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MODELLING AND OPTIMAL CONTROL ANALYSIS OF VIOLENCE AGAINST WOMEN WITH MEDIA IMPACT

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Abstract. Ensuring the safety and rights of women is essential, regardless of their backgrounds. It's about creating a fair and secure society for everyone. In this study, we construct a deterministic model that targets explicitly vulnerable women and those responsible for the violence. The model exhibits two equilibrium points, and we compute the basic reproduction number, denoted as $\mathcal{R}_0$. When $\mathcal{R}_0 < 1$, multiple endemic equilibrium points emerge, which we scrutinize using backward bifurcation analysis. Additionally, we conduct a sensitivity analysis to identify the most influential parameters affecting $\mathcal{R}_0$. Subsequently, we expand our model into an optimal control problem by introducing social media awareness campaigns and enhanced law enforcement interventions. Our findings indicate that raising awareness through campaigns and enforcing stricter laws can reduce violence against women.

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Key words and phrases. stability analysis; optimal control; Pontryagin's maximum principle; PRCC technique.

1. Introduction

The World Health Organization (WHO), in its report [24], characterizes violence against women as "any form of gender-based violence leading to, or having the potential to lead to, physical, sexual, or psychological harm or distress in women. This encompasses threats of such acts, forceful actions, or unjust restrictions on personal freedom, publicly or privately." The global prevalence of violence against women is alarmingly high, affecting approximately 35% of women worldwide who have encountered physical or sexual violence at some stage. However, prevalence data merely offers a glimpse of the issue's scope regarding its frequency within the population. A persistent concern pertains to cases of unreported domestic abuse, as highlighted in [8]. A significant factor contributing to under-reporting

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is that victims often perceive such incidents as private matters, as noted in [17]. It is worth noting that, in certain nations, women lack legal protection from domestic abuse, with statistics revealing that one out of every four countries does not have explicit legislation safeguarding women in such situations, as reported by United Nations women [1]. Moreover, the COVID-19 pandemic has presented further difficulties for women, leading to a sharp rise in global violence against them during the lockdowns that were introduced [22]. Several nations have put into effect measures, rules, and support systems aimed at tackling domestic violence against women, encompassing psychological, physical, or sexual abuse inflicted by their partners or other members of the family [11]. Studies have assessed the prevalence of physical and mental health symptoms in victims of violence against women [7,9]. Besides this, the empowerment of women is inevitable for achieving gender equality. Governments should provide critical resources and funding to combat violence against women and devise policies to make these resources available during physical distancing interventions. Health providers should locate locally accessible social resources for survivors and direct women to them when they need health care [18]. Law enforcement response to victims, criminals, and the activities of law enforcement agencies should be implemented to combat violence against women [10].

In recent years, there has been a growing utilization of mathematical modeling to gain insights into and forecast the dynamics of violence. Mathematical models offer a distinctive perspective by enabling researchers to quantify the interplay among various factors contributing to violence. These models also serve as testing grounds for diverse interventions and policies to mitigate violence. Numerous research studies have delved into violence-related topics. For instance, in one study [21], researchers found that augmenting negotiation and reconciliation rates can foster peaceful coexistence and reduce violence, regardless of whether negotiations are formally involved. Another study [19] highlighted that the most effective strategy for reducing violence involves a combination of preventive measures and de-radicalization efforts. Additionally, a mathematical model was employed to examine the coexistence of racism and corruption within specific societies [20]. The study in [2] delved into the dynamics of rape and strategies to control it, suggesting that a multifaceted approach comprising government interventions, parental guidance, and supportive environments can significantly curtail rape prevalence. Intimate partner violence has been scrutinized through mathematical modeling in [6], wherein authors developed a model employing a system of linear differential equations. Their central achievement lies in identifying pivotal triggers for intimate partner violence and incorporating them into their initial differential equation model. Despite its apparent simplicity, this model effectively replicates various scenarios of intimate partner violence and offers insights into potential intervention strategies. Furthermore, research presented in [16] grounds itself in a difference equation framework, featuring a probabilistic parameter. This research has uncovered an overlooked group of aggressors that official statistics fail to capture, highlighting a pressing issue often ignored until it becomes a serious concern.

Key contributing factors include the inadequate recognition of early abusive behaviors, social stigma, low self-esteem among victims, and a lack of essential resources, all of which deter women from reporting their abusive partners [12].

Drawing inspiration from these studies, our research focuses on a mathematical model aimed at raising awareness about violence against women. This model includes a dedicated compartment for an awareness population to educate the public about laws and societal actions related to violence against women. Our primary objective is to promote awareness and enforce laws to combat violence against women.

Firstly, Section 2 delves into the problem formation of violence against women. Next, Section 3 discusses the analysis of the model, followed by the derivation of equilibrium points in Section 4. In addition, sensitivity analysis is performed in Section 6. Moreover, optimal control analysis is handled in Section 7, and numerical simulation is performed to support the analytical findings in Section 8. Finally, the results are discussed and concluded in Section 9.

2. Problem Formation

The total women population, N_w, is subdivided into five sub-populations They are susceptible vulnerable, naive women population S_u, victimized vulnerable, naive women population V_u, aware population who aware of laws on violence against activities A, susceptible aware women population S_a, victimized aware women population V_a. So, $N_w = S_u + V_u + A + S_a + V_a$.

In the same way, the total perpetrators of violence N_p have been subdivided into two sub-populations: Namely, susceptible to becoming a perpetrator of violence against women S_p and perpetrator of violence against women V_p. So, $N_p = S_p + V_p$.

The vulnerable class is assumed to recruit susceptible and naive individuals solely through birth and/or immigration, at a rate of Λ_u. The interaction rate between the susceptible and naive, vulnerable women population and perpetrators of violence is denoted by β_1. The rate at which unaware women become aware, through any medium, is represented by ρ_u. In contrast, the victimized women population becomes susceptible to unaware women, due to their lack of awareness at a rate of γ_u.

Similarly, β_1 represents the interaction rate between the susceptible and aware women population and perpetrators of violence. The victimized knowledgeable women population is treated as a susceptible, aware population, at a rate of α_a. The perpetrators of violence are recruited at a rate of Λ_p, and β_2 denotes the interaction rate between the aware susceptible women population and perpetrators of violence against women. The removal rate of perpetrators due to incarceration or any similar form is denoted as ϕ_p.

Based on the above assumptions, a model has been formulated and is presented below. A description of the parameters for this system can be found in Table 1.

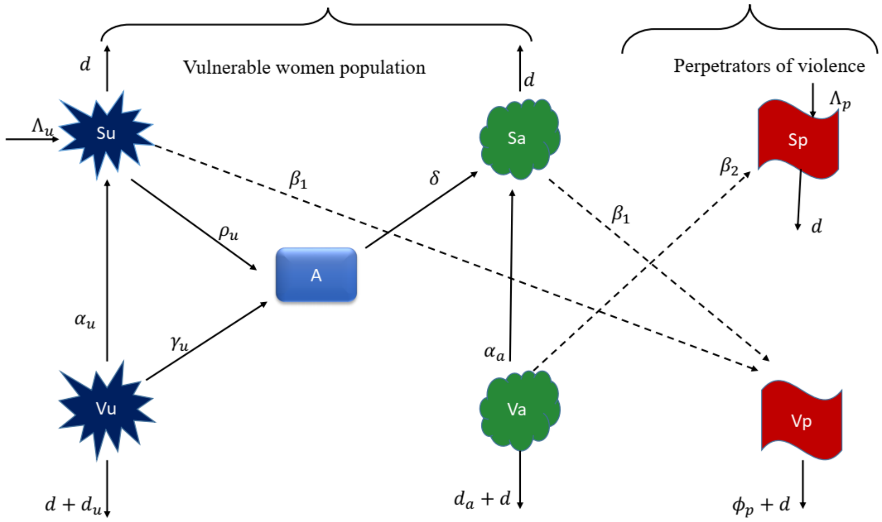

FIGURE 1. Schematic Diagram of Violence against women (1)

$$
\begin{aligned}
\frac{dS_u}{dt} &= \Lambda_u + \alpha_u V_u - \beta_1 V_p S_u - (d+\rho_u) S_u \\
\frac{dV_u}{dt} &= \beta_1 V_p S_u - (\alpha_u + \gamma_u + d_u + d) V_u \\
\frac{dA}{dt} &= \gamma_u V_u + \rho_u S_u - (d+\delta) A \\
\frac{dS_a}{dt} &= \delta A - \beta_1 V_p S_a + \alpha_a V_a - d S_a \\
\frac{dV_a}{dt} &= \beta_1 V_p S_a - (\alpha_a + d_a + d) V_a \\
\frac{dS_p}{dt} &= \Lambda_p - \beta_2 V_a S_p - d S_p \\
\frac{dV_p}{dt} &= \beta_2 V_a S_p - (\phi_p + d) V_p
\end{aligned}
\tag{1}
$$

The above model(1) has rewritten as follows:

$$
\begin{aligned}
\frac{dS_u}{dt} &= \Lambda_u + \alpha_u V_u - \beta_1 V_p S_u - k_1 S_u \\
\frac{dV_u}{dt} &= \beta_1 V_p S_u - k_2 V_u \\
\frac{dA}{dt} &= \gamma_u V_u + \rho_u S_u - k_3 A \\
\frac{dS_a}{dt} &= \delta A - \beta_1 V_p S_a + \alpha_a V_a - d S_a \\
\frac{dV_a}{dt} &= \beta_1 V_p S_a - k_4 V_a \\
\frac{dS_p}{dt} &= \Lambda_p - \beta_2 V_a S_p - d S_p \\
\frac{dV_p}{dt} &= \beta_2 V_a S_p - k_5 V_p
\end{aligned}
\tag{2}
$$

where, $k_1 = (d + \rho_u)$, $k_2 = (\alpha_u + \gamma_u + d_u + d)$, $k_3 = (d + \delta)$, $k_4 = (\alpha_a + d_a + d)$, $k_5 = (\phi_p + d)$

TABLE 1. Table of Parameters

Parameter	Description
Λ_u	Recruitment rate of vulnerable women
Λ_p	Recruitment rate of perpetrators of violence against women
α_u	Rate of transmission from V_u to S_u
β_1	Interaction of S_u and S_a with V_p
β_2	Contact rate between V_a and S_p
ρ_u	Rate of transmission from S_u to A
δ	Rate of transmission from A to S_a
γ_u	Rate of transmission from V_u to A
α_a	Rate of transmission from V_a to S_a
ϕ_p	Removal rate of perpetrators due to police arrest or in any other similar form
d	Natural death rate of women
d_u	Rate of death due to violence on unaware victim population
d_a	Rate of death due to violence on aware victim population

3. Analysis of the Model (1)

3.1. **Boundedness of the solutions.** In this system, we have assumed two types of population: women and perpetrators of violence. Here, the boundedness of the population of vulnerable women and the population of perpetrators of violence has shown. Since the whole women population has assumed as N_w then $N_w = S_u + V_u + A + S_a + V_a$,

$\frac{dN_w}{dt} \le \Lambda_u - dN_w$,

Integrating on both sides we get,

$N_w(t) \le \frac{\Lambda_u}{d} + Ce^{-dt}$

when $t = 0$, $N_w(t) \le \frac{\Lambda_u}{d} + C$.

$N_w(t) \le \frac{\Lambda_u}{d}(1 - e^{-dt}) + N_w(0)e^{-dt}$

when $t \to \infty$ $N_w(t) \le \frac{\Lambda_u}{d}$. Therefore, $N_w(t)$ is bounded above. By applying a similar procedure to the perpetrators of violence population $N_p(t) = S_p(t) + V_p(t)$, we get $N_p(t) \le \frac{\Lambda_p}{d}$.

4. Existence of Equilibrium

The model(1) has exhibited two equilibria: violence-free equilibrium point and violence present equilibrium point.

4.1. **Violence-free equilibrium solution.** Here, violence free equilibrium; $V_u = V_p = 0$ Then the model 2 become,

$$\begin{aligned} \Lambda_u - (d+\rho_u)S_u &= 0 \\ \rho_u S_u - (d+\delta)A &= 0 \\ \delta A - dS_a &= 0 \\ \Lambda_p - dS_p &= 0 \end{aligned} \tag{3}$$

It gives, the violence free equilibrium $\mathcal{E}^0$, where $S_u^0 = \frac{\Lambda_u}{(d+\rho_u)}$,
$A^0 = \frac{\rho_u \Lambda_u}{(d+\rho_u)(\alpha_u+\gamma_u+d_u+d)}$, $S_a^0 = \frac{\delta\rho_u\Lambda_u}{(d+\rho_u)(\alpha_u+\gamma_u+d_u+d)d}$, $S_p^0 = \frac{\Lambda_p}{d}$.

4.2. **Basic reproduction number.** The basic reproduction number $\mathcal{R}_0$ has been determined using the Next generation matrix method [23]. The threshold parameter $\mathcal{R}_0$ has defined as basic reproduction ratio is defined as the number of violent behaviors carried out by a single perpetrator of violence against women in his whole period of life.

$$\mathcal{F}(t) = \begin{pmatrix} \beta_1 V_p S_u \\ \beta_1 V_p S_a \\ \beta_2 V_a S_p \end{pmatrix}$$

$$\mathcal{V}(t) = \begin{pmatrix} (\alpha_u+\gamma_u+d_u+d)V_u \\ (\alpha_a+d_a+d)V_a \\ (\phi_p+d)V_p \end{pmatrix}$$

where $\mathcal{F}(t)$ is the new infection terms and $\mathcal{V}(t)$ is the transition terms of the system (1). The Jacobian matrix of $\mathcal{F}(t)$ and $\mathcal{V}(t)$ are denoted by F and V respectively evaluated at $\mathcal{E}_0$. The spectral radius of the matrix FV^{-1} is called the basic reproduction number $\mathcal{R}_0$ of the system (1).

$$\mathcal{R}_0 = \sqrt{\frac{\beta_1\beta_2\Lambda_p\Lambda_u\delta\rho_u}{(d+\rho_u)(\alpha_u+\gamma_u+d_u+d)(\alpha_a+d_a+d)(\phi_p+d)d^2}} \tag{4}$$

4.3. **Violence present equilibrium solution.** To get the violence present equilibrium solution $\mathcal{E}^*$, set the right-hand side of (2)equal to zero.

$$\begin{aligned} \Lambda_u + \alpha_u V_u - \beta_1 V_p S_u - k_1 S_u &= 0 \\ \beta_1 V_p S_u - k_2 V_u &= 0 \\ \gamma_u V_u + \rho_u S_u - k_3 A &= 0 \\ \delta A - \beta_1 V_p S_a + \alpha_a V_a - dS_a &= 0 \end{aligned} \tag{5}$$

$$\beta_1 V_p S_a - k_4 V_a = 0$$

$$\Lambda_p - \beta_2 V_a S_p - dS_p = 0$$

$$\beta_2 V_a S_p - k_5 V_p = 0$$

Then we get, $\mathcal{E}^*(S_u^*, V_u^*, A^*, S_a^*, V_a^*, S_p^*, V_p^*)$. Where
$S_u^* = \frac{k_2\Lambda_u}{(k_1k_2+(\gamma_u+d_u+d)\beta_1V_p^*)}$,
$V_u^* = \frac{\Lambda_u\beta_1V_p^*}{k_1k_2+(\gamma_u+d_u+d)\beta_1V_p^*}$,
$A^* = \frac{\Lambda_u(\beta_1V_p^*\gamma_u+\rho_uk_2)}{k_3(k_1k_2+(\gamma_u+d_u+d)\beta_1V_p^*)}$
$S_a^* = \frac{k_4\delta\Lambda_u(\beta_1V_p^*\gamma_u+\rho_uk_2)}{k_3(k_1k_2+(\gamma_u+d_u+d)\beta_1V_p^*)((d_a+d)\beta_1V_p^*+k_4d)}$
$V_a^* = \frac{\delta\Lambda_u(\beta_1V_p^*\gamma_u+\rho_uk_2)\beta_1V_p^*}{k_3(k_1k_2+(\gamma_u+d_u+d)\beta_1V_p^*)((d_a+d)\beta_1V_p^*+k_4d)}$
$S_p^* = \frac{\Lambda_p}{d+\beta_2V_a^*}$
by substituing the value of S_p^* into last equation of (5) we get the quadratic equation interms of V_p^*. To get the violence present equilibrium point we need to solve this quadratic equation $F_2(V_p^*)^2+F_1(V_p^*)+F_0 = 0$ where
$F_2 = -k_5k_3d(\gamma_u + d_u + d)(d_A + d)\beta_1^2 - \beta_2\delta\Lambda_u\beta_1^2\gamma_uk_5$,
$F_1 = \beta_2\delta\Lambda_u\gamma_u\Lambda_p\beta_1^2 - \beta_2\delta\Lambda_u\beta_1\rho_uk_5k_2 - k_5dk_3k_2k_1(d_a + d)\beta_1 - k_5dk_3(\gamma_u + d_u + d)k_4d\beta_1$,
$F_0 = k_2(\mathcal{R}_0^2 - 1)$. Using Descart's rule for signs, we get the following Table 2

Table 2. Existence of positive roots depends on value of $\mathcal{R}_0$

Cases	F_2	F_1	F_0	$\mathcal{R}_0$ value	No.of sign changes	No.of positive roots
1	-	-	+	$\mathcal{R}_0 > 1$	1	Unique
2	-	+	+	$\mathcal{R}_0 > 1$	1	Unique
3	-	-	-	$\mathcal{R}_0 < 1$	0	No root
4	-	+	-	$\mathcal{R}_0 < 1$	2	Two roots

This shows that the existence of multiple equilibria when $\mathcal{R}_0 < 1$. It leads to backward bifurcation analysis.

5. Bifurcation Analysis

To study the bifurcation analysis we assume the state variables as $S_u = z_1$, $V_u = z_2$, $A = z_3$, $S_a = z_4$, $V_a = z_5$, $S_p = z_6$, $V_p = z_7$ which leads to the system (2) as $\frac{dZ}{dt} = (h_1, h_2, h_3, h_4, h_5, h_6, h_7)^T$

$$\begin{aligned}\frac{dz_1}{dt} &= \Lambda_u + \alpha_u z_2 - \beta_1 z_7 z_1 - k_1 z_1\\ \frac{dz_2}{dt} &= \beta_1 z_7 z_1 - k_2 z_2\\ \frac{dz_3}{dt} &= \gamma_u z_2 + \rho_u z_1 - k_3 z_3\end{aligned} \tag{6}$$

$$\frac{dz_4}{dt} = \delta z_3 - \beta_1 z_7 z_4 + \alpha_a z_5 - dz_4$$
$$\frac{dz_5}{dt} = \beta_1 z_7 z_4 - k_4 z_5$$
$$\frac{dz_6}{dt} = \Lambda_p - \beta_2 z_5 z_6 - dz_6$$
$$\frac{dz_7}{dt} = \beta_2 z_5 z_6 - k_5 z_7$$

Here, we consider β_1 as a bifurcation parameter when $\mathcal{R}_0 = 1$ is also a more sensitive parameter than others. Solving for β_1 we get,

$$\beta_1^* = \frac{k_1 k_2 k_4 k_5 d^2}{\beta_2 \Lambda_p \Lambda_u \delta \rho_u}$$

The right and left eigenvalues corresponding to zero eigen vector calculated from the Jacobian matrix of the system (6) at $(\mathcal{E}_0, \beta_1^*)$, as follows $\omega = (\omega_1, \omega_2, \omega_3)^T$, where $\omega_1 = \frac{k_3 k_4 k_5 d^2 (\rho_u + d_u + d)}{\beta_2 \Lambda_p \delta \rho_u k_1 k_2}$, $\omega_2 = \frac{d^2 k_3 k_4 k_5}{\beta_2 \Lambda_p \delta \rho_u k_2}$ and $\omega_3 = \frac{d^2 k_4 k_5 (\gamma_u + \rho_u)}{\beta_2 \Lambda_p \Lambda_u \delta \rho_u k_2}$, $\omega_4 = \frac{k_4 k_5 (d\gamma_u - d_a \rho_u)}{\Lambda_p \rho_u \beta_2}$, $\omega_5 = \frac{dk_5}{\Lambda_p \beta_2}$, $\omega_6 = \frac{-k_5}{d}$, $\omega_7 = 1$ and $\nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5, \nu_6, \nu_7)^T$ where $\nu_1 = 0$,$\nu_2 = 0$,$\nu_3 = 0$,$\nu_4 = 0$,$\nu_5 = \frac{\Lambda_p \beta_2}{dk_4}$,$\nu_6 = 0$,$\nu_7 = 1$ respectively. Now we employ the following theorem [4] to study its stability:

Theorem 5.1. *Consider the following general system of ordinary differential equations with a parameter β_1^*, $\frac{dx}{dt} = f(x, \beta_1^*)$, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and $f \in \mathbb{C}^2(\mathbb{R}^n \times \mathbb{R})$,*
where 0 is an equilibrium of the system that is $f(0, \beta_1^*) = 0\ \forall$
$beta_1^*$ and assume:

- *i $D = D_x f(0,0)$=$(\frac{\partial f_i}{\partial x_j})$ is the linearization matrix of the system around the equilibrium 0 with f evaluated at 0;*
- *ii Zero is a simple eigenvalue of D and other eigenvalue of D have negative real parts;*
- *iii Matrix D has a right eigenvector ωand a left eigenvector ν corresponding to the zero eigenvalue.*

Let f_k bet the k^{th} component of f and

$$a^* = \sum_{k,i,j=1}^{n} \nu_k \omega_i \omega_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0),\ b^* = \sum_{k,i=1}^{n} \nu_k \omega_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_1^*}(0,0)$$

then the local dynamics of the system around the equilibrium point 0 is totally determined by the signs of a^* and b^*.

5.1. Computation of a^* and b^*.

$$a^* = 2\nu_5 \omega_4 \omega_7 \beta_1 + 2\nu_7 \omega_5 \omega_6 \beta_2$$
$$b^* = \nu_5 \omega_7 \frac{\rho_u \Lambda_u}{k_1 k_2}$$

It shows that $a^* > 0$ and $b^* > 0$ when $\frac{\beta_1^* d\gamma_u}{\rho_u} > d_a \beta_1^* + \frac{k_5}{\Lambda_p}$. Therefore, by [4] Theorem 5.1, the model undergoes backward bifurcation.

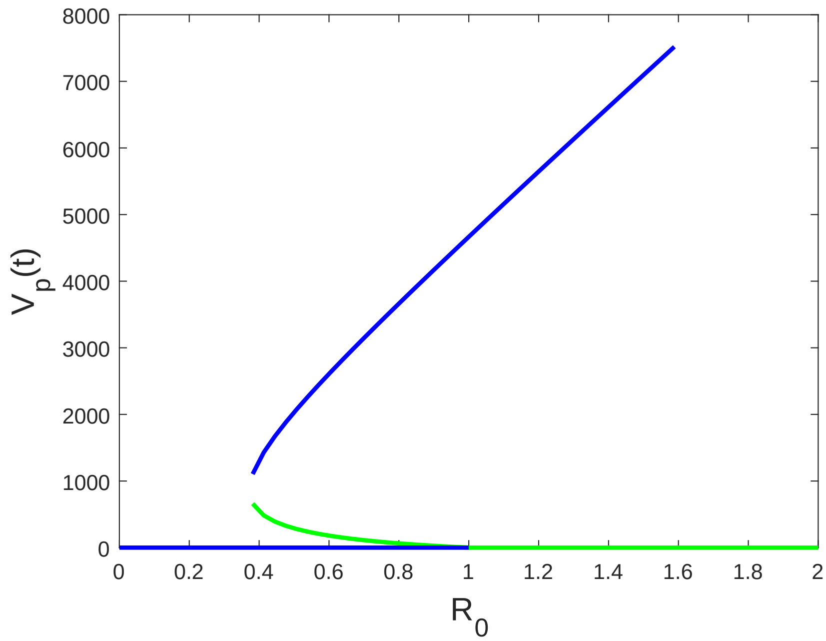

FIGURE 2. Existence of backward bifurcation of the model (2) when $\phi_p = 0.0004; \alpha_a = 0.3; \beta_1 = 0.00015; \beta_2 = 0.000019; d = 0.129; d_a = 0.21; \rho_u = 0.1101; \delta = 0.004257; \Lambda_p = 1000; \Lambda_u = 200000; d_u = 0.02826; \gamma_u = 0.4178; \alpha_u = 0.237;$

6. Sensitivity Analysis

In this section, sensitivity analysis for the (threshold parameter) basic reproduction number $\mathcal{R}_0$ has been conducted. It is used to identify parameters with a significant impact on $\mathcal{R}_0$ that should be focused on by intervention strategies. Sensitivity indices help analyze a variable's relative change whenever a parameter changes. Here, we use the forward sensitivity index of a variable with respect to a given parameter, which will be calculated as the proportion of the variable's relative change to the parameter's relative change. A variable involved in $\mathcal{R}_A$ is partially differentiable with respect to the parameter, then the sensitivity index is defined using partial derivatives [5]. The formula for the forward sensitivity index is

$$\Gamma_{\alpha}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \alpha} \frac{\alpha}{\mathcal{R}_0}$$

and it is defined for the parameter Γ and it is denoted by $\Gamma_{\alpha}^{\mathcal{R}_0}$. Now we find the forward sensitivity index for the parameters involved in $\mathcal{R}_0$.

Also, we performed a sensitivity analysis using Latin hypercube sampling (LHS) and partial rank correlation coefficients (PRCC) to evaluate the influence of uncertainty and the sensitivity of the numerical simulation results to variations in each model parameter [3, 14]. To produce the LHS matrices, It has assumed that almost all the model parameters are distributed uniformly within the specified range. The partial rank correlation coefficients (PRCCs) between the parameters β_1, β_2, ϕ_p, α_a,

d, α_u,d_a,ρ_u,δ, Λ_p, Λ_u,d_u, γ_u of the system (1) are computed. Also, it has fixed $\pm 30\%$ as the baseline for the respective parameters and generated 1000 random samples from the chosen parameter distributions. The bar diagram of the PRCC values is portrayed in 3. Here the parameters β_1, β_2, δ, Λ_p, Λ_u and ρ_u have positive indices with $\mathcal{R}_0$ whereas ϕ_p, α_a, d ,d_a,d_u, γ_u and α_u have negative indices with $\mathcal{R}_0$. It has shown in the Figure 3.

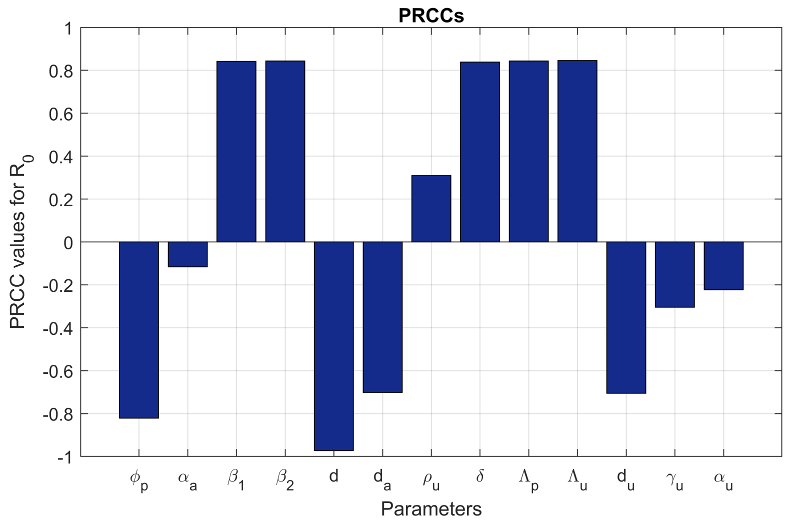

FIGURE 3. Normalized forward sensitivity indices of $\mathcal{R}_0$ with respect to model parameters with the assumed base values $\phi_p = 0.265$; $\alpha_a = 0.00385$; $\beta_1 = 0.0321$; $\beta_2 = 0.0235$; $d = 0.0143$; $d_a = 0.034$; $\rho_u = 0.057$; $\delta = 0.36$; $\Lambda_p = 100$; $\Lambda_u = 1000$; $d_u = 0.248$; $\gamma_u = 0.0678$; $\alpha_u = 0.0538$;

7. Optimal control analysis

Here, the system has extended to optimal control problems. Optimal control theory is one of the finest tools to reach this goal. It is helpful to minimize the total number of victims over a finite interval of time at a minimal cost of effort. we assume two control measures $b_1(t)$ and $b_2(t)$.

Control variable $b_1(t)$: Social media awareness helps increase awareness of susceptible, vulnerable, unaware women population. Also, with the help of this control, violent activities against women will be controlled over a period of time.

Control variable $b_2(t)$: The control measure $b_2(t)$ represents the strong law implementation in the society, which helps reduce the crime rate.

The parameters $mb_1(t)$ and $lb_2(t)$ are additional time-dependent parameters.

$$\frac{dS_u}{dt} = \Lambda_u + \alpha_u V_u - \beta_1 V_p S_u - (d + \rho_u) S_u - \underbrace{mb_1(t) S_u}_{\textit{Media awareness on } S_u \textit{ population}}$$

$$\frac{dV_u}{dt} = \beta_1 V_p S_u - (\alpha_u + \gamma_u + d_u + d)V_u$$
$$\frac{dA}{dt} = \gamma_u V_u + \rho_u S_u - (d+\delta)A + mb_1(t)S_u$$
$$\frac{dS_a}{dt} = \delta A - \beta_1 V_p S_a + \alpha_a V_a - dS_a$$
$$\frac{dV_a}{dt} = \beta_1 V_p S_a - (\alpha_a + d_a + d)V_a$$
$$\frac{dS_p}{dt} = \Lambda_p - \beta_2 V_a S_p - dS_p - \underbrace{lb_2(t)S_p}_{Law\ enforcement\ on\ S_p\ population}$$
$$\frac{dV_p}{dt} = \beta_2 V_a S_p - (\phi_p + d)V_p + lb_2(t)S_p$$

7.1. **The Optimal Control Problem.** This section deals with the behavior of the given system by using optimal control theory. We formulated an optimal control problem with the fixed time t_f of objective functional is given by

$$J = \int_0^{t_f} (B_1 S_u + B_2 S_p + B_3 \frac{b_1^2}{2} + B_4 \frac{b_2^2}{2}) \quad (7)$$

Where the weight constants B_1, B_2, B_3 and B_4 are non-negative. Here B_3 represents the cost associated with spreading awareness of susceptible vulnerable women through media. B_4 describes the cost associated with the additional efforts to increase the law enforcement on the perpetrator population. subject to the state system given by (7).

Our aim is to find a control b_1 and b_2 such that

$$J(b_1^*, b_2^*) = \min_{b_1, b_2 \in \Omega} J(b_1, b_2) \quad (8)$$

where $\Omega = \{b_1, b_2 : \text{measurable and } 0 \leq b_1(t) \leq 1, 0 \leq b_2(t) \leq 1 \text{ for } t \in [0, t_f]\}$ is the set for the controls.

7.2. **Hamiltionian and Adjoint Equations.** The Lagrangian of this problem is given by

$$(S_u, S_p, b_1, b_2) = B_1 A + B_2 I + B_3 \tfrac{b_1^2}{2} + B_4 \tfrac{b_2^2}{2}.$$

Hamiltonian form $\mathcal{H}$ of this problem is given by

$$\begin{aligned}\mathcal{H} = \quad & B_1 S_u + c_2 S_p + \frac{1}{2} B_3 b_1^2 + \frac{1}{2} B_4 b_2^2 + \lambda_1 [\Lambda_u + \alpha_u V_u - \beta_1 V_p S_u - k_1 S_u - mb_1 S_u] \\ & + \lambda_2 [\beta_1 V_p S_u - k_2 V_u] + \lambda_3 [\gamma_u V_u + \rho_u S_u - k_3 A + mb_1 S_u] \\ & + \lambda_4 [\delta A - \beta_1 V_p S_a + \alpha_a V_a - dS_a] + \lambda_5 [\beta_1 V_p S_a - k_4 V_a] \\ & + \lambda_6 [\Lambda_p - \beta_2 V_a S_p - dS_p - lb_2 S_p] + \lambda_7 [\beta_2 V_p S_p - k_5 V_p + lb_2 S_p]\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial S_u} &= B_1 + \lambda_1 \left[-\beta_1 V_p - k_1 - mb_1\right] + \lambda_2 \left[\beta_1 V_p\right] + \lambda_3 \left[\rho_u + mb_1 S_u\right] \\
\frac{\partial \mathcal{H}}{\partial V_u} &= \lambda_1 \left[\alpha_u\right] + \lambda_2 \left[-k_2\right] + \lambda_3 \left[\gamma_u\right] \\
\frac{\partial \mathcal{H}}{\partial A} &= \lambda_3 \left[-k_3\right] + \lambda_4 \left[\delta\right] \\
\frac{\partial \mathcal{H}}{\partial S_a} &= \lambda_4 \left[\beta_1 V_p - d\right] + \lambda_5 \left[\beta_1 V_p\right] \\
\frac{\partial \mathcal{H}}{\partial V_a} &= \lambda_4 \left[\alpha_a\right] + \lambda_5 \left[-k_4\right] + \lambda_6 \left[-\beta_2 S_p\right] + \lambda_7 \left[\beta_2 S_p\right] && (9) \\
\frac{\partial \mathcal{H}}{\partial S_p} &= B_2 + \lambda_6 \left[-\beta_2 V_a\right] + \lambda_7 \left[\beta_2 V_p + lb_2\right] \\
\frac{\partial \mathcal{H}}{\partial V_p} &= \lambda_1 \left[-\beta_1 S_u\right] + \lambda_2 \left[\beta_1 S_u\right] + \lambda_4 \left[-\beta_1 S_a\right] + \lambda_5 \left[\beta_1 S_a\right] + \lambda_7 \left[-k_5\right]
\end{aligned}$$

(10)

satisfying the transversality condition $\lambda_i(T) = 0$, for i=1,2...7. Let $\widetilde{S_u}$, $\widetilde{V_u}$, $\widetilde{A}$,$\widetilde{S_a}$, $\widetilde{V_a}$, $\widetilde{S_p}$ and $\widetilde{V_p}$ be the solution of the corresponding state system (7). Also let $\{\widetilde{\lambda_1}, \widetilde{\lambda_2}, \widetilde{\lambda_3}, \widetilde{\lambda_4}, \lambda_5, \lambda_6, \lambda_7\}$ be the solution (7).

7.3. **Optimal Control Theorems.**

Theorem 7.1. *There exist optimal controls* b_1^* , $b_2^* \in \Omega$ *such that* $J(b_1^*, b_2^*) = \min_{b_1, b_2 \in \Omega} J(b_1, b_2)$ *subject to the system* (7)

Proof. we adopted a method used in [13] to solve this theorem. From the observation, we easily conclude that all the control variables and state variables are non-negative. Also, the necessary convexity of our objective functional in b_1 and b_2 is satisfied for this minimizing problem. Also, we can conclude that the control variable set $b_1, b_2 \in \Omega$ is closed and convex. The necessary condition for the existence of optimal control is in the system (7) with control parameters b_1 and b_2 applied is bounded, which brings the condition of compactness. In addition, the integrand in the functional (7), $B_1 S_u + B_2 S_p + B_4 \frac{b_1^2}{2} + B_4 \frac{b_2^2}{2}$ is convex on the control set Ω and state variables are bounded. Hence, the theorem. □

Since there exists an optimal control for minimizing the functional subject to equations (7) and transversality condition, Pontryagin's Maximum Principle to derive the necessary conditions to find the optimal solution as follows: Let (x, u) be an optimal solution of an optimal control problem. Then there exists a non trivial vector function $\lambda = (\lambda_1, \lambda_2.....\lambda_n)$ satisfying the following equalities.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\partial \mathcal{H}(t, x, u, \lambda)}{\partial \lambda} \\
0 &= \frac{\partial \mathcal{H}(t, x, u, \lambda)}{\partial u} && (11) \\
\frac{d\lambda}{dt} &= -\frac{\partial \mathcal{H}(t, x, u, \lambda)}{\partial x}
\end{aligned}$$

by using Pontryagin's Maximum Principle [15] and the theorem (7.1), we now state and prove the following theorem.

Theorem 7.2. *The optimal controls b_1^* ,b_2^* minimizes J over the region Ω given by*

$$b_1^* = \quad max\{0, min(\widetilde{b_1}, 1)\}$$
$$b_2^* = \quad max\{0, min(\widetilde{b_2}, 1)\}$$

where

$$\widetilde{\beta_2} = \frac{(\lambda_1 - \lambda_3)mS_u}{B_3}$$
$$\widetilde{u} = \frac{(\lambda_6 - \lambda_7)lS_p}{B_4}$$

Proof. Using the optimality condition $\frac{\partial \mathcal{H}}{\partial b_1} = 0$ and $\frac{\partial \mathcal{H}}{\partial b_2} = 0$. We get $\widetilde{\beta_2} = \frac{(\lambda_1 - \lambda_3)mS_u}{B_3}$, $\widetilde{u} = \frac{(\lambda_6 - \lambda_7)lS_p}{B_4}$ These controls are bounded, and the upper and lower bounds are 0 and 1, respectively. $b_1 = 0$ if $\widetilde{b_1} < 0$ and $b_1 = 1$ if $\widetilde{b_1} > 1$ and $b_1 = 0$ if $\widetilde{b_2} < 0$ and $b_2 = 1$ if $\widetilde{b_2} > 1$ otherwise $b_1 = \widetilde{b_1}$ and $b_2 = \widetilde{b_2}$. Hence for this controls (b_1^*) and (b_2^*) we get optimum value of the functional J given by equation (7). Hence, the theorem. □

8. Simulation

In this section, we performed numerical simulation for the system (1). The stability of both the equilibrium points has shown in Figures 4 and 5 respectively. For a violence-free equilibrium point, the parameter values are $\phi_p = 0.0004; \alpha_a = 0.3; \beta_1 = 0.00015; \beta_2 = 0.000019; d = 0.129; d_a = 0.21; \rho_u = 0.1101; \delta = 0.004257; \Lambda_p = 1000; \Lambda_u = 200000; d_u = 0.02826; \gamma_u = 0.4178; \alpha_u = 0.237;$ and $\mathcal{R}_0 = 0.6225 < 1$. For the violence present equilibrium point, the parameter values are $\Lambda_u = 5000, \Lambda_p = 500, \alpha_u = 0.00538, \beta_1 = 0.00091, \beta_2 = 0.000235, \gamma_u = 0.778, \rho_u = 0.077, \delta = 0.07, d_u = 0.248, d = 0.143, d_a = 0.34, \phi_p = 0.765.$ and $\mathcal{R}_0 = 1.069 > 1$.These parameter values have been used for the analysis of optimal control simulations. In our study, we have implemented two crucial control measures, denoted as b_1 and b_2, designed to address the issues of violence against women in our society. The graphical representation of the control profiles is depicted in Figure 6, and it provides valuable insights into the dynamics of these control measures. The control profile $b_1(t)$ exhibits an intriguing pattern. It initiates with its highest value, 1, and this state is maintained for approximately 10 years. However, as the 28-year control period nears its expiration, $b_1(t)$ gradually diminishes, ultimately reaching a value of zero. On the other hand, the control profile $b_2(t)$ takes a different approach. It peaks at its maximum value for approximately 20 years, and then, for the 40-year control period, it gradually diminishes to zero. This contrast in control profiles is strategically designed to work in tandem to create a society that is free from violence against women. To rigorously assess the influence of these control profiles, we conducted simulations that compare the outcomes in various populations, both with and without the effect of these control measures. The results of these simulations are illustrated in Figures 7 through 13, and they provide compelling evidence of the effectiveness of these controls on each population. With the implementation of control measures, we observed a significant reduction in the susceptibility of violence against vulnerable women over a span of approximately 2 years, and this reduction is sustained when compared to scenarios without these controls (refer to Figure 7). In the population of victimized vulnerable women, the implementation of control measures resulted in a notable decrease in the number of victims over a period of approximately 2 years, as opposed to the scenario without these controls (see Figure 8). Furthermore, the population of aware individuals also experienced positive growth over a period of approximately 6 years, as illustrated in Figure 9. The impact of these control measures extends beyond the vulnerable population. Susceptibility to violence in susceptible, aware women decreased over a period of approximately 3 years (Figure 10), while the population of aware, victimized women experienced a significant reduction (Figure 11). After 15 years, the victims of violence against women have completely eradicated. Notably, the population susceptible to becoming perpetrators of violence against women witnessed a decrease in their propensity over approximately one year (Figure 12). Similarly, the population of perpetrators of violence against women decreased over 3 years (approximately) (Figure 13) and produced better results to eradicate violence in society. These findings underscore the profound impact of the control measures b_1 and b_2 on mitigating violence against women and demonstrate the effectiveness of our approach in creating a safer and more equitable society.

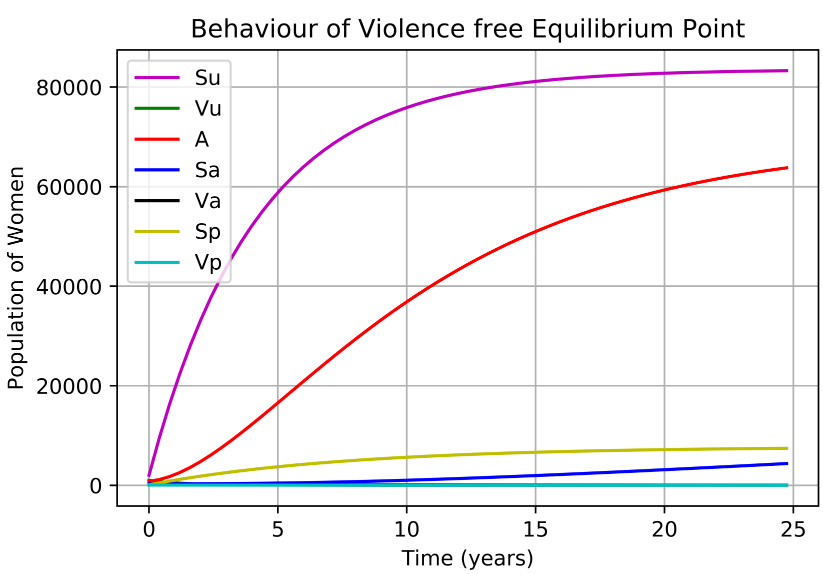

FIGURE 4. Behavior of violence free equilibrium point of the system (1) for the parameter values $\phi_p = 0.0004; \alpha_a = 0.3; \beta_1 = 0.00015; \beta_2 = 0.000019; d = 0.129; d_a = 0.21; \rho_u = 0.1101; \delta = 0.004257; \Lambda_p = 1000; \Lambda_u = 200000; d_u = 0.02826; \gamma_u = 0.4178; \alpha_u = 0.237;$

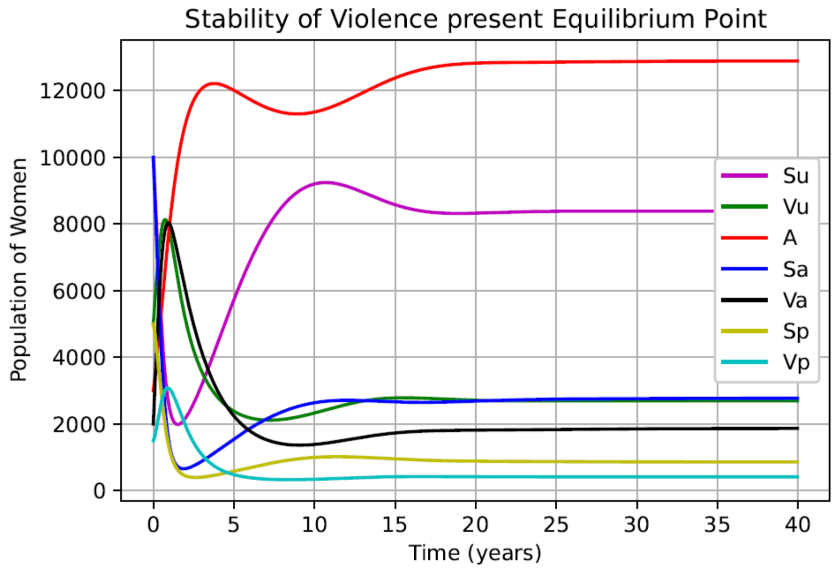

Figure 5. Behavior of violence present equilibrium point of the system (1) for the parameter values $\Lambda_u = 5000, \Lambda_p = 500, \alpha_u = 0.00538, \beta_1 = 0.00091, \beta_2 = 0.000235, \gamma_u = 0.778, \rho_u = 0.077, \delta = 0.07, d_u = 0.248, d = 0.143, d_a = 0.34, \phi_p = 0.765$.

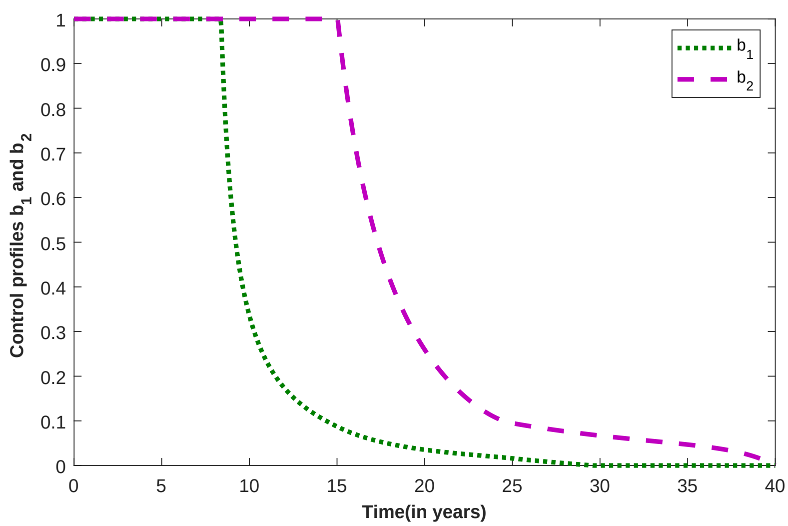

Figure 6. Control profiles of optimal controls b_1 and b_2.

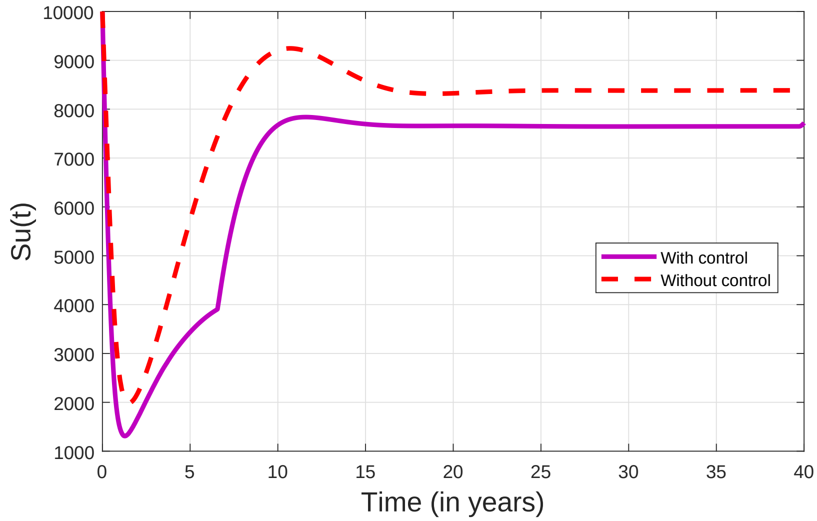

FIGURE 7. The effect with and without optimal control on S_u

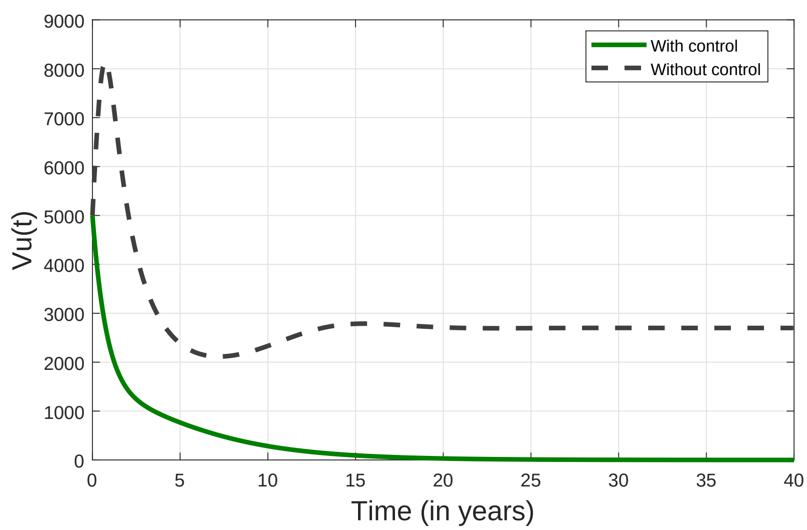

FIGURE 8. The effect with and without optimal control on V_u

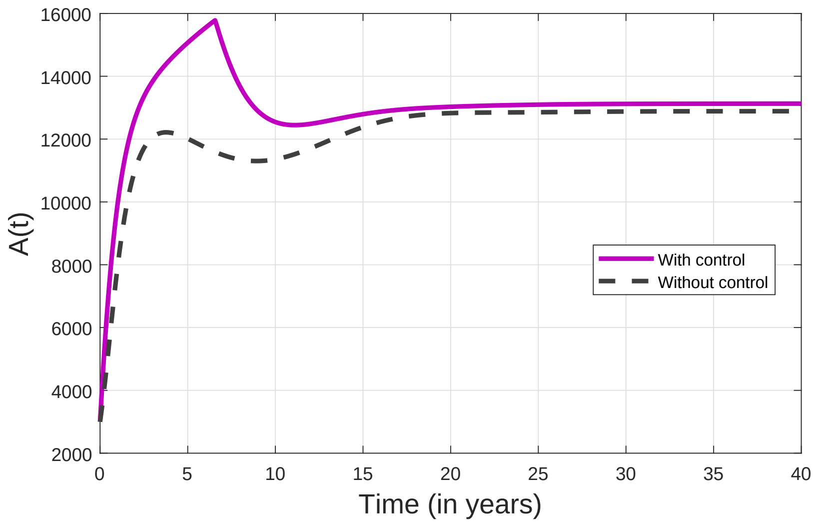

FIGURE 9. The effect with and without optimal control on A

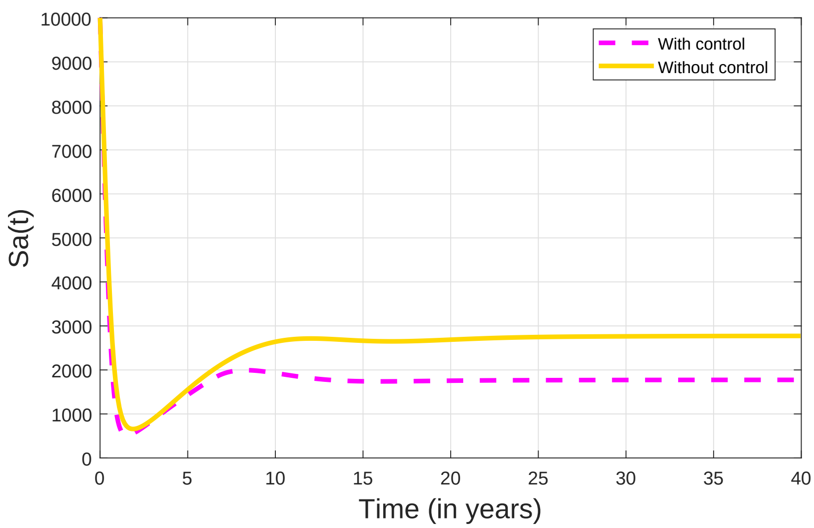

FIGURE 10. The effect with and without optimal control on S_a

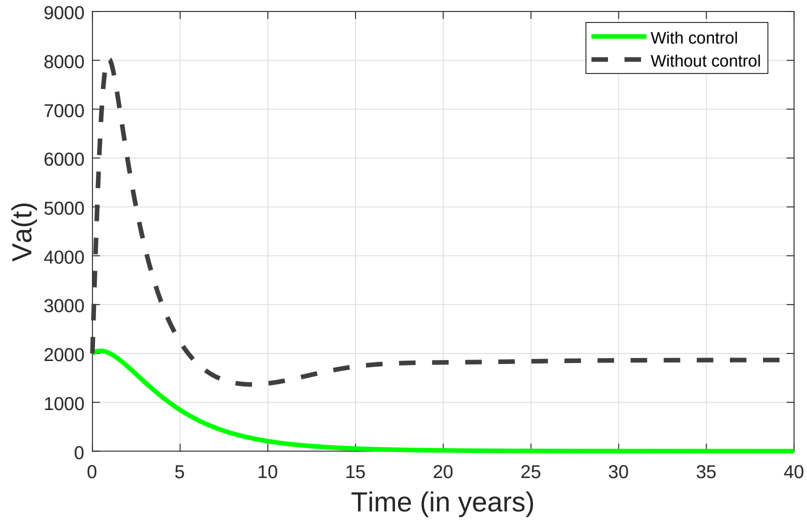

FIGURE 11. The effect with and without optimal control on V_a

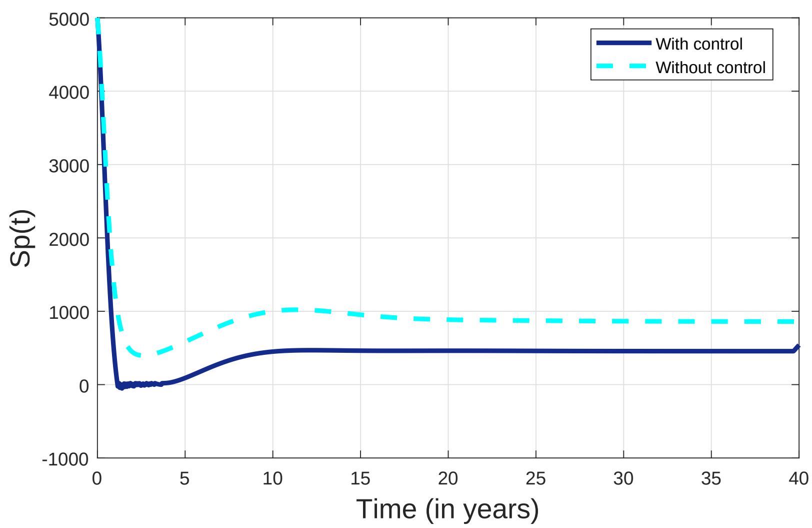

FIGURE 12. The effect with and without optimal control on S_p

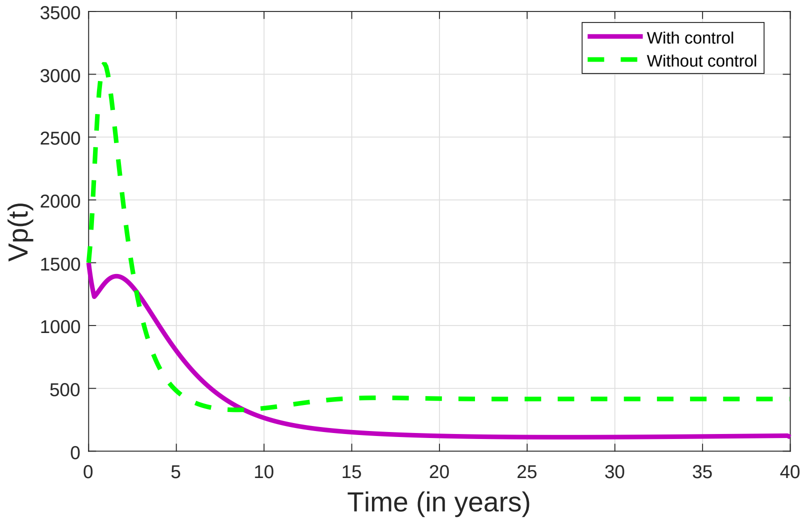

FIGURE 13. The effect with and without optimal control on V_p

9. Results and Discussions

In this paper, we have framed a model using the major compartments as the population of vulnerable women and the population of perpetrators of violence population. There are five sub-populations in the vulnerable women population: the naive susceptible women population, the naive victimized women population, the aware population, the knowledgeable women population, and the victimized knowledgeable women population. Similarly, there are two sub-populations in the perpetrators of violence population, namely susceptible to becoming a perpetrator of violence against women and perpetrator of violence against women. The basic reproduction number has been found for the system (1) to investigate the spread of violence against women in society. Moreover, sensitivity analysis has been done with the help of PRCC (Partial Rank Correlation Coefficient) and LHS (Latin Hypercube Sampling) techniques. It produces the sensitive parameter of this model. β_1, β_2, δ, Λ_p, Λ_u and ρ_u are positively effects on $\mathcal{R}_0$. In the existence equilibrium solutions, multiple endemic equilibrium points exist when $\mathcal{R}_0 < 1$, Which leads to backward bifurcation analysis. Further, we extend our model to optimal control problem by introducing two control measures, social media awareness $b_1(t)$ on naive, susceptible women population and law enforcement on susceptible to become perpetrators of violence $b_2(t)$. And performed the numerical simulation with and without the use of control measures. It shows better results in reducing the victimized populations and perpetrators of violence against women through increasing the awareness of the women population. These two control measures played a vital role in controlling the violence in the women population of our model analysis. Finally, governments should introduce awareness programs and enforce laws to reduce violence against women cases and take strong measures to address the issue. Reducing violence against women is a crucial step in creating a more equitable and just society for all people.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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