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ON CLOSED ELEMENTS IN AN ALMOST SEMI-HEYTING ALGEBRA

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Abstract. In this paper, we introduce and study closed elements in an almost semi-Heyting algebra and explore the fundamental characteristics of closed elements in terms of an implication within the framework of almost semi-Heyting algebras. We rigorously study the nature of closed elements in an almost semi-Heyting algebra as well as in a semi-Heyting almost distributive lattice in various aspects. Mainly, we derive a class of pseudo-complemented lattices, a class of Heyting algebras and a class of Boolean algebras in an almost semi-Heyting algebra. Finally, we prove the class of closed elements in an almost semi-Heyting algebra forms a Boolean algebra.

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1. Introduction

One of the interesting mathematical structures that lies at the intersection of all posets which are neither lattices nor distributive is known as an almost distributive lattice [11]. Let's take a deeper look at the introduction to understand almost distributive lattices fully. According to [1,3], a poset is referred to as a lattice when each pair of elements has the least upper bound (usually abbreviated as join $\veebar$) and a greatest lower bound (usually abbreviated as meet $\barwedge$). This pair of operations, meet and join allows us to compare and combine elements within the lattice systematically. What sets distributive

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lattices [1,3] apart is their adherence to the distributive law a fundamental principle in mathematics. The distributive law states that for any elements $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{r}$ within the lattice $\mathfrak{p} \curlywedge (\mathfrak{q} \curlyvee \mathfrak{r}) = (\mathfrak{p} \curlywedge \mathfrak{q}) \curlyvee (\mathfrak{p} \curlywedge \mathfrak{r})$, the meet and join operations interact in a way that distributes over one another, ensuring a consistent and orderly structure. Here almost distributive lattices [11] add a unique criteria, they preserve the fundamental lattice characteristics of meets and joins while allowing small deviations from the distributive law. All of the features of a distributive lattice, with the possible exception of commutativity of $\curlyvee$, commutativity of $\curlywedge$, associativity of $\curlyvee$ and the right distributivity of $\curlyvee$ over $\curlywedge$ were found to be satisfied by an almost distributive lattice.

An almost semi-Heyting algebra [2,8] is a mathematical structure that extends the framework of almost distributive lattices by introducing a binary operation $\twoheadrightarrow$ on the lattice. This operation is defined in such a way that it adheres to a set of intriguing conditions, which include interactions with a maximal element u. The result is a rich algebraic structure where meet ($\curlywedge$), join ($\curlyvee$), implication ($\twoheadrightarrow$) and the maximal element u harmoniously coexist.

McKinsey and Tarski initiated to characterisation of the class of closure algebras [4] through closed elements in 1946 and continued the same by many authors in different structures of almost distributive lattices [5–7].

Almost semi-Heyting algebras, denoted as $(\mathbb{A}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, u)$, represent a unique and intriguing class of algebraic structures with distinct properties. One of the defining characteristics of almost semi-Heyting algebras is the concept of closed elements, which play a fundamental role in understanding algebra's behaviour. In this context, a closed element often denoted as $x^{\circledcirc}$ (where x is an element of the algebra), is defined as $(x \twoheadrightarrow 0) \curlywedge u$. These closed elements exhibit remarkable properties and significance within this algebraic framework. Closed elements are a fundamental concept within almost semi-Heyting algebras, and they are defined by the interplay of two crucial components the implication operation ($\twoheadrightarrow$) and the algebra's maximal element u. The expression $(x \twoheadrightarrow 0) \curlywedge u$ captures the essence of a closed element, and it represents a specific relationship between an element x and the constant 0 both under the influence of the maximal element u. To provide a more comprehensive understanding, let's break down the definition:

Implication ($\twoheadrightarrow$): The implication operation, denoted as $\twoheadrightarrow$, signifies the relationship between two elements within the algebra. It represents a form of logical implication, where $x \twoheadrightarrow 0$ expresses the idea that x implies 0.

Maximal Element (u): The maximal element u serves as a cornerstone in almost semi-Heyting algebras, connecting various algebraic operations. It plays a pivotal role in defining and understanding closed elements.

Closed Element ($x^{\circledcirc}$): The closed element $x^{\circledcirc}$ is formed by taking the conjunction ($\curlywedge$) of $x \twoheadrightarrow 0$ and u. This conjunction captures a unique property of x within the algebra, indicating that x is closed under the operation of implication, with the maximal element u influencing this closure.

The study of closed elements within almost semi-Heyting algebras has far-reaching implications in various mathematical, logical, and computational applications. Understanding closed elements helps in analyzing the algebra's structure and its behaviour when subjected to logical operations and constraints.

In this exploration, we will delve deeper into the concept of closed elements and uncover their properties, significance and applications within almost semi-Heyting algebras. These closed elements offer valuable insights into algebra's inner workings and provide a foundation for investigating the broader implications of this unique algebraic structure in the realms of formal logic and mathematics. Mainly we have obtained a class of pseudo-complemented almost distributive lattices [12], a class of pseudo-complemented lattices, another class of almost semi-Heyting algebras, a class of Boolean algebras [10] in an almost semi-Heyting algebra.

2. Preliminaries

A few definitions and findings that are important for the sequel will be reviewed here.

Definition 2.1. [11] The term Almost Distributive Lattice,or ADL for short, refers to $(\mathbb{A}, \curlyvee, \curlywedge, 0)$, an algebra of type $(2, 2, 0)$ with the conditions: for all $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathbb{A}$,

(1) $\mathfrak{p} \curlyvee 0 = \mathfrak{p}$
(2) $0 \curlywedge \mathfrak{p} = 0$
(3) $(\mathfrak{p} \curlyvee \mathfrak{q}) \curlywedge \mathfrak{r} = (\mathfrak{p} \curlywedge \mathfrak{r}) \curlyvee (\mathfrak{q} \curlywedge \mathfrak{r})$
(4) $\mathfrak{p} \curlywedge (\mathfrak{q} \curlyvee \mathfrak{r}) = (\mathfrak{p} \curlywedge \mathfrak{q}) \curlyvee (\mathfrak{p} \curlywedge \mathfrak{r})$
(5) $\mathfrak{p} \curlyvee (\mathfrak{q} \curlywedge \mathfrak{r}) = (\mathfrak{p} \curlyvee \mathfrak{q}) \curlywedge (\mathfrak{p} \curlyvee \mathfrak{r})$
(6) $(\mathfrak{p} \curlyvee \mathfrak{q}) \curlywedge \mathfrak{q} = \mathfrak{q}$.

Example 2.2. [11] Upon defining $\mathfrak{p} \curlyvee \mathfrak{q} = \mathfrak{p}$ and $\mathfrak{p} \curlywedge \mathfrak{q} = \mathfrak{q}$, for any $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}$ and $\mathbb{A}$ is a non-empty set, we observed that $(\mathbb{A}, \curlyvee, \curlywedge)$ is an ADL and is considered as a discrete ADL.

The term $\mathbb{A}$ refers to an almost distributive lattice $(\mathbb{A}, \curlyvee, \curlywedge)$, unless otherwise specified, throughout the preliminaries section.

A relation $\leq$ on $\mathbb{A}$ is defined by $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p} = \mathfrak{p} \curlywedge \mathfrak{q}$ or alternatively $\mathfrak{p} \curlyvee \mathfrak{q} = \mathfrak{q}$, for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}$. As a result, $\leq$ represents a partial ordering on $\mathbb{A}$. If there is no element $\mathfrak{p}$ such that $\mathfrak{u} < \mathfrak{p}$, then an element $\mathfrak{u}$ is regarded as maximum.

Theorem 2.3. *[11] The following conditions are interchangeable for any $\mathfrak{u} \in \mathbb{A}$,* *(1)* $\mathfrak{u}$ *a Maximal element*

(2) $\mathfrak{u} \curlyvee \mathfrak{p} = \mathfrak{u}, \forall \mathfrak{p} \in \mathbb{A}$

(3) $\mathfrak{u} \curlywedge \mathfrak{p} = \mathfrak{p}, \forall \mathfrak{p} \in \mathbb{A}$.

Definition 2.4. [12] If a unary operation $\circledcirc$ on $\mathbb{A}$, satisfies the following conditions, it is referred to be a pseudo-complementation: for all $\mathfrak{p}, \mathfrak{q} \in \mathbb{A}$,

(1) $\mathfrak{p} \curlywedge \mathfrak{p}^{\circledcirc} = 0$

(2) $\mathfrak{p} \curlywedge \mathfrak{q} = 0 \Leftrightarrow \mathfrak{p}^{\circledcirc} \curlywedge \mathfrak{q} = \mathfrak{q}$

(3) $(\mathfrak{p} \curlyvee \mathfrak{q})^{\circledcirc} = \mathfrak{p}^{\circledcirc} \curlywedge \mathfrak{q}^{\circledcirc}$.

For any binary operation $\twoheadrightarrow$ in an ADL $(\mathbb{A}, \curlyvee, \curlywedge, 0)$ with a maximal element u, let us denote the identities, given $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathbb{A}$, Considering an ADL $(\mathbb{A}, \curlyvee, \curlywedge, 0)$ with a maximal element u, let us indicate the identities for any binary operation $\twoheadrightarrow$ given $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \mathbb{A}$,

PI(1) $[(\mathfrak{p} \curlywedge \mathfrak{q}) \twoheadrightarrow \mathfrak{q}] \curlywedge \mathfrak{u} = \mathfrak{u}$

PI(2) $\mathfrak{p} \curlywedge (\mathfrak{p} \twoheadrightarrow \mathfrak{q}) = \mathfrak{p} \curlywedge \mathfrak{q} \curlywedge \mathfrak{u}$

PI(3) $\mathfrak{p} \curlywedge (\mathfrak{q} \twoheadrightarrow \mathfrak{r}) = \mathfrak{p} \curlywedge [(\mathfrak{p} \curlywedge \mathfrak{q}) \twoheadrightarrow (\mathfrak{p} \curlywedge \mathfrak{r})]$

PI(4) $(\mathfrak{p} \twoheadrightarrow \mathfrak{q}) \curlywedge \mathfrak{u} = (\mathfrak{p} \curlywedge \mathfrak{u}) \twoheadrightarrow (\mathfrak{q} \curlywedge \mathfrak{u})$

PI(5) $\mathfrak{p} \twoheadrightarrow \ \mathfrak{p} = \mathfrak{u}$

PI(6) $(\mathfrak{p} \twoheadrightarrow \mathfrak{q}) \curlywedge \mathfrak{q} = \mathfrak{q}$

PI(7) $\mathfrak{p} \twoheadrightarrow (\mathfrak{q} \curlywedge \mathfrak{r}) = (\mathfrak{p} \twoheadrightarrow q) \curlywedge (\mathfrak{p} \twoheadrightarrow \mathfrak{r})$

PI(8) $(\mathfrak{p} \curlyvee \mathfrak{q}) \twoheadrightarrow r = (\mathfrak{p} \twoheadrightarrow \mathfrak{r}) \curlywedge (\mathfrak{q} \twoheadrightarrow \mathfrak{r})$.

Now, we have the following identities which are the consequences of PI(1), PI(2), PI(3) and PI(4):

CI(1) $(\mathfrak{p} \twoheadrightarrow \mathfrak{p}) \curlywedge \mathfrak{u} = \mathfrak{u}$

CI(2) $[\mathfrak{p} \curlywedge (\mathfrak{p} \twoheadrightarrow \mathfrak{q})] \curlywedge \mathfrak{u} = \mathfrak{p} \curlywedge \mathfrak{q} \curlywedge \mathfrak{u}$

CI(3) $[\mathfrak{p} \curlywedge (\mathfrak{q} \twoheadrightarrow \mathfrak{r})] \curlywedge \mathfrak{u} = [\mathfrak{p} \curlywedge [(\mathfrak{p} \curlywedge \mathfrak{q}) \twoheadrightarrow (\mathfrak{p} \curlywedge \mathfrak{r})]] \curlywedge \mathfrak{u}$

CI(4) $(\mathfrak{p} \twoheadrightarrow \mathfrak{q}) \curlywedge \mathfrak{u} = [(\mathfrak{p} \curlywedge \mathfrak{u}) \twoheadrightarrow (\mathfrak{q} \curlywedge \mathfrak{u})] \curlywedge \mathfrak{u}$.

Definition 2.5. [7] $\mathbb{A}$ is a semi-Heyting almost distributive lattice (abbreviated: SHADL) if $\mathbb{A}$ holds CI(1), PI(2), PI(3) and PI(4) with a maximal element u.

Definition 2.6. [8] $\mathbb{A}$ is an almost semi-Heyting algebra (abbreviated: ASHA) if $\mathbb{A}$ holds PI(1), CI(2), CI(3) and CI(4) with a maximal element u.

3. Closed elements in almost semi-Heyting algebras

In this section, we define closed elements resembling those found in a semi-Heyting almost distributive lattice and demonstrate some fundamental algebraic conditions on them.

The authors defined two algebras on an almost distributive lattice in [7] and [8], one was called a semi-Heyting almost distributive lattice, an abstraction from semi-Heyting algebra, and the other was called an almost semi-Heyting algebra, which is a generalisation of a Heyting algebra. The authors also noted that these two algebras are distinct from one another by using examples in [8].

From the perspective of the variations between these two algebras, we began to discuss closed elements in an almost semi-Heyting algebra that are similar to those found in a semi-Heyting almost distributive lattice.

Let us denote $x^{\circledcirc} = (x \twoheadrightarrow 0) \curlywedge u$ for any element x in $\mathbb{A}$, where u is the maximal element in $\mathbb{A}$.

Definition 3.1. If $x^{\circledcirc\circledcirc} = x$, then the element x is a *closed element* in an almost semi-Heyting algebra $\mathbb{A}$.

Remark 3.2. 1. Zero element is always closed in $\mathbb{A}$.

2. Every maximal element is closed in $\mathbb{A}$.

3. There are some elements in $\mathbb{A}$ which are not closed. See the illustration below as verification.

Example 3.3. Let $\mathbb{A} = \{0, \mathfrak{x}, \mathfrak{u}\}$ be a discrete ADL of 3-elements, where the binary operation $\twoheadrightarrow$ given as:

$\twoheadrightarrow$	0	$\mathfrak{x}$	$\mathfrak{u}$
0	m	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{x}$	0	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{u}$	0	$\mathfrak{x}$	$\mathfrak{u}$

Then $(L, \curlyvee, \curlywedge, \twoheadrightarrow, 0, m)$ is an ASHA. Here $\mathfrak{x}^{\circledcirc\circledcirc} = u \neq \mathfrak{x}$.

The properties discussed in Theorem 3.4 provide mathematical relationships and insights into the behaviour of elements within an almost semi-Heyting algebra, straightening out how the algebraic operations interact and influence one another. They are essential for understanding the algebra's structure and properties and can be applied in various computational contexts.

Theorem 3.4. *For any $x, y, z \in \mathbb{A}$, where $\mathbb{A}$ is an ASHA and u represents the maximal element, the following hold:*

(1) $x \curlywedge (x \twoheadrightarrow y)^{\circledcirc} = x \curlywedge y^{\circledcirc}$

(2) $x \curlywedge y = 0 \Leftrightarrow x \curlywedge u \leq y^{\circledcirc}$

(3) $u^{\circledcirc} = 0$

(4) $x^{\circledcirc} = u \Leftrightarrow x = 0$

(5) $x \curlywedge x^{\circledcirc} = 0 \Rightarrow x \curlywedge u \leq x^{\circledcirc\circledcirc}$

(6) $x \leq y \Rightarrow y^{\circledcirc} \leq x^{\circledcirc}, x^{\circledcirc\circledcirc} \leq y^{\circledcirc\circledcirc}$

(7) $x \curlywedge y^{\circledcirc} = x \curlywedge (x \curlywedge y)^{\circledcirc}$

(8) $(x \curlywedge y)^{\circledcirc} = (y \curlywedge x)^{\circledcirc}$*. In particular,* $(x \curlywedge u)^{\circledcirc} = x^{\circledcirc}$

(9) $x \curlywedge x^{\circledcirc\circledcirc} = x \curlywedge u$ *and* $x^{\circledcirc\circledcirc} \curlywedge x = x$

(10) $x^{\circledcirc} = x^{\circledcirc\circledcirc\circledcirc}$

(11) $(x \dot{\vee} y)^{\circledcirc} = x^{\circledcirc} \dot{\wedge} y^{\circledcirc}$

(12) $y \dot{\wedge} x = x \Rightarrow x \dot{\wedge} y^{\circledcirc} = 0$

(13) $x^{\circledcirc} \dot{\wedge} y^{\circledcirc} = y^{\circledcirc} \dot{\wedge} x^{\circledcirc}$

(14) $x^{\circledcirc} \dot{\vee} y^{\circledcirc} = y^{\circledcirc} \dot{\vee} x^{\circledcirc}$

(15) $x \dot{\wedge} y = 0 \Rightarrow x^{\circledcirc\circledcirc} \dot{\wedge} y = 0$

(16) $(x \dot{\wedge} y)^{\circledcirc\circledcirc} \leq x^{\circledcirc\circledcirc}$

(17) $x^{\circledcirc\circledcirc} \dot{\wedge} (x \twoheadrightarrow y)^{\circledcirc\circledcirc} = x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc}$

(18) *If* $x \dot{\wedge} y = 0 \Leftrightarrow x^{\circledcirc} \dot{\wedge} y = y$

(19) $(0 \twoheadrightarrow u) \dot{\wedge} u = 0 \Leftrightarrow (0 \twoheadrightarrow x) \dot{\wedge} u \leq x^{\circledcirc}$ *for all* $x \in \mathbb{A}$

(20) $x^{\circledcirc} \leq (0 \twoheadrightarrow x) \dot{\wedge} u$, *In particular* $x^{\circledcirc} \leq (0 \twoheadrightarrow x^{\circledcirc\circledcirc}) \dot{\wedge} u$

(21) $x \dot{\wedge} u \leq (0 \twoheadrightarrow x^{\circledcirc}) \dot{\wedge} u$

(22) $(x \twoheadrightarrow x^{\circledcirc}) \dot{\wedge} u \leq x^{\circledcirc} \leq (x^{\circledcirc\circledcirc} \twoheadrightarrow x) \dot{\wedge} u$

(23) *If* $x^{\circledcirc} \leq 0 \hookrightarrow x^{\circledcirc}$, *then* $(x \hookrightarrow x^{\circledcirc}) \dot{\wedge} u = x^{\circledcirc}$

(24) $x \dot{\wedge} u \leq (x^{\circledcirc\circledcirc} \twoheadrightarrow x) \dot{\wedge} u$

(25) $x \dot{\wedge} u \leq (x \twoheadrightarrow x^{\circledcirc\circledcirc}) \dot{\wedge} u$

(26) $(x^{\circledcirc\circledcirc} \hookrightarrow x^{\circledcirc}) \dot{\wedge} u \leq x^{\circledcirc} \leq (x \twoheadrightarrow x^{\circledcirc\circledcirc}) \dot{\wedge} u$

(27) $z \leq x \Rightarrow z \dot{\wedge} (x \hookrightarrow y^{\circledcirc}) \dot{\wedge} u = z \dot{\wedge} y^{\circledcirc}$

(28) $x \dot{\wedge} u \leq (0 \twoheadrightarrow x^{\circledcirc\circledcirc}) \dot{\wedge} u \Leftrightarrow x \dot{\wedge} u \leq (0 \twoheadrightarrow x) \dot{\wedge} u$

(29) $y \dot{\wedge} (x \twoheadrightarrow y^{\circledcirc}) \dot{\wedge} u = y \dot{\wedge} x^{\circledcirc}$

(30) $y^{\circledcirc} \dot{\wedge} (x \twoheadrightarrow y) \dot{\wedge} u = y^{\circledcirc} \dot{\wedge} x^{\circledcirc}$

(31) *If* $x \dot{\wedge} y = 0$, *then* $(x \twoheadrightarrow y) \dot{\wedge} u \leq x^{\circledcirc}$

(32) $(x^{\circledcirc} \dot{\vee} y^{\circledcirc})^{\circledcirc\circledcirc} = (x \dot{\wedge} y)^{\circledcirc} = x^{\circledcirc} \underline{\dot{\vee}} y^{\circledcirc}$, *where* $x \underline{\dot{\vee}} y = (x^{\circledcirc} \dot{\wedge} x^{\circledcirc})^{\circledcirc}$

(33) $(x \dot{\wedge} y)^{\circledcirc} = (x \twoheadrightarrow y^{\circledcirc}) \dot{\wedge} u$.

Remark 3.5. If both x and y are closed, then $x \dot{\wedge} y$ is also closed ($(x \dot{\wedge} y)^{\circledcirc\circledcirc} = x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc}$). Because it is simple to see that $(x \dot{\wedge} y)^{\circledcirc\circledcirc} \leq x^{\circledcirc\circledcirc}$ and $(x \dot{\wedge} y)^{\circledcirc\circledcirc} \leq y^{\circledcirc\circledcirc}$. Consequently, $(x \dot{\wedge} y)^{\circledcirc\circledcirc} \leq x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc}$ (by (16) of Theorem 3.4). The fact that

$$\begin{aligned}
x \dot{\wedge} y \dot{\wedge} (x \dot{\wedge} y)^{\circledcirc} = 0 \quad &\Rightarrow \quad x^{\circledcirc\circledcirc} \dot{\wedge} y \dot{\wedge} (x \dot{\wedge} y)^{\circledcirc} = 0 && \text{(by (15) of Theorem 3.4)} \\
&\Rightarrow \quad y \dot{\wedge} x^{\circledcirc\circledcirc} \dot{\wedge} (x \dot{\wedge} y)^{\circledcirc} = 0 && \\
&\Rightarrow \quad y^{\circledcirc\circledcirc} \dot{\wedge} x^{\circledcirc\circledcirc} \dot{\wedge} (x \dot{\wedge} y)^{\circledcirc} = 0 && \text{(by (15) of Theorem 3.4)} \\
&\Rightarrow \quad x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc} \dot{\wedge} (x \dot{\wedge} y)^{\circledcirc} = 0 && \\
&\Rightarrow \quad x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc} \leq (x \dot{\wedge} y)^{\circledcirc\circledcirc}. && \text{(by (2) of Theorem 3.4)}
\end{aligned}$$

Therefore, $(x \dot{\wedge} y)^{\circledcirc\circledcirc} = x^{\circledcirc\circledcirc} \dot{\wedge} y^{\circledcirc\circledcirc}$.

Remark 3.6. If both x and y are closed, then $x \dot{\vee} y$ does not need to be closed $(x \dot{\vee} y)^{\circledcirc\circledcirc} \neq x^{\circledcirc\circledcirc} \dot{\vee} y^{\circledcirc\circledcirc}$.

Example 3.7. Let $\mathbb{A} = \{0, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{u}\}$ be a distributive lattice, in which $\curlyvee$, $\curlywedge$ and $\hookrightarrow$ are as follows.

$\curlyvee$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$
0	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$
$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{u}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$
$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathfrak{u}$
$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$

$\curlywedge$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$
0	0	0	0	0	0
$\mathfrak{a}$	0	$\mathfrak{a}$	0	$\mathfrak{a}$	$\mathfrak{a}$
$\mathfrak{b}$	0	0	$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{b}$
$\mathfrak{c}$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{c}$
$\mathfrak{u}$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$

$\twoheadrightarrow$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$
0	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{u}$	$\mathfrak{b}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{b}$	$\mathfrak{a}$	$\mathfrak{a}$	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{c}$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{u}$	0	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{u}$

Clearly, $(\mathbb{A}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, \mathfrak{u})$ is an ASHA. It is observed that $0, \mathfrak{a}, \mathfrak{b}, \mathfrak{u}$ are closed elements and $\mathfrak{c}$ is not ($\mathfrak{c}^{\circledcirc\circledcirc} = \mathfrak{u} \neq \mathfrak{c}$). Now, put $x = \mathfrak{a}$ and $y = \mathfrak{b}$, then $(x \curlyvee y)^{\circledcirc\circledcirc} = (\mathfrak{a} \curlyvee \mathfrak{b})^{\circledcirc\circledcirc} = \mathfrak{c}^{\circledcirc\circledcirc} = \mathfrak{u}$ and $x^{\circledcirc\circledcirc} \curlyvee y^{\circledcirc\circledcirc} = \mathfrak{a}^{\circledcirc\circledcirc} \curlyvee \mathfrak{b}^{\circledcirc\circledcirc} = \mathfrak{a} \curlyvee \mathfrak{b} = \mathfrak{c}$. Therefore, $(x \curlyvee y)^{\circledcirc\circledcirc} \neq x^{\circledcirc\circledcirc} \curlyvee y^{\circledcirc\circledcirc}$.

Remark 3.8. In an SHADL, the property (33) from Theorem 3.4 does not hold. Consider the example below.

Example 3.9. Suppose $\mathbb{A} = \{0, \mathfrak{a}, \mathfrak{u}\}$ is a chain, on which $\twoheadrightarrow$ is specified as;

$\twoheadrightarrow$	0	$\mathfrak{a}$	$\mathfrak{u}$
0	$\mathfrak{u}$	$\mathfrak{u}$	$\mathfrak{u}$
$\mathfrak{a}$	0	$\mathfrak{u}$	$\mathfrak{a}$
$\mathfrak{u}$	0	$\mathfrak{a}$	$\mathfrak{u}$

It is observed, $(\mathbb{A}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, \mathfrak{u})$ is an SHADL but not an ASHA (since $[(\mathfrak{a} \curlywedge \mathfrak{u}) \hookrightarrow \mathfrak{u}] \curlywedge \mathfrak{u} \neq \mathfrak{u}$). If we take $x = \mathfrak{a}$ and $y = 0$, then $[\mathfrak{a} \curlywedge 0]^{\circledcirc} = 0^{\circledcirc} = \mathfrak{u}$ and $[\mathfrak{a} \twoheadrightarrow 0^{\circledcirc}] \curlywedge \mathfrak{u} = \mathfrak{a} \curlywedge \mathfrak{u} = \mathfrak{a}$. Therefore, $(x \curlywedge y)^{\circledcirc} \neq (x \twoheadrightarrow y^{\circledcirc}) \curlywedge \mathfrak{u}$.

Let us represent the collection of all closed elements of an almost semi-Heyting algebra $\mathbb{A}$ by $\mathbb{C}$ and for any $x, y \in \mathbb{C}$, let us define $x \underline{\curlyvee} y = (x^{\circledcirc} \curlywedge y^{\circledcirc})^{\circledcirc}$. Also, we can observe that $x^{\circledcirc} \underline{\curlyvee} x = u$ for $x \in \mathbb{C}$.

Remark 3.10. C does not form a filter in $\mathbb{A}$.

Theorem 3.11 quotes three properties that are valid in an ASHA but invalid in an SHADL.

Theorem 3.11. *For any $x, y \in \mathbb{A}$, where $\mathbb{A}$ is an ASHA the following hold.*

(1) $(x^{\circledcirc\circledcirc} \curlywedge y)^{\circledcirc} = (x \curlywedge y)^{\circledcirc}$

(2) $(x \twoheadrightarrow y^{\circledcirc})^{\circledcirc\circledcirc} = (x^{\circledcirc\circledcirc} \twoheadrightarrow y^{\circledcirc}) \curlywedge u$

(3) $(x \curlywedge (y^{\circledcirc} \curlyvee y))^{\circledcirc} = x^{\circledcirc}$.

If $p \twoheadrightarrow q = p^{\circledcirc} \curlyvee q$, then it is obvious that a Boolean algebra is a Heyting algebra. Here is what we have in an ASHA as an alternative.

Theorem 3.12. $(x \twoheadrightarrow y)^{\circledcirc\circledcirc} \leq x^{\circledcirc} \underline{\curlyvee}\, y^{\circledcirc\circledcirc}$, *for* $x, y \in \mathbb{A}$, *where* $\mathbb{A}$ *is an ASHA.*

The next Theorems 3.13 and 3.14 allow us to derive some important features of an ASHA using the operation $\circledcirc$.

Theorem 3.13. $(x \curlyvee x^{\circledcirc}) \curlywedge (x \twoheadrightarrow y) \curlywedge u \leq (x^{\circledcirc} \curlyvee y) \curlywedge u$*., for* $x, y \in \mathbb{A}$, *where* $\mathbb{A}$ *is an ASHA.*

Proof. $(x \curlyvee x^{\circledcirc}) \curlywedge (x \twoheadrightarrow y) \curlywedge u = [[x \curlywedge (x \twoheadrightarrow y)] \curlyvee [x^{\circledcirc} \curlywedge (x \twoheadrightarrow y)]] \curlywedge u$
$= [(x \curlywedge y \curlywedge u) \curlyvee (x^{\circledcirc} \curlywedge (x \twoheadrightarrow y))] \curlywedge u$
$= [(x \curlywedge y \curlywedge u) \curlyvee x^{\circledcirc}] \curlywedge [(x \curlywedge y \curlywedge u) \curlyvee (x \twoheadrightarrow y)] \curlywedge u$
$\leq [x^{\circledcirc} \curlyvee (x \curlywedge y \curlywedge u)]$
$= (x^{\circledcirc} \curlyvee x) \curlywedge (x^{\circledcirc} \curlyvee (y \curlywedge u)) \leq (x^{\circledcirc} \curlyvee y) \curlywedge u$. □

Theorem 3.14. *For* $x, y, z \in \mathbb{A}$, *where* $\mathbb{A}$ *is an ASHA, we have*

(1) $x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc} = x^{\circledcirc} \underline{\curlyvee} z^{\circledcirc} \Leftrightarrow x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge z^{\circledcirc}$

(2) $x^{\circledcirc\circledcirc} \curlywedge (x \twoheadrightarrow y)^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc}$

(3) $x^{\circledcirc\circledcirc} \curlywedge (y \twoheadrightarrow z)^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge [(x \curlywedge y) \twoheadrightarrow (x \curlywedge z)]^{\circledcirc}$

(4) $y^{\circledcirc} \curlywedge (x \twoheadrightarrow y) \curlywedge u = y^{\circledcirc} \curlywedge x^{\circledcirc}$.

Proof. Let $x, y, z \in \mathbb{A}$.

(1) Suppose $x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc} = x^{\circledcirc} \underline{\curlyvee} z^{\circledcirc}$, then $x^{\circledcirc\circledcirc} \curlywedge (x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc}) = x^{\circledcirc\circledcirc} \curlywedge (x^{\circledcirc} \underline{\curlyvee} z^{\circledcirc})$
$\Rightarrow (x^{\circledcirc\circledcirc} \curlywedge x^{\circledcirc}) \underline{\curlyvee} (x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc}) = (x^{\circledcirc\circledcirc} \curlywedge x^{\circledcirc}) \underline{\curlyvee} (x^{\circledcirc\circledcirc} \curlywedge z^{\circledcirc})$
$\Rightarrow x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge z^{\circledcirc}$. On the other hand, if $x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge z^{\circledcirc}$, then $x^{\circledcirc} \underline{\curlyvee} (x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc}) = x^{\circledcirc} \underline{\curlyvee} (x^{\circledcirc\circledcirc} \curlywedge z^{\circledcirc})$
$\Rightarrow (x^{\circledcirc} \underline{\curlyvee} x^{\circledcirc\circledcirc}) \curlywedge (x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc}) = (x^{\circledcirc} \underline{\curlyvee} x^{\circledcirc\circledcirc}) \curlywedge (x^{\circledcirc} \underline{\curlyvee} z^{\circledcirc})$
$\Rightarrow x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc} = x^{\circledcirc} \underline{\curlyvee} z^{\circledcirc}$.

(2) Consider, $x^{\circledcirc} \underline{\curlyvee} (x \twoheadrightarrow y)^{\circledcirc}) = (x^{\circledcirc\circledcirc} \curlywedge (x \twoheadrightarrow y)^{\circledcirc\circledcirc})^{\circledcirc} = (x \curlywedge (x \twoheadrightarrow y))^{\circledcirc} = (x \curlywedge y \curlywedge m)^{\circledcirc} = (x \curlywedge y)^{\circledcirc} = x^{\circledcirc} \underline{\curlyvee} y^{\circledcirc}$. Therefore, $x^{\circledcirc\circledcirc} \curlywedge (x \twoheadrightarrow y)^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge y^{\circledcirc}$ (by (i) above).

(3) $x^{\circledcirc} \underline{\curlyvee} (y \twoheadrightarrow r)^{\circledcirc} = [x \curlywedge (y \twoheadrightarrow z)]^{\circledcirc} = [x \curlywedge \{(x \curlywedge y) \twoheadrightarrow (x \curlywedge z)\}]^{\circledcirc} = x^{\circledcirc} \underline{\curlyvee}\, [(x \curlywedge y) \twoheadrightarrow (x \curlywedge z)]^{\circledcirc}$. Therefore, $x^{\circledcirc\circledcirc} \curlywedge (y \twoheadrightarrow z)^{\circledcirc} = x^{\circledcirc\circledcirc} \curlywedge [(x \curlywedge y) \twoheadrightarrow (x \curlywedge z)]^{\circledcirc}$ (by (i) above).

(4) $y^{\circledcirc} \curlywedge (x \twoheadrightarrow y) \curlywedge u = y^{\circledcirc} \curlywedge [(y^{\circledcirc} \curlywedge x) \hookrightarrow (y^{\circledcirc} \curlywedge y)] \curlywedge u = y^{\circledcirc} \curlywedge [(y^{\circledcirc} \curlywedge x) \hookrightarrow 0] \curlywedge u = y^{\circledcirc} \curlywedge (x \twoheadrightarrow 0) \curlywedge u = y^{\circledcirc} \curlywedge x^{\circledcirc}$. □

Theorem 3.15 demonstrates a relationship between intersections, implications, and the maximal element u within an ASHA. It provides insight into how the implication of 0 to various elements in the algebra relates to their intersections with the maximal element u, and how this relationship can be represented and compared within the algebraic framework.

Theorem 3.15. *For $x \in \mathbb{A}$, where $\mathbb{A}$ is an ASHA, we have, $(0 \twoheadrightarrow u) \curlywedge u = u \Leftrightarrow x \curlywedge u \leq (0 \twoheadrightarrow x) \curlywedge u$.*

Proof. Suppose $(0 \twoheadrightarrow u) \curlywedge u = u \Rightarrow x \curlywedge (0 \twoheadrightarrow u) \curlywedge u = x \curlywedge u \Rightarrow (x \curlywedge u) \curlywedge (0 \twoheadrightarrow x) \curlywedge u = x \curlywedge u \Rightarrow x \curlywedge u \leq (0 \twoheadrightarrow x) \curlywedge u$. The converse can obtained by replacing x by u. □

Finally, we extract some equivalent conditions for an ASHA to satisfy the condition $(0 \twoheadrightarrow u) \curlywedge u = 0$, before that we illustrate the following.

Lemma 3.16. *If $\mathbb{A}$ is an ASHA satisfying the condition $(0 \twoheadrightarrow u) \curlywedge u = 0$, and $x, y \in \mathbb{A}$, then*

(1) $x \curlywedge y = 0 \Rightarrow (x \twoheadrightarrow y) \curlywedge u = x^{\circledcirc} \curlywedge y^{\circledcirc}$
(2) $(0 \twoheadrightarrow y) \curlywedge u = y^{\circledcirc}$
(3) $(x \hookrightarrow x^{\circledcirc}) \curlywedge u = 0$
(4) $(x^{\circledcirc} \twoheadrightarrow x) \curlywedge u = 0$.

Proof. Suppose $x \curlywedge y = 0$. Then $y \curlywedge (x \twoheadrightarrow y) \curlywedge u = y \curlywedge [(y \curlywedge x) \twoheadrightarrow (y \curlywedge y)] \curlywedge u = y \curlywedge (0 \twoheadrightarrow u) \curlywedge u = 0 \Rightarrow (x \twoheadrightarrow y) \curlywedge u \leq y^{\circledcirc}$. Again, since $x \curlywedge (x \twoheadrightarrow y) \curlywedge u = x \curlywedge y \curlywedge u = 0 \Rightarrow (x \twoheadrightarrow y) \curlywedge u \leq x^{\circledcirc}$. Thus $(x \twoheadrightarrow y) \curlywedge u \leq x^{\circledcirc} \curlywedge y^{\circledcirc}$. Now, $y^{\circledcirc} \curlywedge (x \twoheadrightarrow y) \curlywedge u = y^{\circledcirc} \curlywedge x^{\circledcirc} \Rightarrow x^{\circledcirc} \curlywedge y^{\circledcirc} \curlywedge (x \twoheadrightarrow y) \curlywedge u = y^{\circledcirc} \curlywedge x^{\circledcirc} \Rightarrow y^{\circledcirc} \curlywedge x^{\circledcirc} \leq (x \twoheadrightarrow y) \curlywedge u$. Hence, $(x \twoheadrightarrow y) \curlywedge u = x^{\circledcirc} \curlywedge y^{\circledcirc}$. The rest follows immediately from (1). □

Theorem 3.17. *The following are identical: if $\mathbb{A}$ is an ASHA and $x, y \in \mathbb{A}$, then*

(1) $(0 \twoheadrightarrow u) \curlywedge u = 0$
(2) $(0 \twoheadrightarrow x) \curlywedge u \leq x^{\circledcirc}$
(3) $(0 \twoheadrightarrow x) \curlywedge u = x^{\circledcirc}$
(4) $x \curlywedge (x^{\circledcirc} \twoheadrightarrow y) \curlywedge u = x \curlywedge y^{\circledcirc}$
(5) $(x^{\circledcirc} \twoheadrightarrow u) \curlywedge u = x^{\circledcirc}$
(6) $(x^{\circledcirc} \twoheadrightarrow x) \curlywedge u = 0$.

Proof. Let $(0 \twoheadrightarrow u) \curlywedge m = 0$. Then $0 = x \curlywedge (0 \twoheadrightarrow u) \curlywedge u = x \curlywedge [0 \twoheadrightarrow (x \curlywedge u)] \curlywedge u = x \curlywedge (0 \twoheadrightarrow x) \curlywedge u$. Thus, $(0 \twoheadrightarrow x) \curlywedge u \leq x^{\circledcirc}$. Now, assume (2). Then $(0 \twoheadrightarrow x) \curlywedge u = x^{\circledcirc} \curlywedge (0 \twoheadrightarrow x) \curlywedge u = x^{\circledcirc} \curlywedge (0 \twoheadrightarrow 0) \curlywedge u = x^{\circledcirc}$. Suppose (3) holds. Then $x \curlywedge (x^{\circledcirc} \twoheadrightarrow y) \curlywedge u = x \curlywedge [0 \twoheadrightarrow (x \curlywedge y)] \curlywedge m = x \curlywedge (x \curlywedge y)^{\circledcirc} = x \curlywedge [(x \curlywedge y) \twoheadrightarrow 0)] \curlywedge u = x \curlywedge (y \twoheadrightarrow 0) \curlywedge u = x \curlywedge y^{\circledcirc}$. Suppose (4) holds. Replacing y by u in (4), we get $x \curlywedge (x^{\circledcirc} \twoheadrightarrow u) \curlywedge m = 0$ and hence, $(x^{\circledcirc} \twoheadrightarrow u) \leq x^{\circledcirc}$. From the definition of an almost semi-Heyting algebra, $x^{\circledcirc} = x^{\circledcirc} \curlywedge u = x^{\circledcirc} \curlywedge (x^{\circledcirc} \twoheadrightarrow u)$. Thus $x^{\circledcirc} \leq (x^{\circledcirc} \twoheadrightarrow u) \curlywedge u$. Now assume (5). $x \curlywedge (x^{\circledcirc} \twoheadrightarrow x) \curlywedge u = x \curlywedge [0 \twoheadrightarrow (x \curlywedge u)] \curlywedge u = x \curlywedge (x^{\circledcirc} \twoheadrightarrow u) \curlywedge u = x \curlywedge x^{\circledcirc} \curlywedge u = 0$ and hence, $(x^{\circledcirc} \twoheadrightarrow x) \curlywedge u \leq x^{\circledcirc}$. Also $x^{\circledcirc} \curlywedge (x^{\circledcirc} \twoheadrightarrow x) \curlywedge u = x^{\circledcirc} \curlywedge x \curlywedge u = 0$. So that $(x^{\circledcirc} \twoheadrightarrow x) \curlywedge u \leq x^{\circledcirc\circledcirc}$. Hence, $(x^{\circledcirc} \twoheadrightarrow x) \curlywedge u = 0$. Finally, by replacing x by u in (6), we get (1). □

Here, we make out a class of pseudo-complemented almost distributive lattices through the class of closed elements in an ASHA.

Theorem 3.18. *If $(\mathbb{A}, \curlyvee, \curlywedge, \hookrightarrow, 0, u)$ is an ASHA and $x^{\circledcirc} = (x \twoheadrightarrow 0) \curlywedge u$ is defined for each $x \in \mathbb{A}$, then $\circledcirc$ is a pseudo-complementation on $\mathbb{A}$.*

Proof. For $x, y \in \mathbb{A}$ and $x^{\circledcirc} = (x \twoheadrightarrow 0) \curlywedge u$. Conditions (1) and (2) of pseudo-complementation follow from (18) and (11) of Theorem 3.4 and $x \curlywedge x^{\circledcirc} = 0$. Hence, $\circledcirc$ is a pseudo-complementation on $\mathbb{A}$. □

Theorem 3.19. *If $\mathbb{A}$ is an ASHA and $x, y \in \mathbb{A} \ni x \curlywedge u \leq y \curlywedge u$, for $z \in [x \curlywedge u, y \curlywedge u]$, define $z^{\circledcirc xy} = (z \twoheadrightarrow x) \curlywedge y \curlywedge u$, then $([x \curlywedge u, y \curlywedge u], \curlyvee, \curlywedge, ^{\circledcirc xy}, x \curlywedge u, y \curlywedge u)$ is a pseudo-complemented lattice.*

Proof. It is sufficient to show that $a \curlywedge b = x \curlywedge u \Leftrightarrow a \leq b^{\circledcirc xy}$ for all $a, b \in [x \curlywedge u, y \curlywedge u]$. Let $z \in [x \curlywedge u, y \curlywedge u]$. Since $x \curlywedge u \leq z \curlywedge u$, we have $x \curlywedge u \leq (z \twoheadrightarrow x) \curlywedge u \Rightarrow x \curlywedge y \curlywedge u \leq (z \twoheadrightarrow x) \curlywedge y \curlywedge u \Rightarrow x \curlywedge u \leq (z \twoheadrightarrow x) \curlywedge y \curlywedge u \leq y \curlywedge u$. Therefore, $z^{\circledcirc xy} \in [x \curlywedge u, y \curlywedge u]$. Let $a, b \in [x \curlywedge u, y \curlywedge u]$. Assume that $a \curlywedge b = x \curlywedge u$. Then $b^{\circledcirc xy} = (b \twoheadrightarrow x) \curlywedge y \curlywedge u$, $a \curlywedge b^{\circledcirc xy} = a \curlywedge (b \twoheadrightarrow x) \curlywedge y \curlywedge u = a \curlywedge (b \twoheadrightarrow x) \curlywedge u = a \curlywedge (a \curlywedge b \twoheadrightarrow x) \curlywedge u = a \curlywedge (x \twoheadrightarrow x) \curlywedge u = a \curlywedge u = a$. Therefore, $a \leq b^{\circledcirc xy}$. Conversely, suppose $a \leq b^{\circledcirc xy} \Rightarrow a = a \curlywedge (b \twoheadrightarrow x) \curlywedge y \curlywedge u$. Now, $b \curlywedge a = b \curlywedge a \curlywedge (b \twoheadrightarrow x) \curlywedge y \curlywedge u = a \curlywedge b \curlywedge x \curlywedge y \curlywedge u = x \curlywedge u$. Hence, $([x \curlywedge u, y \curlywedge u], \curlyvee, \curlywedge, ^{\circledcirc xy}, x \curlywedge u, y \curlywedge u)$ is a pseudo-complemented lattice. □

Corollary 3.20. *The algebra $([0, u], \curlyvee, \curlywedge, ^{\circledcirc}, 0, u)$ is a pseudo-complemented lattice, where $z^{\circledcirc} = (z \twoheadrightarrow 0) \curlywedge u$ for $z \in [0, u]$, if $\mathbb{A}$ is an ASHA.*

Theorem 3.21. *Assume $\mathbb{A}$ is an ASHA and $x, y \in \mathbb{A}$ with $x \curlywedge u \leq y \curlywedge u$. For $z, t \in [x \curlywedge u, y \curlywedge u]$ define $z \twoheadrightarrow^{xy} t = (z \twoheadrightarrow t) \curlywedge y \curlywedge u$, then the algebra $\mathbb{A}_0 = [[x \curlywedge u, y \curlywedge u], \curlyvee, \curlywedge, \twoheadrightarrow^{xy}, x \curlywedge u, y \curlywedge u]$ is a Heyting algebra. Additionally, if L is an SHADL, then $\mathbb{A}_0$ is a semi-Heyting algebra as well.*

Proof. Let $z, t, s \in [x \curlywedge u, y \curlywedge u]$. First we note that $z \twoheadrightarrow^{xy} t \in [x \curlywedge u, y \curlywedge u]$. Since $x \curlywedge u \leq z \curlywedge u, x \curlywedge u \leq t \curlywedge u$, we get $x \curlywedge u \leq (z \twoheadrightarrow t) \curlywedge u$ and hence, $c \twoheadrightarrow^{xy} t \in [x \curlywedge u, y \curlywedge u]$. Now, $z \curlywedge (z \twoheadrightarrow^{xy} t) = z \curlywedge (z \twoheadrightarrow t) \curlywedge y \curlywedge u = z \curlywedge t \curlywedge y \curlywedge u = z \curlywedge t$. $s \curlywedge (z \twoheadrightarrow^{xy} t) = s \curlywedge (z \twoheadrightarrow t) \curlywedge y \curlywedge u = s \curlywedge [(s \curlywedge z) \twoheadrightarrow (s \curlywedge t)] \curlywedge y \curlywedge u = s \curlywedge [(s \curlywedge z) \twoheadrightarrow^{xy} (s \curlywedge t)]$. Finally, $[(z \curlywedge y) \twoheadrightarrow^{xy} y] = [(z \curlywedge y) \twoheadrightarrow y] \curlywedge u \curlywedge y \curlywedge u = u \curlywedge y \curlywedge u = y \curlywedge u$. Hence, $\mathbb{A}_0$ is a Heyting algebra. □

Here it can be easily proven that the set $\mathbb{C} = \{x^{\circledcirc} \mid x \in \mathbb{A}\}$ of all closed elements in an ASHA, forms a Boolean algebra (Heyting algebra) with the operations $\underline{\curlyvee}$ ($\twoheadrightarrow$) given by $x \underline{\curlyvee} y = (x^{\circledcirc} \curlywedge y^{\circledcirc})^{\circledcirc}$ ($x \twoheadrightarrow y = x^{\circledcirc} \curlyvee\ y$) for every $x, y \in \mathbb{C}$.

Theorem 3.22. $(\mathbb{C}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, u)$ *is a Heyting algebra, with $x \twoheadrightarrow y = x^{\circledcirc} \curlyvee y$ for all $x, y \in \mathbb{C}$, provided* $(\mathbb{A}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, u)$ *is an ASHA.*

Theorem 3.23. $(\mathbb{C}, \underline{\curlyvee}, \curlywedge, \circledcirc, 0, u)$ *is a Boolean algebra, where $x \underline{\curlyvee} y = (x^{\circledcirc} \curlywedge y^{\circledcirc})^{\circledcirc}$, for all $x, y \in \mathbb{C}$, provided* $(\mathbb{A}, \curlyvee, \curlywedge, \twoheadrightarrow, 0, u)$ *is an ASHA.*

4. Conclusions

In this paper, we have introduced and studied closed elements in an almost semi-Heyting algebra and explored the fundamental characteristics of closed elements in terms of an implication within the framework of almost semi-Heyting algebras. We have studied the nature of closed elements in an almost semi-Heyting algebra as well as in a semi-Heyting almost distributive lattice in various aspects. Mainly, we derived a class of pseudo-complemented lattices, a class of Heyting algebras and a class of Boolean algebras in an almost semi-Heyting algebra. Finally, we proved that the class of closed elements in an almost semi-Heyting algebra forms a Boolean algebra.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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