. *A submodule S of a QTAG-module M is called a center of h-purity in M if every S-high submodule of M is h-pure in M.*

Now, we will verify the validity of the following proposition.

Proposition 2.2. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that S_2 is S_1-high in M. If ϕ is a homomorphism on M such that*
(i) $Soc^t\,(S_1) \subset ker\phi$ for some t;
(ii) $H_M(u) = H_M(\phi(v))$ for all $u \in S_2$.
Then $S_2 \cap H_t(M) = H_t\,(S_2)$.

Proof. For $u \in S_2$ and $v \in S_1$, we have

$$H_M(u) = H_M(\phi(u)) = H_M(\phi(u - v)) \geq H_M(u - v) = H_M(v - u),$$

so that the condition in Theorem 2.1 alternative to h-purity of S_2 cannot hold. Hence $S_2 \cap H_t(M) = H_t\,(S_2)$, and we are done. □

The following corollary extends [12, Theorem 6] to $QTAG$-modules.

Corollary 2.2. *Let M be a QTAG-module M with $H_\infty(M) = \cap_t H_t(M)$ for some t, and $H_{\infty+1}(M) = 0$. Then any submodule S of M such that $H_k(M) \supseteq S \supseteq H_{k+1}(M)$ for some k, $0 \leq k \leq \infty$ is a center of h-purity in M.*

Proof. Let $\phi : M \to M/S$ be the canonical homomorphism, so that $H_{k+1}(M/S) = 0$, and hence $H_M(\phi(u)) \leq k$ for all $u \in M, u \notin S$. Suppose that $H_n(\phi(v)) = \phi(u)$ for some integer n and $u \notin S$. Then $H_n(v) + w = u$ for some $w \in S$. Since $S \subseteq H_k(M)$, $n \leq k$, and for $n < \infty, \exists\, x \in M$ such that $x' = w$ where $d\,(xR/x'R) = n$. Hence, $y' = u$ where $d\,(yR/y'R) = n$ and $y = v + x$. Thus $H_M(u) \geq H_M(\phi(u))$. By hypothesis, $H_M(u) \leq H_M(\phi(u))$. Therefore, $H_M(u) = H_M(\phi(u))$ for all $u \in M, u \notin S$. The proof is over. □

Next, we concentrate on the following easy observation.

Lemma 2.1. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that $S_2 \subseteq S_1$ and let S_2 be the maximal submodule of M. Then S_1 is a center of h-purity in M if and only if for all $u \in M, uR \cap S_1 = 0$ and $H_M(u) = 0$ imply $H_M(u + v) = 0$ for all $v \in S_2$.*

Proof. Suppose S_1 satisfies the condition. Then by Proposition 2.1, S_1 is a center of h-purity in M.

Next, we deal with the converse implication. Then there exists a submodule S_3 of M maximal disjoint from S_1 containing u. For $v \in S_2$, let us assume that $H_M(v) > 0$, it is clear that $H_M(u+v) = H_M(u) = 0$. We next assume that $v \in S_2$ and $H_M(v) = 0$, then $v = v_1 + v_2$ where $e\,(v_1) = k$ for some $k \geq 0$ and $e\,(v_2) = 1$. Therefore $H_M\,(v_2) = \infty$, and hence $H_M(v) = H_M\,(v_1) = 0$.

In this connection, observe that $H_M(u+v) = H_M\,(u + v_1) \geq n$ (say). Let $w \in M$ such that $w' = u+v_1$ where $d\,(wR/w'R) = n$. Then $w' = u'$ where $d\,(wR/w'R) = t$, $d\,(uR/u'R) = k$ and $t = n + k$. From the h-pureness of $S_3, \exists\, a \in S_3$ such that $a' = u'$ where $d\,(aR/a'R) = t$ and $d\,(uR/u'R) = k$. Hence $H_k\,((a' - u)\,R) = 0$ where $d\,(aR/a'R) = n$. Since $S_2 \subseteq S_1$ and $S_3 \cap S_1 = 0$, we obtain $a' = u$ where $d\,(aR/a'R) = n$. Therefore, by hypothesis on u, we have $n = 0$. Thus. $H_M\,(u + v_1) = H_M(u + v) = 0$, as desired. □

Motivated by center of h-purity, we introduce the following.

Definition 2.2. *A submodule S of a QTAG-module M containing maximal submodule is called a special center of h-purity in M if S is a center of h-purity and $\exists\, u \in M$ such that $u \neq 0, uR \cap S = 0$.*

So, we will now argue the following theorem.

Theorem 2.2. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that $0 \neq S_2 \neq M$ and let S_2 be the maximal submodule of M. The following are equivalent.*

(i) S_2 is h-divisible;

(ii) Every submodule of M is a center of h-purity in M;

(iii) M contains a special center of h-purity.

Proof. Clearly (i) implies (ii). Assuming (ii) then to verify (iii), let M be a $QTAG$ module such that $0 \neq S_2 \neq M$. It follows that S_2 is a special center of h-purity.

Finally, we assume (iii) is true and verify (i). Let S_2 be a special center of h-purity in M. Assume that S_2 is not h-divisible, then $Soc^t(S_2)$ is not h-divisible for some t. Therefore, there exists $a \in Soc^t(S_2)$ such that $H_M(a) = 0$. Let $0 \neq u \in M, uR \cap S_1 = 0$ and $v = u' + a$ where $d(uR/u'R) = 1$. Clearly $vR \cap S_1 = 0$ and $H_M(v) = 0$. Hence $H_M(v - a) \geq 1$. This a contradiction by Lemma 2.1 and completes the proof. □

If S_1 is maximal disjoint from S_2 in M, we consider here certain circumstances under which we can reduce the assumption of the h-purity of S_1 in M to an analogous assumption in a submodule of M, or in a quotient submodule of M.

We can now give the second main characterization of this paper.

Theorem 2.3. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that $M = S_1 + S_2$. Then*
(i) For a submodule $S_3 \subseteq S_2, S_1$ is S_3-high in M if and only if $S_1 \cap S_2$ is S_3-high in S_2.
(ii) If $(S_1 \cap S_2) \cap H_t(S_2) = H_t(S_1 \cap S_2)$ then $S_1 \cap H_t(M) = H_t(S_1)$.
(iii) If S_4 is a maximal submodule of S_2 and $S_1 \cap H_t(M) = H_t(S_1)$ then $(S_1 \cap S_2) \cap H_t(S_2) = H_t(S_1 \cap S_2)$.

Proof. (i) If S_1 is S_3-high in M and $u \in S_2, u \notin S_1 \cap S_2$ then $(S_1 + u) \cap S_3 \neq 0$. Hence there exist $v \in S_1, t \geq 0$ such that $v + tu \in S_3$ and $v + tu \neq 0$. Since $S_3 \subseteq S_2$, we obtain $v + tu \in S_2$, and so that $v \in S_1 \cap S_2$. Thus, $[(S_1 \cap S_2) + u] \cap S_3 \neq 0$.

Conversely, if $S_1 \cap S_2$ is S_3-high in S_2 and $x \in M, x \notin S_1$ then by hypothesis $x = v + u$ for some $v \in S_1, u \in S_2$. Since $x \notin S_1$, we get $u \notin S_1 \cap S_2$, and hence $[(S_1 \cap S_2) + u] \cap S_3 \neq 0$. Let $v \in S_1$ and n be any integer, $\exists\, 0 \neq z \in M$ such that $z = y + nu \in S_3$. This, in tern, implies that $y + nx = z + nv$, and so $y - nv + nx = z \in S_3$. Henceforth, $y - nv \in S_1$; that is, $(S_1 + x) \cap S_3 \neq 0$. Since $0 = S_1 \cap S_2 \cap S_3 = S_1 \cap S_3, S_1$ is S_3-high in M, as stated.
(ii) If $(S_1 \cap S_2) \cap H_t(S_2) = H_t(S_1 \cap S_2)$ and let $kx = v \in S_1, x = y + u$ for some $y \in S_1$, $u \in S_2$ and $k \geq 0$. Then $v = kx = ky = ku$, and so that $v - ky = ku \in S_1 \cap S_2$. From the h-purity of $S_1 \cap S_2$ in $S_2, \exists\, w \in S_1 \cap S_2$ such that $kw = v - ky$. Hence $k(y + w) = v$ such that $y + w \in S_1$, as needed.
(iii) If $S_4 \subseteq S_2$ and $S_1 \cap H_t(M) = H_t(S_1)$, let $ku = v \in S_1 \cap S_2$ for some $u \in S_2, k \geq 0$. From the h-purity of S_1 in M $\exists\, y \in S_1$ such that $ky = v$. Then $k(y - u) = 0$, and so that $y - u \in S_4 \subseteq S_2$. Consequently, $y \in S_1 \cap S_2$. Therefore, $(S_1 \cap S_2) \cap H_t(S_2) = H_t(S_1 \cap S_2)$, as required. □

The following characterization is a slight variation on Theorem 2.1 and 2.3.

Theorem 2.4. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a QTAG-module M such that S_2 is S_1-high in M, and let S_3 be the maximal submodule of M such that $S_3 \subseteq S_1$. The following are equivalent.*
(i) $S_2 \cap H_t(M) = H_t(S_2)$ for some $t > 0$;
(ii) $(S_2 + S_3)/S_3$ is h-pure in M/S_3;
(iii) $(S_2 + S_3)/S_3$ is (S_1/S_3)-high in M/S_3.

Proof. $(i) \Rightarrow (ii)$. It is evident that if $S_2 \cap H_t(M) = H_t(S_2)$ then $(S_2 + S_3) \cap H_t(M) = H_t(S_2 + S_3)$. Since $S_2 + S_3 = S_2 \oplus S_3$ and $S_3 \cap H_t(M) = H_t(S_3)$, then $(S_2 + S_3) \cap H_t(M) = H_t(S_2 + S_3)$ if and only if $(S_2 + S_3)/S_3$ is h-pure in M/S_3.

$(iii) \Rightarrow (ii)$. If $(S_2 + S_3)/S_3$ is (S_1/S_3)-high in M/S_3, then it plainly follows that $(S_2 + S_3)/S_3$ is h-pure in M/S_3.

$(i) \Rightarrow (iii)$. If $S_2 \cap H_t(M) = H_t(S_2)$ and let $x + S_3 \in M/S_3, x + S_3 \notin (S_2 + S_3)/S_3$. Then $x \notin S_2$, and there exist $y \in S_2, t \geq 0$ such that $y + tx \in S_1$ and $y + tx \neq 0$. If $y + tx - a \in S_3$, then $na = 0$ for some integer n. Hence $ny = -ntx$, and from h-purity of S_2 in $M, \exists\, b \in S_2$ such that $ntb = ny$. Then $n(tb - y) = 0$, and so that $tb = y$ because $S_2 \cap S_3 = 0$. Now, $a = y + tx = t(b + x) \in S_3$, therefore, $b + x \in S_3$. But this contradicts that $x + S_3 \notin (S_2 + S_3)/S_3$. Henceforth, $(y + tx) + S_3 \in S_1/S_3$ and $y + tx + S_3 \neq S_3$. By the hypothesis of disjointness, we get that $(S_2 + S_3)/S_3$ is (S_1/S_3)-high in M/S_3. □

We recollect that a $QTAG$-module M is defined to be bounded if $\exists\, t \geq 0$ such that $H_M(u) \leq t$, for some $u \in M$ (see [18] for more details).

Besides, we state the following.

Definition 2.3. *Let S be the maximal submodule of a $QTAG$-module M with $0 \neq S \neq M$. We say that M satisfies the maximal element condition if each coset of S in M contains an uniform element u such that $H_M(u) = H_M(u + S)$.*

The following assertion relates above concept to our investigation.

Theorem 2.5. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a $QTAG$-module M such that $0 \neq S_2 \neq M$ and let S_2 be the maximal submodule of M, and suppose that*
(i) M satisfies the maximal element condition;
(ii) M/S_2 is h-divisible;
(iii) $\cap_t \cap_k H_k(S_2)$ is bounded for some $t \geq 0$.
Then $\overline{M}$ is a summand of M.

Proof. In each coset of S_2 in M, we choose an element u such that $H_M(u) = H_M(u + S_2)$. Then, by (ii) $H_M(u) = \infty$. Let S_1 be the submodule of M generated by the elements in M such that $H_M(u) = H_M(u + S_2)$, for some $u \in M$. Then it is fairly to see that $S_1 \cap S_1 \subseteq \cap_t \cap_k H_k(S_2)$, and hence by (iii) $S_1 \cap S_2$ is bounded. Therefore, $\overline{M}$ is a summand of M, as required. □

3. Open Problems

We shall pose in this section some questions that remain unanswered yet.

Problem 3.1. *Suppose $\{S_1, S_2\}$ is a pair of submodules of a $QTAG$-module M such that S_2 is S_1-high in M. What are the conditions under which $S_1 \cong S_2$?*

Problem 3.2. *What are the conditions under which any center of h-purity between the $QTAG$-module M and $Soc(M)$ is special center of h-purity?*

Problem 3.3. *Does it follow that the Theorem 2.5 remains true without the given conditions (i), (ii), and (iii)?*

Acknowledgments

The authors express their appreciation to the referee and to the editor, for their valuable assistance in processing our work. Also, the first author thanks to University of Johannesburg, South Africa for supporting the financial assistance (APC) for the publication of this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Danchev, Nice bases for primary abelian groups, Ann. Univ. Ferrara. 53 (2007), 39–50. https://doi.org/10.1007/s11565-007-0004-2.

[2] P. Danchev, On primary abelian groups modulo finite subgroups, Comm. Algebra. 37 (2009), 933–947. https://doi.org/10.1080/00927870802278610.

[3] P. Danchev, P. Keef, The nice basis rank of a primary abelian group, Stud. Sci. Math. Hung. 48 (2011), 247–256. https://doi.org/10.1556/sscmath.48.2011.2.1168.

[4] L. Fuchs, Infinite Abelian Groups, Volume I, Pure and Applied Mathematics, 36, Academic Press, New York, 1970 .

[5] L. Fuchs, Infinite Abelian Groups, Volume II, Pure and Applied Mathematics, 36, Academic Press, New York, 1973.

[6] A. Hasan, On quasi-large submodules of $QTAG$-modules, Asian-European J. Math. 09 (2016), 1650033. https://doi.org/10.1142/s1793557116500339.

[7] A. Hasan, Structures of $QTAG$-modules determined by their submodules, Uzbek Math. J. 65 (2021), 38–45. https://doi.org/10.29229/uzmj.2021-3-4.

[8] A. Hasan, J.C. Mba, On $QTAG$-modules having all N-high submodules h-pure, Mathematics. 10 (2022), 3523. https://doi.org/10.3390/math10193523.

[9] A. Hasan, Rafiquddin, M.N. Ali, Imbeddedness and direct sum of uniserial modules, Bol. Soc. Parana. Math. Accepted.

[10] M.Z. Khan, Modules behaving like torsion abelian groups II, Math. Japon. 23 (1979), 509–516.

[11] M.Z. Khan, On h-purity in $QTAG$-modules, Comm. Algebra. 16 (1988), 2649–2660. https://doi.org/10.1080/00927879808823707.

[12] M.Z. Khan, Complement submodules and quasi-essentail submodules, Tamkang J. Math. 19 (1988), 23–28.

[13] A. Mehdi, F. Mehdi, N-high submodules and h-topology, South East Asian J. Math. Math. Sci. 1 (2002), 83–85.

[14] F. Sikander, T. Fatima, A. Hasan, Nice bases of $QTAG$-modules, Sci. J. King Faisal Univ.: Basic Appl. Sci. 22 (2021), 51–56. https://doi.org/10.37575/b/sci/210031.

[15] R.S. Pierce, Centers of purity in abelian groups, Pac. J. Math. 13 (1963), 215–219. https://doi.org/10.2140/pjm.1963.13.215.

[16] J.D. Reid, On subgroups of an abelain group maxiaml disjoint from a given subgroup, Pac. J. Math. 13 (1963), 657–664. https://doi.org/10.2140/pjm.1963.13.657.

[17] H.A. Mehran, S. Singh, On σ-pure submodules of $QTAG$-modules, Arch. Math. 46 (1986), 501–510. https://doi.org/10.1007/BF01195018.

[18] S. Singh, Some decomposition theorems in abelian groups and their generalizations, in: Ring Theory: Proceedings of Ohio University Conference, Marcel Dekker, New York, 25 (1976), 183–189.

[19] S. Singh, Abelian groups like modules, Act. Math. Hung. 50 (1987), 85–95. https://doi.org/10.1007/BF01903367.