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EXPLORING NEW DIRECTIONAL CURVES OF A SPACELIKE CURVE IN MINKOWSKI $3-$SPACE

SHAYMAA S. A. AL-OBAIDI[1,*], GHASSAN A. MAHMOOD MAHMOOD[2], UFUK ÖZTÜRK[3]

[1]Department of Mathematics, University of Tikrit, Iraq

[2]Department of Mathematics, College of Basic Education, University of Telafer, Iraq

[3]Department of Mathematics, Bolu Abant İzzet Baysal University, Turkey

[*]Corresponding author: shaymaashihab9@gmail.com

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Abstract. In this study, we explore new associated curves in Minkowski $3-$space $\mathbb{E}_1^3$ by using the Darboux frame $\{T, \zeta, \eta\}$ instead of Frenet frame $\{T, N, B\}$ of the spacelike curve α having a spacelike principal normal lying on a timelike surface M. These associated curves, denoted as $\bar{D}_n$, $\bar{D}_r$, and $\bar{D}_o$, lie in planes defined by $\{\zeta, \eta\}$, $\{T, \eta\}$, and $\{T, \zeta\}$, respectively. We establish relationships between the Darboux frame and the curvatures k_g, k_n, τ_g of the curve α as well as the Frenet apparatus of the associated curves. Furthermore, we derive necessary and sufficient conditions for these associated curves to exhibit helical or spherical characteristics. Finally, we present relevant examples.

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1. INTRODUCTION

Minkowski space geometry, integral to both differential geometry and physics, particularly in general relativity, introduces $\mathbb{E}_1^3$ as a non-degenerate (*spacelike* or *timelike*) surface M due to its unique metric properties ([13]). For any arbitrary curve α on M, a Darboux frame $\{T, \zeta, \eta\}$, comprising the unit tangent vector field T of α, the unit vector field η representing the surface's unit normal restricted to α, and the timelike vector field $\zeta = \eta \times T$, can be constructed.

Various helical types, including general helices, isophote curves, and relatively normal-slant helices, play crucial roles in diverse fields such as Computer Aided Geometric Design ([2]), medical sciences ([3]), engineering ([5]), and biology ([11]). Notably, regular curves in Minkowski space are classified as *general helices* if and only if their conical curvature τ/κ is constant ([12]). Dogan ([1]) explored

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isophote curves on timelike surfaces, providing necessary and sufficient conditions for their existence and introduced a specific function,

$$\delta_o = \frac{k_n \tau_g' - \tau_g k_n'}{\left(k_n^2 - \tau_g^2\right)^{3/2}} - \frac{k_g}{\left(k_n^2 - \tau_g^2\right)^{1/2}}, \tag{1.1}$$

which remains constant. These curves are valuable in surface interrogation for detecting and visualizing small irregularities ([4]).

Nešović *et al.* ([6]) defined relatively normal-slant helices on non-degenerate surfaces in Minkowski 3−space and introduced another constant function,

$$\delta_r = \frac{k_g \tau_g' - \tau_g k_g'}{\left(k_g^2 + \tau_g^2\right)^{3/2}} - \frac{k_n}{\left(k_g^2 + \tau_g^2\right)^{1/2}}, \tag{1.2}$$

expressing these helices in terms of geodesic curvature, normal curvature, and geodesic torsion.

In this paper, we adopt the Darboux frame $\{T, \zeta, \eta\}$ instead of Frenet frame $\{T, N, B\}$ in Minkowski 3−space $\mathbb{E}1^3$ to introduce new associated curves, denoted as $\bar{D}n$, $\bar{D}r$, and $\bar{D}o$. These curves lie in planes spanned by $\{T, \zeta\}$, $\{T, \eta\}$, and $\{T, \zeta\}$, respectively, for the spacelike curve α with a spacelike principal normal on a timelike surface M. Our study establishes relationships between the Darboux frame and the curvatures k_g, k_n, τ_g of α, and the Frenet apparatus of the associated curves. Consequently, we derive necessary and sufficient conditions for the associated curves to exhibit helical or spherical characteristics. Relevant examples are also presented to illustrate our findings.

2. FUNDAMENTAL CONCEPTS

Minkowski space $\mathbb{E}_1^3$ is a real vector space, denoted as $\mathbb{E}^3$, equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$, defined as:

$$\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. Because this metric is indefinite, any vector x can have one of three causal characters: it can be *spacelike*, *timelike*, or *null* (*lightlike*), if $\langle x, x \rangle > 0$,$\langle x, x \rangle < 0$, or $\langle x, x \rangle = 0$ and $x \neq 0$, respectively ([7]). In particular, the vector $x = 0$ is considered spacelike. The norm (length) of a vector $x \in \mathbb{E}_1^3$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. If $\|x\| = 1$, we refer to it as a *unit vector*.

The *vector product* of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$ is defined as:

$$u \times v = (u_3 v_2 - u_2 v_3,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$$

Let u, v, and w be the vectors in $\mathbb{E}_1^3$. The following relations hold:

$$\begin{aligned} (i) &\quad \langle u \times v, w \rangle = \det(u, v, w) = [u, v, w], \\ (ii) &\quad u \times (v \times w) = -\langle u, w \rangle v + \langle u, v \rangle w, \\ (iii) &\quad \langle u \times v, u \times v \rangle = -\langle u, u \rangle \langle v, v \rangle + \langle u, v \rangle^2. \end{aligned}$$

An arbitrary curve $\alpha : I \to \mathbb{E}_1^3$ can be locally classified as *spacelike*, *timelike*, or *null* (*lightlike*) if all of its velocity vectors α' are spacelike, timelike, or null, respectively ([7]). A spacelike or timelike curve is also referred to as *non-null curve*.

The Frenet formulae for a unit-speed spacelike or timelike curve α with a non-null principal normal N in $\mathbb{E}_1^3$ are as follows ([14]):

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1\kappa(s) & 0 \\ -\epsilon_0\kappa(s) & 0 & -\epsilon_0\epsilon_1\tau(s) \\ 0 & -\epsilon_1\tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.1}$$

where $\kappa(s)$ and $\tau(s)$ are the *first curvature* and the *second curvature* of α, and it holds:

$$\langle T, T\rangle = \epsilon_0 = \pm 1, \quad \langle N, N\rangle = \epsilon_1 = \pm 1, \quad \langle B, B\rangle = -\epsilon_0\epsilon_1, \tag{2.2}$$

$$T \times N = -\epsilon_0\epsilon_1 B, \quad N \times B = \epsilon_0 T, \quad B \times T = \epsilon_1 N. \tag{2.3}$$

To determine whether the curve α lies on a Lorentzian sphere in Minkowski space, we give the following Theorems.

Theorem 1. ([8,9]) *Let $\alpha(s)$ be a unit-speed spacelike curve in Minkowski space $\mathbb{E}_1^3$, with the non-null principal normal N, and with curvature $\kappa(s)$ and torsion $\tau(s)$ satisfying $\frac{1}{\kappa} \neq 0$ and $\frac{1}{\tau} \neq 0$ for each $s \in I \subseteq \mathbb{R}$. The image of α lies on a Lorentzian sphere of radius $r > 0$ if and only if the following conditions hold:*

$$\left(\frac{1}{\kappa}\right)^2 - \left[\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right]^2 = \epsilon_1 r^2,$$

and

$$\frac{\tau}{\kappa} = \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]',$$

where $\epsilon_1 = \langle N, N\rangle = \pm 1$.

Theorem 2. ([10]) *Let $\alpha(s)$ be a unit-speed timelike curve in Minkowski space $\mathbb{E}_1^3$, with curvature $\kappa(s) \neq 0$ and torsion $\tau(s) \neq 0$ for each $s \in I \subseteq \mathbb{R}$. The image of α lies on a Lorentzian sphere of radius $r \in \mathbb{R}^+$ if and only if the following conditions hold:*

$$\left(\frac{1}{\kappa}\right)^2 + \left[\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right]^2 = r^2,$$

and

$$\frac{\tau}{\kappa} = -\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'.$$

Let M be a timelike surface in Minkowski $3-$space $\mathbb{E}_1^3$, parameterized as follows:

$$X\left(u,t\right)=\left(x_1\left(u,t\right),x_2\left(u,t\right),x_3\left(u,t\right)\right).$$

where x_1, x_2, and x_3 represent differentiable functions. The unit spacelike normal vector field on M is denoted as

$$n(u,t)=\frac{X_u\times X_t}{||X_u\times X_t||},$$

which plays a critical role on this surface M. Let $\alpha:I\subset\mathbb{R}\to M$ be a spacelike curve with a spacelike principal normal lying on M. The *Darboux frame*, comprising the vector fields $\{T,\zeta,\eta\}$ is orthonormal and consists of a unit tangential vector field: $T=\alpha'$, a unit spacelike normal vector field: $\eta=n(u,t)|_\alpha$, and a unit timelike vector field $\zeta=\pm\eta\times T$. The sign in front of ζ is chosen in such a way that the determinant $\det(T,\zeta,\eta)=1$. The Darboux frame satisfies the following relationships:

$$\langle T,T\rangle=1,\quad\langle\zeta,\zeta\rangle=-1,\quad\langle\eta,\eta\rangle=1,\tag{2.4}$$

$$\langle T,\zeta\rangle=\langle T,\eta\rangle=\langle\zeta,\eta\rangle=0,\tag{2.5}$$

and

$$T\times\zeta=\eta,\quad\zeta\times\eta=T\,,\quad\eta\times T=-\zeta.\tag{2.6}$$

The *normal curvature* $k_n(s)$, *geodesic curvature* $k_g(s)$ and *geodesic torsion* $\tau_g(s)$ of α are respectively defined as:

$$k_n\left(s\right)=\left\langle\eta\left(s\right),T'\left(s\right)\right\rangle,\quad k_g(s)=\left\langle\zeta\left(s\right),T'\left(s\right)\right\rangle,\quad\tau_g(s)=\left\langle\eta\left(s\right),\zeta'\left(s\right)\right\rangle,\tag{2.7}$$

where s is the arc-length parameter of α. Hence Darboux frame's equations have are as follows:

$$\begin{bmatrix}T'(s)\\\zeta'(s)\\\eta'(s)\end{bmatrix}=\begin{bmatrix}0&-k_g(s)&k_n(s)\\-k_g(s)&0&\tau_g(s)\\-k_n(s)&\tau_g(s)&0\end{bmatrix}\begin{bmatrix}T(s)\\\zeta(s)\\\eta(s)\end{bmatrix}.\tag{2.8}$$

Also, the Frenet frame and Darboux frame of the spacelike curve α are interrelated through a composition of hyperbolic rotation for an angle $-\theta$ and symmetry with respect to the null straight line $x_1=-x_2$, which is expressed as:

$$\begin{bmatrix}T\\\zeta\\\eta\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cosh(-\theta)&\sinh(-\theta)\\0&\sinh(-\theta)&\cosh(-\theta)\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix},\tag{2.9}$$

where $\theta(s)=\angle(\eta,N)$ is an angle between a timelike and a spacelike vector. This set of relations, as described in (2.1), (2.8) and (2.9), yields valuable expressions:

$$k_g=\kappa\sinh\theta,\quad k_n=-\kappa\cosh\theta,\quad\tau_g=\theta'-\tau,\quad\kappa^2=k_n^2-k_g^2.\tag{2.10}$$

Lemma 1. *Let α be an arbitrary curve lying on a surface M in $\mathbb{E}_1^3$ with the geodesic curvature k_g, normal curvature k_n and geodesic torsion τ_g. Then the following statements hold:*
(i) *α is a geodesic curve on M if and only if $k_g = 0$;*
(ii) *α is an asymptotic curve on M if and only if $k_n = 0$;*
(iii) *α is a line of principal curvature on M if and only if $\tau_g = 0$.*

Throughout the following sections, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

3. ON DARBOUX DIRECTIONAL CURVES IN MINKOWSKI $3-$SPACE

In this section, we introduce the concept of new associated curves for a spacelike curve lying on a timelike surface in Minkowski $3-$space. We establish conditions for a curve to be a relatively normal-slant helix and an isophote curve based on the geodesic curvature, normal curvature, and geodesic torsion of spacelike curves on the lightlike surface. Throughout this section, we assume that all curves and surfaces are smooth and regular unless stated otherwise.

Let M be a timelike surface in Minkowski space $\mathbb{E}_1^3$, and α be a unit spacelike curve lying on M with the Darboux frame $\{T, \zeta, \eta\}$. The Darboux frame can be seen as a dynamic entity with an axis of rotation determined by the Darboux vector (Arslan *et al.* 2016, Hartl 1993, Nešović *et al.* 2016, Öztürk and Nešović 2016, Scofield 1995). This vector, which plays a pivotal role in this framework, is defined as

$$T' = D \times T, \quad \zeta' = D \times \zeta, \quad \eta' = D \times \eta. \tag{3.1}$$

By using the relations (2.6), (2.8) and (3.1), we obtain the expression for the Darboux vector:

$$D = \tau_g T - k_n \zeta + k_g \eta. \tag{3.2}$$

To understanding of the geometric properties associated with the Darboux frame and the curve α, we introduce three vector fields along the spacelike curve α, each lying in the normal plane $T^{\perp} = Sp\{\zeta, \eta\}$, the rectifying plane $\zeta^{\perp} = Sp\{T, \eta\}$ and the osculating pane $\eta^{\perp} = Sp\{T, \zeta\}$:

- The normal Darboux vector field $D_n = -k_n \zeta + k_g \eta$,
- The rectifying Darboux vector field $D_r = \tau_g T + k_g \eta$,
- The osculating Darboux vector field $D_o = \tau_g T - k_n \zeta$,

respectively.

Additionally, we have a vector field k along the curve α. There is an interesting relationship between the vector field k and the Darbox frame $\{T, \zeta, \eta\}$ of α. Specifically, we can express k as follows:

$$k(s) = k_1(s)T(s) + k_2(s)\zeta(s) + k_3(s)\eta(s), \tag{3.3}$$

where $k_1(s)$, $k_2(s)$ and $k_3(s)$ are scalar functions in the arc length parameter s of α, and ϵ_k is a value from the set $\{-1, 0, 1\}$ such that:

$$\langle k, k\rangle = k_1^2 - k_2^2 + k_3^2 = \epsilon_k. \tag{3.4}$$

With the help of α, we can define a curve $\beta = \beta(s)$ with the same parameter as α, such that β is an integral curve of k (i.e., $\beta'(s) = k$). We then provide the following definition:

Definition 1. *Let α be a unit spacelike curve lying on a timelike surface M in $\mathbb{E}_1^3$, and let k be a vector field as given in (3.3). A curve β is called a k-*directional Darboux curve *of α if β is an integral curve of k (i.e., $\beta' = k$).*

3.1. $\bar{D}_n-$**direction curve.** The unit normal Darboux vector, denoted as $\bar{D}_n$, is defined as follows:

$$\bar{D}_n = \frac{-k_n\zeta + k_g\eta}{\sqrt{|k_g^2 - k_n^2|}} \quad \text{if } (k_g, k_n) \neq (0,0).$$

Now, consider a curve β as a $\bar{D}_n-$*direction curve* of the curve α. According to Definition 1, we can establish the following relationship:

$$\beta' = \bar{D}_n.$$

By using the relation (2.10), β' is a unit timelike vector, which implies that β is a unit speed timelike curve with the arc length equal to s. The unit tangent vector T_β of β is then given by:

$$T_\beta = \frac{-k_n\zeta + k_g\eta}{\sqrt{k_n^2 - k_g^2}}. \tag{3.5}$$

Next, by differentiating equation (3.5) with respect to s and using the relation (2.8), we can deduce that:

$$T_\beta' = -\left[\frac{k_g k_n' - k_n k_g'}{k_n^2 - k_g^2} + \tau_g\right] \frac{T'}{\sqrt{k_n^2 - k_g^2}},$$

and since $\|T'\| = \sqrt{k_n^2 - k_g^2}$,

$$T_\beta' = -\varepsilon_{\bar{D}_n} \frac{T'}{\|T'\|}, \tag{3.6}$$

where $\varepsilon_{\bar{D}_n} = \frac{k_g k_n' - k_n k_g'}{k_n^2 - k_g^2} + \tau_g$.

By taking the norm of equation (3.6), we obtain $\left\|T_\beta'\right\| = \epsilon\varepsilon_{\bar{D}_n}$, where $\epsilon = \pm 1$ ($\epsilon\varepsilon_{\bar{D}_n} > 0$). Then, the unit normal vector field N_β of β is given by

$$N_\beta = -\epsilon \frac{T'}{\|T'\|}, \tag{3.7}$$

which is a spacelike vector. Then, form the relations (3.6) and (3.7), the curvature κ_β of β is given by

$$\kappa_\beta = \left\|T_\beta'\right\| = \left\|\bar{D}_n\right\| = \epsilon\varepsilon_{\bar{D}_n}. \tag{3.8}$$

Furthermore, using equations (2.8), (3.5), and (3.7), the binormal vector field B_β of β can be found as

$$B_\beta = T_\beta \wedge N_\beta = \epsilon T.$$

Moreover, the torsion of β is given by

$$\tau_\beta = -\left\langle B'_\beta, N_\beta \right\rangle = \sqrt{k_n^2 - k_g^2}, \tag{3.9}$$

respectively. Then, we can state the following theorem:

Theorem 3. *Let α be a unit speed spacelike curve with the spacelike principal normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let β be a $\bar{D}_n$-direction curve of α. Then, β is a unit speed timelike curve, and its Frenet vectors are given by*

$$T_\beta = \bar{D}_n, \quad N_\beta = -\epsilon \frac{T'}{\|T'\|}, \quad B_\beta = \epsilon T,$$

and its curvatures are given by

$$\kappa_\beta = \epsilon\varepsilon_{\bar{D}_n}, \quad \tau_\beta = \sqrt{k_n^2 - k_g^2},$$

where $\varepsilon_{\bar{D}_n} = \frac{k_g\kappa_n' - \kappa_n\kappa_g'}{k_n^2 - \kappa_g^2} + \tau_g$, and $\epsilon = \pm 1$.

From the relation (2.10) and Thereom 3, we can give the following corollary.

Corollary 1. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$. Then, $\bar{D}_o-$direction curve is a helix if and only if the curve α is a helix.*

The curve β is also the spherical image of the normal Darboux vector field. Namely, β lies on a Lorentzian sphere with radius $r \in \mathbb{R}^+$. According to Theorem 2, we have the following relationship:

$$\left(\frac{1}{\kappa_\beta}\right)^2 + \left[\frac{1}{\tau_\beta}\left(\frac{1}{\kappa_\beta}\right)'\right]^2 = r^2 = constant.$$

This implies that

$$\kappa_\beta' = \mp\tau_\beta\kappa_\beta\sqrt{r^2\kappa_\beta^2 - 1}.$$

By utilizing the expressions for κ_β and τ_β as defined in Theorem 3, we can derive the following relationship:

$$\varepsilon_{\bar{D}_n}' = \mp\varepsilon_{\bar{D}_n}\sqrt{k_n^2 - k_g^2}\sqrt{r^2\varepsilon_{\bar{D}_n}^2 - 1}.$$

If $r^2\varepsilon_{\bar{D}_n}^2 - 1 = 0$ for all s, then $\varepsilon_{\bar{D}_n}' = 0$, leading to $\kappa_\beta = \frac{1}{r} = constant$. However, given that β is a spherical curve, Theorem 2, meaning $\frac{\tau_\beta}{\kappa_\beta} = 0$. This aligns with the condition $\tau_\beta = \sqrt{k_n^2 - k_g^2} = 0$ which is coincide with $(k_g, k_n) \neq (0,0)$. Therefore, $r^2\varepsilon_{\bar{D}_n}^2 - 1 \neq 0$ for all s, and we can express the above relation as follows:

$$\frac{\frac{\varepsilon_{\bar{D}_n}'}{r\varepsilon_{\bar{D}_n}^2}}{\sqrt{1 - \frac{1}{r^2\varepsilon_{\bar{D}_n}^2}}} = \sqrt{k_n^2 - k_g^2}. \tag{3.10}$$

Taking integrate the relation (3.10), we have

$$\varepsilon_{\bar{D}_n} = \frac{1}{r}\sec\left(\int\sqrt{k_n^2 - k_g^2}ds\right)$$

Conversely, assuming $\varepsilon_{\bar{D}_n} = \frac{1}{r}\sec\left(\int\sqrt{k_n^2 - k_g^2}ds\right)$ and $r \in \mathbb{R}^+$, we can deduce the following relationships from Theorem 3

$$\frac{1}{\kappa_\beta} = \frac{1}{\epsilon\varepsilon_{\bar{D}_n}} = \epsilon r \cos\left(\int\sqrt{k_n^2 - k_g^2}ds\right),$$

and

$$\frac{1}{\tau_\beta} = \frac{1}{\sqrt{k_n^2 - k_g^2}},$$

where $(k_g, k_n) \neq (0, 0)$ for all s. These relationships, in turn, lead to:

$$\frac{\tau_\beta}{\kappa_\beta} + \left[\frac{1}{\tau_\beta}\left(\frac{1}{\kappa_\beta}\right)'\right]' = 0.$$

From Theorem 2, we can conclude that the curve β lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$. Therefore, we can state the following theorem:

Theorem 4. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let β be a $\bar{D}_n-$direction curve of α. Then, β lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$ if and only if*

$$\varepsilon_{\bar{D}_n} = \frac{1}{r}\sec\left(\int\sqrt{k_n^2 - k_g^2}ds\right),$$

everywhere $(k_n, k_g) \neq (0, 0)$.

3.2. $\bar{D}_r-$**direction curve.** The unit rectifying Darboux rector is defined as:

$$\bar{D}_r = \frac{\tau_g T + k_g \eta}{\sqrt{k_g^2 + \tau_g^2}} \quad \text{if } (k_g, \tau_g) \neq (0, 0).$$

Consider $\bar{\beta}$ as a $\bar{D}_r-$*direction curve* of the curve α. Then, from Definition 1, we have

$$\bar{\beta}' = \bar{D}_r.$$

which is a unit spacelike vector. Thus, the curve $\bar{\beta}$ is a unit-speed spacelike curve.with the arc length equal to s. Then, the unit tangent vector $T_{\bar{\beta}}$ of $\bar{\beta}$ is given by:

$$T_{\bar{\beta}} = \frac{\tau_g T + k_g \eta}{\sqrt{k_g^2 + \tau_g^2}}. \tag{3.11}$$

Differentiating equation (3.11) with respect to s and using the relation (2.8), we get

$$T_{\bar{\beta}}' = \left[\frac{\left(k_g\tau_g' - \tau_g k_g'\right)}{k_g^2 + \tau_g^2} - k_n\right]\frac{\zeta'}{\sqrt{k_g^2 + \tau_g^2}}.$$

Since $\|\zeta'\| = \sqrt{k_g^2 + \tau_g^2}$, we obtain

$$T_{\bar{\beta}}' = -\varepsilon_{\bar{D}_r}\frac{\zeta'}{\|\zeta'\|}, \tag{3.12}$$

where $\varepsilon_{\bar{D}_r} = \frac{k_g\tau_g' - \tau_g k_g'}{k_g^2+\tau_g^2} - k_n$. Then, the normal vector field of the curve $\bar{\beta}$ is given by:

$$N_{\bar{\beta}} = -\epsilon \frac{\zeta'}{\|\zeta'\|}, \tag{3.13}$$

is a spacelike vector, where $\epsilon = \pm 1$ such that $\epsilon\varepsilon_{\bar{D}_r} > 0$. Therefore, the curvature of $\bar{\beta}$ can be fond as $\kappa_{\bar{\beta}} = \epsilon\varepsilon_{\bar{D}_r}$. Also, from the relations (2.6), (2.8), (3.11) and (3.13), the binormal vector field $B_{\bar{\beta}}$ of $\bar{\beta}$ is given by

$$B_{\bar{\beta}} = -T_{\bar{\beta}} \wedge N_{\bar{\beta}} = \epsilon\zeta, \tag{3.14}$$

Furthermore, the torsion of the curve $\bar{\beta}$ is given by

$$\tau_{\bar{\beta}} = -\left\langle B_{\bar{\beta}}', N_{\bar{\beta}} \right\rangle = \sqrt{k_g^2 + \tau_g^2},$$

respectively. The following theorem can now be presented:

Theorem 5. *Let α be a unit speed spacelike curve with the spacelike principal normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\beta}$ be a $\bar{D}_r$−direction curve of α. Then, $\bar{\beta}$ is a unit speed spacelike curve with a spacelike principal normal, and its Frenet appartus $\left\{T_{\bar{\beta}}, N_{\bar{\beta}}, B_{\bar{\beta}}, \kappa_{\bar{\beta}}, \tau_{\bar{\beta}}\right\}$ is given by*

$$T_{\bar{\beta}} = \bar{D}_r, \quad N_{\bar{\beta}} = -\epsilon\frac{\zeta'}{\|\zeta'\|}, \quad B_{\bar{\beta}} = \epsilon\zeta,$$

and

$$\kappa_{\bar{\beta}} = \epsilon\varepsilon_{\bar{D}_r}, \quad \tau_{\bar{\beta}} = \sqrt{k_g^2 + \tau_g^2},$$

where $\varepsilon_{\bar{D}_r} = \frac{k_g\tau_g' - \tau_g k_g'}{k_g^2+\tau_g^2} - k_n$, and $\epsilon = \pm 1$.

From Thereom 5 and the relation (1.2), the following corollary can be given.

Corollary 2. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$. Then,the $\bar{D}_r$−direction curve $\bar{\beta}$ is a helix if and only if the curve α is a relatively normal-slant helix.*

Moreover, since the curve $\bar{\beta}$ is also the spherical image of the rectifying Darboux vector field, $\bar{\beta}$ lies on a Lorentzian sphere with radius $r \in \mathbb{R}^+$. Using Theorem 1, we have

$$\left(\frac{1}{\kappa_{\bar{\beta}}}\right)^2 - \left[\frac{1}{\tau_{\bar{\beta}}}\left(\frac{1}{\kappa_{\bar{\beta}}}\right)'\right]^2 = r^2 = constant.$$

It follows that

$$\kappa_{\bar{\beta}}' = \mp\tau_{\bar{\beta}}\kappa_{\bar{\beta}}\sqrt{1 - r^2\kappa_{\bar{\beta}}^2},$$

and by using the expression of $\kappa_{\bar{\beta}}$ and $\tau_{\bar{\beta}}$ in Theorem 5, we obtain

$$\frac{\varepsilon_{\bar{D}_r}'}{\varepsilon_{\bar{D}_r}\sqrt{1 - r^2\varepsilon_{\bar{D}_r}^2}} = \mp\sqrt{k_g^2 + \tau_g^2}.$$

Taking integrate the above relation, we get

$$\varepsilon_{\bar{D}_r} = \frac{1}{r}\operatorname{sech}\left(\int \sqrt{k_g^2+\tau_g^2}ds\right),$$

where $r \in \mathbb{R}^+$.

Conversely, we assume that $\varepsilon_{\bar{D}_r} = \frac{1}{r}\operatorname{sech}\left(\int \sqrt{k_g^2+\tau_g^2}ds\right)$ and $r \in \mathbb{R}^+$. Then, from the Theorem 5 we have

$$\frac{1}{\kappa_{\bar{\beta}}} = \frac{1}{\epsilon\varepsilon_{\bar{D}_r}} = \epsilon r\cosh\left(\int \sqrt{k_g^2+\tau_g^2}ds\right),$$

and

$$\frac{1}{\tau_{\bar{\beta}}} = \frac{1}{\sqrt{k_g^2+\tau_g^2}},$$

where $(k_g,\tau_g) \neq (0,0)$ for all s. By using the above equations, we get

$$\frac{\tau_{\bar{\beta}}}{\kappa_{\bar{\beta}}} - \left[\frac{1}{\tau_{\bar{\beta}}}\left(\frac{1}{\kappa_{\bar{\beta}}}\right)'\right]' = 0,$$

and from Theorem 1, the curve $\bar{\beta}$ lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$. Therefore, the following theorem can be given.

Theorem 6. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\beta}$ be the $\bar{D}_r-$direction curve of α. Then, $\bar{\beta}$ lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$ if and only if*

$$\varepsilon_{\bar{D}_r} = \frac{1}{r}\operatorname{sech}\left(\int \sqrt{k_g^2+\tau_g^2}ds\right),$$

everywhere $(k_g,\tau_g) \neq (0,0)$.

3.3. $\bar{D}_o-$**direction curve.** The unit osculating Darboux rector is given by

$$\bar{D}_o = \frac{\tau_g T - k_n\zeta}{\sqrt{|\tau_g^2-k_n^2|}} \quad \text{if } (k_n,\tau_g) \neq (0,0).$$

Consider $\bar{\bar{\beta}}$ as a $\bar{D}_o-$*direction curve* of the curve α. Then, from Definition 1, we have

$$\bar{\bar{\beta}}' = \bar{D}_o.$$

Next, we consider three subcases: (I) $\tau_g^2-k_n^2>0$; (II) $\tau_g^2-k_n^2<0$; (III) $\tau_g=\pm k_n$.

(I) If $\tau_g^2-k_n^2>0$ for all s, then β is a unit-speed spacelike curve with the arc length equal to s, and the unit tangent vector $T_{\bar{\bar{\beta}}}$ of $\bar{\bar{\beta}}$ is given by

$$T_{\bar{\bar{\beta}}} = \frac{\tau_g T - k_n\zeta}{\sqrt{\tau_g^2-k_n^2}}. \tag{3.15}$$

Differentiating equation (3.15) with respect to s and using (2.8), we get

$$T'_{\bar{\bar{\beta}}} = \varepsilon_{\bar{D}_o}\frac{\eta'}{\|\eta'\|},$$

where $\varepsilon_{\bar{D}_o} = \frac{k_n\tau_g' - \tau_g k_n'}{\tau_g^2 - k_n^2} - k_g$, with $\epsilon = \pm 1$. Then, the normal vector field of the curve $\bar{\bar{\beta}}$ is a timelike vector given by

$$N_{\bar{\bar{\beta}}} = \epsilon \frac{\eta'}{\|\eta'\|}, \tag{3.16}$$

where $\epsilon = \pm 1$ such that $\epsilon\varepsilon_{\bar{D}_r} > 0$. Therefore, the curvature of $\bar{\bar{\beta}}$ can be fond as $\kappa_{\bar{\bar{\beta}}} = \epsilon\varepsilon_{\bar{D}_o}$. Also, from the relations (2.6), (2.8), (3.15) and (3.16), the binormal vector field $B_{\bar{\bar{\beta}}}$ of $\bar{\bar{\beta}}$ is given by

$$B_{\bar{\bar{\beta}}} = T_{\bar{\bar{\beta}}} \wedge N_{\bar{\bar{\beta}}} = \epsilon\eta \tag{3.17}$$

Moreover, the torsion of the curve $\bar{\bar{\beta}}$ is given by

$$\tau_{\bar{\bar{\beta}}} = -\left\langle B_{\bar{\bar{\beta}}}', N_{\bar{\bar{\beta}}} \right\rangle = \sqrt{\tau_g^2 - k_n^2},$$

respectively. Therefore, we can give the following theorem.

Theorem 7. *Let α be a unit speed spacelike curve with the spacelike principal normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\bar{\beta}}$ be a $\bar{D}_o-$direction curve of α. If $\tau_g^2 - k_n^2 > 0$, then $\bar{\bar{\beta}}$ is a spacelike curve with the timelike normal, and its Frenet appartus $\{T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_g\}$ is given by*

$$T_{\bar{\bar{\beta}}} = \bar{D}_o, \quad N_{\bar{\bar{\beta}}} = \epsilon \frac{\eta'}{\|\eta'\|}, \quad B_{\bar{\bar{\beta}}} = \epsilon\eta,$$

and

$$\kappa_{\bar{\bar{\beta}}} = \epsilon\varepsilon_{\bar{D}_o}, \quad \tau_{\bar{\bar{\beta}}} = \sqrt{\tau_g^2 - k_n^2},$$

where $\varepsilon_{\bar{D}_o} = \frac{k_n\tau_g' - \tau_g k_n'}{\tau_g^2 - k_n^2} - k_g$, and $\epsilon = \pm 1$ such that $\epsilon\varepsilon_{\bar{D}_o} > 0$.

From Thereom 7 and the relation (1.1), the following corollary can be given.

Corollary 3. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$. Then, the $\bar{D}_o-$direction curve $\bar{\bar{\beta}}$ is a helix if and only if the curve α is an isophote curve.*

Using Theorem 1, Theorem 7 and following a similar calculation as in Section 3.2, we can prove the following statement.

Theorem 8. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\bar{\beta}}$ be a unit speed spacelike curve with the timelike normal as a $\bar{D}_o-$direction curve of α. Then, $\bar{\bar{\beta}}$ lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$ if and only if*

$$\varepsilon_{\bar{D}_o} = \frac{1}{r}\operatorname{sech}\left(\int \sqrt{\tau_g^2 - k_n^2} ds\right),$$

where $(k_n, \tau_g) \neq (0,0)$ and $\tau_g^2 - k_n^2 > 0$ for all s.

(II) If $\tau_g^2 - k_n^2 < 0$ for all s, then $\bar{\bar{\beta}}$ is a unit-speed timelike curve with the arc length equal to s, and the unit tangent vector $T_{\bar{\bar{\beta}}}$ of $\bar{\bar{\beta}}$ is given by

$$\bar{\bar{\beta}}' = T_{\bar{\bar{\beta}}} = \frac{\tau_g T - k_n \zeta}{\sqrt{k_n^2 - \tau_g^2}}. \tag{3.18}$$

Differentiating equation (3.18) with respect to s and using (2.8), we get

$$T'_{\bar{\bar{\beta}}} = -\varepsilon_{\bar{D}_o} \frac{\eta'}{\|\eta'\|},$$

where $\varepsilon_{\bar{D}_o} = \frac{k_n \tau_g' - \tau_g k_n'}{\tau_g^2 - k_n^2} - k_g$. Then, its normal vector field is a spacelike vector given by

$$N_{\bar{\bar{\beta}}} = -\epsilon \frac{\eta'}{\|\eta'\|}, \tag{3.19}$$

where $\epsilon = \pm 1$ such that $\epsilon\varepsilon_{\bar{D}_r} > 0$. Thus, the curvature of $\bar{\bar{\beta}}$ is $\kappa_\beta = \epsilon\varepsilon_{\bar{D}_o}$. Also, from the relations (2.6), (2.8), (3.18) and (3.19), the binormal vector field $B_{\bar{\bar{\beta}}}$ of $\bar{\bar{\beta}}$ is given by

$$B_{\bar{\bar{\beta}}} = T_{\bar{\bar{\beta}}} \wedge N_{\bar{\bar{\beta}}} = \epsilon\eta. \tag{3.20}$$

The torsion of the curve $\bar{\bar{\beta}}$ is given by

$$\tau_{\bar{\bar{\beta}}} = -\left\langle B'_{\bar{\bar{\beta}}}, N_{\bar{\bar{\beta}}} \right\rangle = \sqrt{k_n^2 - \tau_g^2},$$

respectively. Therefore, we can give the following theorem.

Theorem 9. *Let α be a unit speed spacelike curve with the spacelike principal normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\bar{\beta}}$ be a $\bar{D}_o-$direction curve of α. If $\tau_g^2 - k_n^2 < 0$, then $\bar{\bar{\beta}}$ is a timelike curve, and its Frenet apparatus $\left\{T_{\bar{\bar{\beta}}}, N_{\bar{\bar{\beta}}}, B_{\bar{\bar{\beta}}}, \kappa_{\bar{\bar{\beta}}}, \tau_{\bar{\bar{\beta}}}\right\}$ is given by*

$$T_{\bar{\bar{\beta}}} = \bar{D}_o, \quad N_{\bar{\bar{\beta}}} = -\epsilon \frac{\eta'}{\|\eta'\|}, \quad B_{\bar{\bar{\beta}}} = \epsilon\eta,$$

and

$$\kappa_{\bar{\bar{\beta}}} = \epsilon\varepsilon_{\bar{D}_o}, \quad \tau_{\bar{\bar{\beta}}} = \sqrt{k_n^2 - \tau_g^2},$$

where $\varepsilon_{\bar{D}_o} = \frac{\tau_g k_n' - k_n \tau_g'}{k_n^2 - \tau_g^2} + k_g$, and $\epsilon = \pm 1$ such that $\epsilon\varepsilon_{\bar{D}_o} > 0$.

Corollary 4. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$. Then, the $\bar{D}_o-$direction curve $\bar{\bar{\beta}}$ is a helix if and only if the curve α is an isophote curve.*

Using Theorem 2, Theorem 9 and following a similar calculation as in Section 3.1, we can prove the following statement.

Theorem 10. *Let α be a unit speed spacelike curve with the spacelike normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\bar{\beta}}$ be a unit speed timelike curve as a $\bar{D}_o-$direction curve of α. Then, $\bar{\bar{\beta}}$ lies on a Lorentzian sphere with the radius $r \in \mathbb{R}^+$ if and only if*

$$\varepsilon_{\bar{D}_o} = -\frac{1}{r}\operatorname{sech}\left(\int \sqrt{\tau_g^2 - k_n^2 ds}\right),$$

where $(k_n, \tau_g) \neq (0,0)$ and $\tau_g^2 - k_n^2 < 0$ for all s.

(III) If $\tau_g = \pm k_n$ for all s, then $\bar{\bar{\beta}}$ is a null curve ,and its tangent vector is given by

$$T_{\bar{\bar{\beta}}} = D_o = \tau_g T - k_n \zeta.$$

Taking derivative the last relation and using the relation (2.8), we get

$$T'_{\bar{\bar{\beta}}} = \left(\tau_g' + k_n k_g\right) T + \left(-k_n' - k_g \tau_g\right) \zeta.$$

Here, we have the following two subcases:
(i) if $\tau_g = k_n$ for all s, then

$$T'_{\bar{\bar{\beta}}} = \left(k_n' + k_n k_g\right)(T - \zeta),$$

with

$$\left\| T'_{\bar{\bar{\beta}}} \right\| = 0.$$

It means that $\bar{\bar{\beta}}$ is a null straight line.
(ii) if $\tau_g = -k_n$ for all s, then

$$T'_{\bar{\bar{\beta}}} = -\left(k_n' + k_n k_g\right)(T + \zeta),$$

with

$$\left\| T'_{\bar{\bar{\beta}}} \right\| = 0,$$

and so $\bar{\bar{\beta}}$ is a null straight line.Therefore, we can give the following theorem.

Theorem 11. *Let α be a unit speed spacelike curve with the spacelike principal normal lying on a timelike surface M in $\mathbb{E}_1^3$, and let $\bar{\bar{\beta}}$ be a $\bar{D}_o-$direction curve of α. If $\tau_g = \pm k_n$ for all$s$, then the $\bar{D}_o-$direction curve $\bar{\bar{\beta}}$ of α is a straight line.*

4. Examples

Example 1. *Let us consider the timelike ruled surface M in $\mathbb{E}_1^3$ parametrized as shown in Figure 1*

$$X(s,t) = \left(\frac{\sqrt{2}}{2}\sinh s, \frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{6}}{2}s\right) + t\left(\sinh s + \sqrt{3}\cosh s, \cosh s + \sqrt{3}\sinh s, 1\right), \tag{4.1}$$

with its base curve parametrized as

$$\alpha(s) = \left(\frac{\sqrt{2}}{2}\sinh s, \frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{6}}{2}s\right).$$

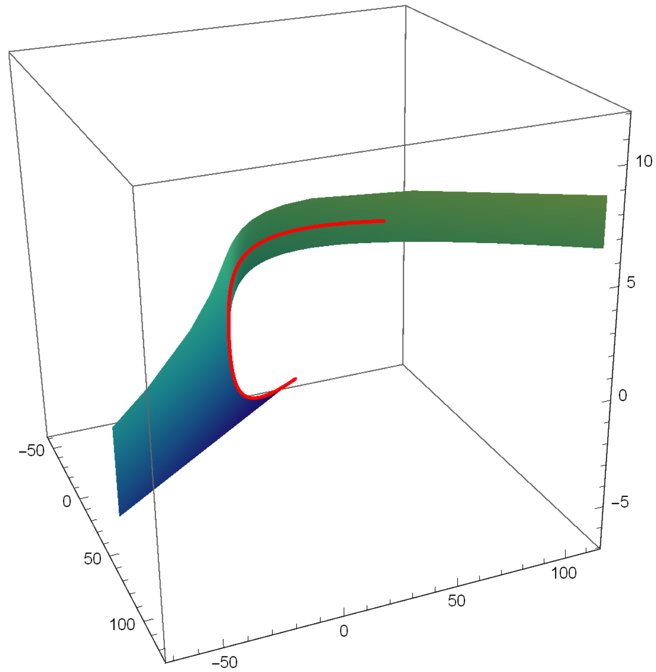


FIGURE 1. The timelike ruled surface M and the curve α

The Frenet frame vectors of α have the form

$$\begin{aligned} T &= \alpha'(s) = (\frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{2}}{2}\sinh s, \frac{\sqrt{6}}{2}), \\ N &= \frac{\alpha''(s)}{\|\alpha''(s)\|} = (\sinh s, \cosh s, 0), \\ B &= T \times N = (\frac{\sqrt{6}}{2}\cosh s, \frac{\sqrt{6}}{2}\sinh s, \frac{\sqrt{2}}{2}) \end{aligned}$$

and the curvatures κ and τ of α read

$$\kappa = \frac{1}{\sqrt{2}}, \quad \tau = \frac{\sqrt{6}}{2}.$$

Consequently, α is a unit spacelike curve with the spacelike principal normal. Also, we find that the Darboux frame of α is given by

$$\begin{aligned} T &= \left(\frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{2}}{2}\sinh s, \frac{\sqrt{6}}{2}\right), \\ \zeta &= \left(\sinh s + \sqrt{3}\cosh s, \cosh s + \sqrt{3}\sinh s, 1\right), \\ \eta &= \left(\sqrt{2}\sinh s + \frac{\sqrt{6}}{2}\cosh s, \sqrt{2}\cosh s + \frac{\sqrt{6}}{2}\sinh s, \frac{\sqrt{2}}{2}\right). \end{aligned}$$

According to the relation (2.7), the curvatures k_g, k_n and τ_g of α read

$$k_g(s) = \frac{\sqrt{2}}{2}, \quad k_n(s) = 1, \quad \tau_g(s) = \frac{\sqrt{6}}{2}.$$

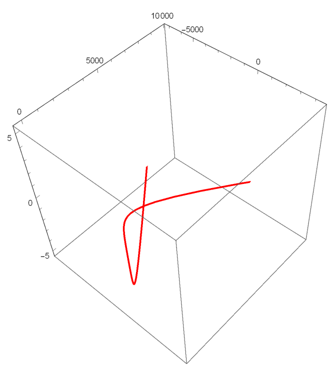
Then, the vector fields D_n, D_r and D_o are given by

$$\begin{aligned} D_n &= \left(-\frac{\sqrt{3}}{2}\cosh s, \frac{\sqrt{3}}{2}\sinh s, -\frac{1}{2}\right), \\ D_r &= \left(\sinh s + \sqrt{3}\cosh s, \cosh s + \sqrt{3}\sinh s, 2\right), \\ D_o &= \left(-\sinh s - \frac{\sqrt{3}}{2}\cosh s, -\cosh s - \frac{\sqrt{3}}{2}\sinh s, \frac{1}{2}\right). \end{aligned}$$

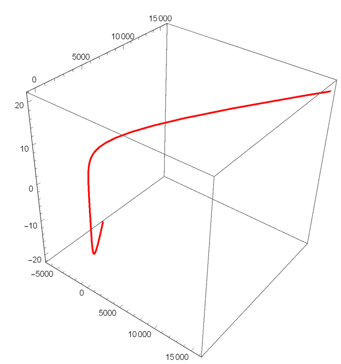
Therefore, the position vectors of D_n-, D_r- and D_o- direction curves of α are given by

$$\begin{aligned} \beta(s) &= \left(-\frac{\sqrt{3}}{2}\sinh s, \frac{\sqrt{3}}{2}\cosh s, -\frac{s}{2}\right) + C_1, \\ \bar{\beta}(s) &= \left(\frac{\sqrt{2}}{2}\cosh s + \frac{\sqrt{6}}{2}\sinh s, \frac{\sqrt{2}}{2}\sinh s + \frac{\sqrt{6}}{2}\cosh s, 2s\right) + C_2, \\ \bar{\bar{\beta}}(s) &= \left(-\sqrt{2}\cosh s - \frac{\sqrt{6}}{2}\sinh s, -\sqrt{2}\sinh s - \frac{\sqrt{6}}{2}\cosh s, \frac{\sqrt{2}}{2}s\right) + C_3, \end{aligned}$$

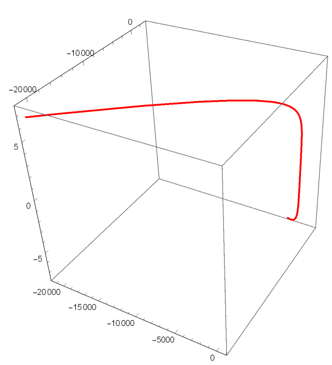
where C_1, C_2, C_3 are constant vectors.



(A) The D_n- direction curves β of α



(B) The D_r- direction curves $\bar{\beta}$ of α



(C) The D_o- direction curves $\bar{\bar{\beta}}$ of α

FIGURE 2. Plots of direction curves

$\beta(s) = \left(-\frac{\sqrt{3}}{2}\sinh s, \frac{\sqrt{3}}{2}\cosh s, -\frac{s}{2}\right) + C_1,$

Example 2. *Let us consider the timelike cylindrical ruled surface M in $\mathbb{E}_1^3$ parametrized as shown in Figure 3*

$$X(u,t)=\left(-\frac{u^5}{40},-\frac{u^5}{40}+u,\frac{u^3}{6}\right)+t(1,1,0). \tag{4.2}$$

with its base curve parametrized as

$$\alpha(u)=\left(-\frac{u^5}{40},-\frac{u^5}{40}+u,\frac{u^3}{6}\right).$$

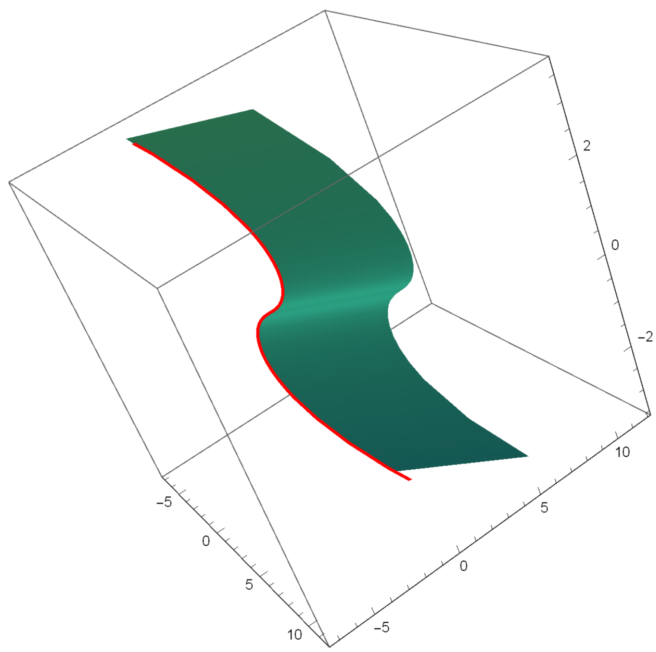


Figure 3. The timelike ruled surface M and the curve α

The Frenet frame vectors of α have the form

$$\begin{aligned} T &= \left(-\frac{u^4}{8},-\frac{u^4}{8}+1,\frac{u^2}{2}\right), \\ N &= \left(-\frac{u^2}{2},-\frac{u^2}{2},1\right), \\ B &= \left(\frac{u^4}{8}+1,\frac{u^4}{8},-\frac{u^2}{2}\right) \end{aligned}$$

and the curvatures κ and τ of α read

$$\kappa=u, \quad \tau=u.$$

Consequently, α is a unit spacelike curve with the spacelike principal normal. By using the relations (2.4), (2.5), (2.6) and (4.2), we find that the Darboux frame of α is given by

$$\begin{aligned} T &= \left(-\frac{u^4}{8}, -\frac{u^4}{8}+1, \frac{u^2}{2}\right), \\ \zeta &= \left(\frac{u^4}{8}+1, \frac{u^4}{8}, -\frac{u^2}{2}\right), \\ \eta &= \left(\frac{u^2}{2}, \frac{u^2}{2}, -1\right). \end{aligned}$$

According to the relation (2.7), the curvatures k_g, k_n and τ_g of α read

$$k_g(u) = 0, \quad k_n(u) = -u, \quad \tau_g(s) = u.$$

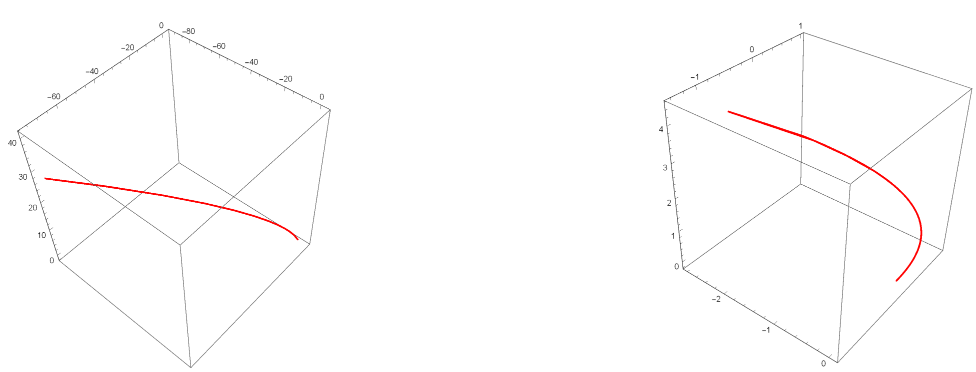
Then, the vector fields D_n, D_r and D_o are given by

$$\begin{aligned} D_n &= \left(-\frac{u^5}{8}-u, -\frac{u^5}{8}, \frac{u^3}{2}\right), \\ D_r &= \left(-\frac{u^5}{8}, -\frac{u^5}{8}+u, \frac{u^3}{2}\right), \\ D_o &= (u, u, 0). \end{aligned}$$

Therefore, from Theorem 11, the D_o- direction curves of α is a straight line, and also the position vectors of D_n- and D_r- direction curves of α are given by

$$\begin{aligned} \beta(s) &= \left(-\frac{u^6}{48}-\frac{u^2}{2}, -\frac{u^6}{48}, \frac{u^4}{8}\right) + \bar{C}_1, \\ \bar{\beta}(s) &= \left(-\frac{u^6}{48}, -\frac{u^6}{48}+\frac{u^2}{2}, \frac{u^4}{8}\right) + \bar{C}_2, \end{aligned}$$

where $\bar{C}_1$ and $\bar{C}_2$ are constant vectors.



(A) The D_n- direction curves β of α

(B) The D_r- direction curves $\bar{\beta}$ of α

Figure 4. Plots of direction curves of α

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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